

Biextensions, ring-like stacks, and their classification

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Categorical rings, informally

A **categorical ring** \mathcal{R} consists of:

- 1 A symmetric monoidal structure $(\mathcal{R}, \boxplus, c, 0_{\mathcal{R}})$
- 2 Group-like:

$$x \boxplus -, - \boxplus x: \mathcal{R} \longrightarrow \mathcal{R}$$

are equivalences for each object x of \mathcal{R}

- 3 $(\mathcal{R}, \boxtimes, 1_{\mathcal{R}})$ second monoidal structure, **distributive** over \boxplus :

$$\lambda_{x,y;z}^1: (x \boxplus y) \boxtimes z \xrightarrow{\sim} (x \boxtimes z) \boxplus (y \boxtimes z)$$

$$\lambda_{x;y,z}^2: x \boxtimes (y \boxplus z) \xrightarrow{\sim} (x \boxtimes y) \boxplus (x \boxtimes z)$$

\mathcal{R} is a **Picard groupoid** (with respect to \boxplus)

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Distributor isomorphisms must be **compatible**

Categorical rings, informally

Compatibility

$$\begin{array}{ccc}
 & \lambda_{x,y;z\boxplus t}^1 & \\
 & \swarrow & \searrow \\
 (x \boxplus y) \boxtimes (z \boxplus t) & & \\
 & \swarrow & \searrow \\
 (x \boxtimes (z \boxplus t)) \boxplus (y \boxtimes (z \boxplus t)) & \circlearrowleft & ((x \boxplus y) \boxtimes z) \boxplus ((x \boxplus y) \boxtimes t) \\
 \lambda_{x;z,t}^2 \boxplus \lambda_{y;z,t}^2 \downarrow & & \downarrow \lambda_{x,y;z}^1 \boxplus \lambda_{x,y;t}^1 \\
 ((x \boxtimes z) \boxplus (x \boxtimes t)) \boxplus ((y \boxtimes z) \boxplus (y \boxtimes t)) & \xrightarrow{\hat{c}} & ((x \boxtimes z) \boxplus (y \boxtimes z)) \boxplus ((x \boxtimes t) \boxplus (y \boxtimes t))
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$\boxtimes: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ is **bi-additive** (with respect to \boxplus)

Presentations by stable modules

Definition (Joyal and Street 1993)

A **stable crossed module** is a crossed module $\partial: R_1 \rightarrow R_0$ with $\{.,.\}: R_0 \times R_0 \rightarrow R_1$ such that the groupoid

$$[R_1 \rtimes R_0 \rightrightarrows R_0]$$

is *braided symmetric*.

Definition

A **presentation**

$$R_1 \xrightarrow{\partial} R_0 \xrightarrow{\pi} \mathcal{R}$$

of $(\mathcal{R}, \boxplus, c, 0_{\mathcal{R}})$ by a stable crossed module $(\partial: R_1 \rightarrow R_0, c)$ is an equivalence

$$[R_1 \rtimes R_0 \rightrightarrows R_0] \overset{\sim}{\longrightarrow} \mathcal{R}.$$

Remark

With $A = \pi_1(R) = \text{Ker } \partial$ and $B = \pi_0(R) = \text{Coker } \partial$, **stable** refers to $k(R) \in H^5(K(B, 3), A)$.

Presentations of categorical rings

Given:

- $\boxtimes: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ biexact
- A presentation $R_1 \rightarrow R_0 \rightarrow \mathcal{R}$

Question

Additional structure

 on $(\partial: R_1 \rightarrow R_0, c)$ so that

$$\begin{array}{ccc}
 [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim \times [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim & \xrightarrow{\quad} & [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim \\
 \downarrow & & \downarrow \\
 \mathcal{R} \times \mathcal{R} & \xrightarrow{\quad \boxtimes \quad} & \mathcal{R}
 \end{array}$$

commutes up to a (coherent) 2-morphism. **Top-row is biadditive.**

Caveat

Not a degree-wise biexact functor on $[R_1 \rtimes R_0 \rightrightarrows R_0]!$

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Playground

- Work over site \mathbf{C}
- **Stable modules** are symmetric crossed modules of $\mathbf{T} = \text{Sh}(\mathbf{C})$
- Picard (=symmetric monoidal, group-like) stacks $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{H}, \mathcal{K}, \mathcal{G}, \mathcal{R}, \dots \rightarrow \mathbf{C}$
objects of a 2-category $\mathfrak{SGrSt}(\mathbf{C})$
- Each object \mathcal{G} admits a presentation $G_1 \rightarrow G_0 \rightarrow \mathcal{G}$ by stable crossed modules
- Stable crossed modules comprise a *bicategory* $\mathfrak{SMod}(\mathbf{C})$ with *butterfly* morphisms



Theorem (Aldrovandi and Noohi (2009))

There is an equivalence of bicategories $\mathfrak{SMod}(\mathbf{C}) \xrightarrow{\sim} \mathfrak{SGrSt}(\mathbf{C})$.

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$$\begin{array}{ccccc}
 H_1 & & & & G_1 \\
 \downarrow & \searrow & & \swarrow & \downarrow \\
 & & E & & \\
 \downarrow & \swarrow & & \searrow & \downarrow \\
 H_0 & & & & G_0
 \end{array}$$

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(Back to) Biexact functors—in general

A bifunctor $F: \mathcal{H} \times \mathcal{K} \rightarrow \mathcal{G}$ in $\mathbf{SGrSt}(\mathbf{C})$ is **biadditive** if:

- 1 There exist functorial (iso)morphisms

$$\lambda_{h,h';k}^1: F(h,k) + F(h',k) \longrightarrow F(h+h',k), \quad \lambda_{h;k,k'}^2: F(h,k) + F(h,k') \longrightarrow F(h,k+k')$$

satisfying the standard associativity conditions and compatibility with the braiding;

- 2 the two morphisms $F(0_{\mathcal{H}}, 0_{\mathcal{K}}) \rightarrow 0_{\mathcal{G}}$ coincide;
- 3 for all objects h, h' of \mathcal{H} and k, k' of \mathcal{K} there exists a functorial

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 (F(h,k) + F(h',k)) + (F(h,k') + F(h',k')) & \xrightarrow{\hat{c}} & (F(h,k) + F(h,k')) + (F(h',k) + F(h',k')) \\
 \downarrow \lambda^1 + \lambda^1 & & \downarrow \lambda^2 + \lambda^2 \\
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 \lambda^{1+\lambda^1} \downarrow & & \downarrow \lambda^2+\lambda^2 \\
 F(h+h',k) + F(h+h',k') & \cup & F(h,k+k') + F(h',k+k') \\
 & \searrow \lambda^2 & \swarrow \lambda^1 \\
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 \end{array}$$

Biextensions (Grothendieck 1972; Mumford 1969)

Let G, H, K be abelian groups of $\mathbf{T} = \text{Sh}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $G_{H \times K}$ -torsor $p: E \rightarrow H \times K$
- **Partial** (abelian) group laws

$$\times_1: E \times_K E \longrightarrow E, \quad \times_2: E \times_H E \longrightarrow E$$

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- \times_1, \times_2 are the group laws of **central extensions**

$$0 \longrightarrow G_K \longrightarrow E \longrightarrow H_K \longrightarrow 0$$

$$0 \longrightarrow G_H \longrightarrow E \longrightarrow K_H \longrightarrow 0$$

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- **Interchange** or **compatibility**

$$\begin{array}{ccc}
 E_{h,k} \wedge^G E_{h',k} \wedge^G E_{h,k'} \wedge^G E_{h',k'} & \longrightarrow & E_{h,k} \wedge^G E_{h,k'} \wedge^G E_{h',k} \wedge^G E_{h',k'} \\
 \downarrow \times_1 \wedge \times_1 & & \downarrow \times_2 \wedge \times_2 \\
 E_{h+h',k} \wedge^G E_{h+h',k'} & \cup & E_{h,k+k'} \wedge^G E_{h',k+k'} \\
 \swarrow \times_2 & & \nwarrow \times_1 \\
 & E_{h+h',k+k'} &
 \end{array}$$

Biextensions, more generally

Let $G = (\partial: G_1 \rightarrow G_0, \{\cdot, \cdot\})$ be a **stable crossed module** of $\mathbf{SxMod}(\mathbf{C})$.

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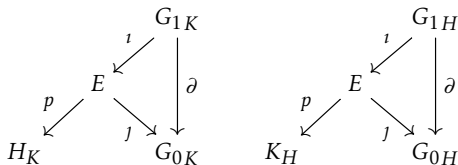
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- \times_1, \times_2 are the group laws of **extensions**



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- The arrow

$$E_{h,k} \wedge^G E_{h',k} \wedge^G E_{h,k'} \wedge^G E_{h',k'} \xrightarrow{\hat{c}} E_{h,k} \wedge^G E_{h,k'} \wedge^G E_{h',k} \wedge^G E_{h',k'}$$

is the braiding of $[G_1 \times G_0 \rightrightarrows G_0]^\sim = \text{Tors}(G_1, G_0)$, induced by $\{\cdot, \cdot\}$.

Groupoid of biextensions

Denote by $\mathbf{BIEXT}(H, K; G)$ the groupoid of biextensions of $H \times K$ by G

Butterflies

$G = (\partial: G_1 \rightarrow G_0)$, $H = (\partial: H_1 \rightarrow H_0)$, $K = (\partial: K_1 \rightarrow K_0)$ stable crossed modules.

Definition (Butterfly from $H \times K$ to G)

- **Biextension** $E \in \text{BIEXT}(H_0, K_0; G)$
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$$s_1(h, z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y, k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi: (E, s_1, s_2) \rightarrow (E', s'_1, s'_2)$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\text{BIEXT}(H, K; G)$. The distinguished object is the trivial biextension.

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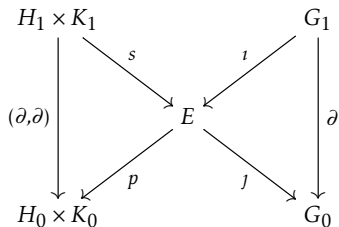
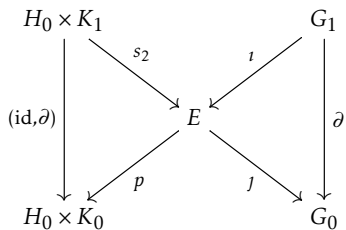
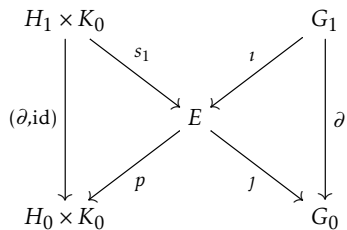
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Question (Name)

Bi-Butterflies?

Butterflies, in diagrams



Butterflies are biadditive functors

Let

- $\mathcal{G}, \mathcal{H}, \mathcal{K}$ objects of $\mathbf{SGrSt}(\mathbf{C})$
- $\mathbf{Hom}(\mathcal{H}, \mathcal{H}; \mathcal{G})$ **groupoid of biadditive functors**:

$$F_U: \mathcal{H}|_U \times \mathcal{H}|_U \rightarrow \mathcal{G}|_U, \quad U \in \mathbf{C}.$$

- presentations by stable crossed modules: $\mathcal{H} \simeq [H_1 \rtimes H_0 \rightrightarrows H_0]^\sim$,
 $\mathcal{K} \simeq [K_1 \rtimes K_0 \rightrightarrows K_0]^\sim$, and $\mathcal{G} \simeq [G_1 \rtimes G_0 \rightrightarrows G_0]^\sim$

Theorem (E.A. TAC (2017))

There exists a (pointed) equivalence

$$\mathbf{BIEXT}(\mathbf{H}, \mathbf{K}; \mathbf{G}) \xrightarrow{\sim} \mathbf{Hom}(\mathcal{H}, \mathcal{K}; \mathcal{G}).$$

Ideas for the proof.

- Systematically exploit the equivalence $[G_1 \rtimes G_0 \rightrightarrows G_0]^\sim \simeq \mathbf{Tors}(G_1, G_0)$, etc.
- Next...



Proof...

From $F: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{G}$ to a butterfly diagram:

$$\begin{array}{ccc}
 H_1 \times K_1 & & G_1 \\
 \downarrow \partial \times \partial & & \downarrow \partial \\
 H_0 \times K_0 & & G_0 \\
 \downarrow \pi \times \pi & & \downarrow \pi \\
 \mathcal{H} \times \mathcal{H} & \xrightarrow{F} & \mathcal{G}
 \end{array}$$

Proof...

From $F: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{G}$ to a butterfly diagram:

$$\begin{array}{ccc}
 H_1 \times K_1 & & G_1 \\
 \downarrow \partial \times \partial & & \downarrow \partial \\
 H_0 \times K_0 & & G_0 \\
 \downarrow \pi \times \pi & \searrow & \downarrow \pi \\
 \mathcal{H} \times \mathcal{H} & \xrightarrow{F} & \mathcal{G}
 \end{array}$$

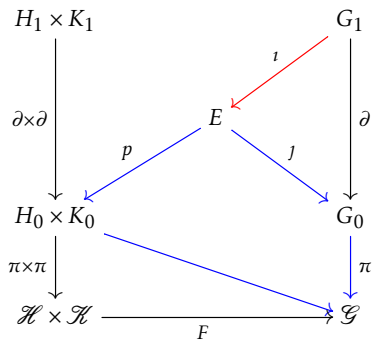
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 & E & \\
 & \swarrow p & \searrow J \\
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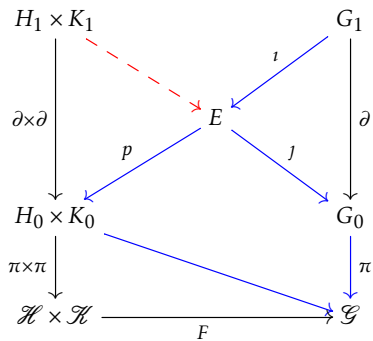
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Upgrade to n -variables

Definition (n -Butterflies)

H_1, \dots, H_n, G stable modules.

- $(G_1, G_0)_{H_1,0 \times \dots \times H_n,0}$ -torsor $E \rightarrow H_{1,0} \times \dots \times H_{n,0}$
- **trivializations** $s_i: H_{1,0} \times \dots \times H_{i,1} \times \dots \times H_{n,0} \rightarrow E$
- “straightforward” generalizations of $n = 2$ data

Theorem

There exists a (pointed) equivalence

$$\underbrace{\text{MExt}(H_{1,\bullet}, \dots, H_{n,\bullet}; G_\bullet)}_{n\text{-Butterflies}} \xrightarrow{\sim} \underbrace{\text{Hom}(\mathcal{H}_1, \dots, \mathcal{H}_n; \mathcal{G})}_{n\text{-Additive functors}}$$

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Multi-variable compositions

Butterflies can be composed

- Stable modules G, H_1, \dots, H_n and $K_{i,1}, \dots, K_{i,m_i}$ of $\mathfrak{S}\mathfrak{X}\mathfrak{M}\mathfrak{o}\mathfrak{d}(\mathbb{C})$

- $K_{i,1,1} \times \dots \times K_{i,m_i,1}$

$$\begin{array}{ccccc}
 & & & H_{i,1} & \\
 & & & \swarrow i & \downarrow \partial \\
 & & & F_i & \\
 & & & \swarrow j & \downarrow \partial \\
 & & & H_{i,0} & \\
 & & & \swarrow p & \\
 & & & K_{i,1,0} \times \dots \times K_{i,m_i,0} & \\
 & & & \downarrow (\partial, \dots, \partial) & \\
 & & & K_{i,1,1} \times \dots \times K_{i,m_i,1} &
 \end{array}$$

$$\begin{array}{ccccc}
 & & & H_{1,1} \times \dots \times H_{n,1} & \\
 & & & \swarrow s & \downarrow \partial \\
 & & & E & \\
 & & & \swarrow j & \downarrow \partial \\
 & & & G_0 & \\
 & & & \swarrow p & \\
 & & & H_{1,0} \times \dots \times H_{n,0} & \\
 & & & \downarrow (\partial, \dots, \partial) & \\
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 & & & K_{1,1,0} \times \dots \times K_{n,m_n,0} & \\
 & & & \downarrow (\partial, \dots, \partial) & \\
 & & & K_{1,1,1} \times \dots \times K_{n,m_n,1} &
 \end{array}
 \quad \times \quad
 \begin{array}{c}
 (F_1 \times \dots \times F_n) \\
 \times \\
 \begin{array}{c}
 H_{1,1} \times \dots \times H_{n,1} \\
 H_{1,0} \times \dots \times H_{n,0}
 \end{array}
 \end{array}$$

Multi-categorical structures

Informal definition

A **Multi-Bicategory** \mathbf{MC} is defined in the same as a multicategory, **except** the **objects**

$$\mathrm{Hom}_{\mathbf{MC}}(x_1, \dots, x_n; y)$$

are groupoids.

Theorem

- $\mathbf{SGrSt}(\mathbf{C})$ and $\mathbf{SXMod}(\mathbf{C})$ are *promoted to multi-bicategories* with Hom-objects $\mathrm{Hom}(\mathcal{H}_1, \dots, \mathcal{H}_n; \mathcal{G})$ and $\mathrm{MExt}(\mathbf{H}_1, \dots, \mathbf{H}_n; \mathbf{G})$, respectively.
- *There is an equivalence of multi-bicategories*

$$\mathbf{Ma}: \mathbf{MSXMod} \xrightarrow{\sim} \mathbf{MSGrSt}$$

induced by the associated stack functor.

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Presentations of categorical rings

Given:

- $\boxtimes: \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ biexact
- A presentation $R_1 \rightarrow R_0 \rightarrow \mathcal{R}$

Question

Additional structure

 on $(\partial: R_1 \rightarrow R_0, c)$ so that

$$\begin{array}{ccc}
 [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim \times [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim & \longrightarrow & [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim \\
 \downarrow & & \downarrow \\
 \mathcal{R} \times \mathcal{R} & \xrightarrow{\quad \boxtimes \quad} & \mathcal{R}
 \end{array}$$

commutes up to a (coherent) 2-morphism. **Top-row is biadditive.**

Caveat

Not a degree-wise biexact functor on $[R_1 \rtimes R_0 \rightrightarrows R_0]!$

Presentations of categorical rings

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Answer

The biadditive functor $\boxtimes: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ is equivalent to a **biextension (=2-butterfly)**

$$\begin{array}{ccccc}
 R_1 \times R_1 & & & & R_1 \\
 \downarrow & \searrow & & \swarrow & \downarrow \\
 & & E_{\boxtimes}^2 & & \\
 & \swarrow & & \searrow & \\
 R_0 \times R_0 & & & & R_0
 \end{array}$$

(∂, ∂) on the left vertical arrow, and ∂ on the right vertical arrow.

In fact, **due to coherence**, we have **n -butterflies**

$$E_{\boxtimes}^2 \times (E_{\boxtimes}^2 \times I) \leftarrow E_{\boxtimes}^3 \rightarrow E_{\boxtimes}^2 \times (I \times E_{\boxtimes}^2), \quad E_{\boxtimes}^4, \dots$$

Classification by Mac Lane cohomology

Given:

- \mathcal{R} ring-like
- A presentation $R_1 \xrightarrow{\partial} R_0 \xrightarrow{\pi} \mathcal{R}$

Postnikov invariant of the stable crossed module

$$k(R) \in H^5(K(B, 3), A),$$

$A = \pi_1 = \text{Ker } \partial$ and $B = \pi_0 = \text{Coker } \partial$

\mathcal{R} ring-like implies B is a ring, and A a B -bimodule of $\text{Sh}(\mathbf{C})$.

Theorem (Jibladze and Pirashvili (2007), Quang (2013), and Aldrovandi (2017))

The data $E_{\boxtimes}^2, E_{\boxtimes}^3, E_{\boxtimes}^4, \dots$ give rise to a cohomology class whose $E^{3,0}$ -term (think $K(B, 3)$ as a simplicial object of $\text{Sh}(\mathbf{C})$) lives in the standard

$$HML^3(B, A)$$

the third *Mac Lane cohomology* of B with coefficients in A .

Thank you

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