Biextensions, ring-like stacks, and their classification

Ettore Aldrovandi

Florida State University

Category Theory Octoberfest 2017

Categorical rings, informally

A categorical ring $\mathscr{R}$ consists of:

1. A symmetric monoidal structure $(\mathscr{R}, \boxplus, c, 0_{\mathscr{R}})$
2. Group-like:

$$x \boxplus -, - \boxplus x \colon \mathscr{R} \longrightarrow \mathscr{R}$$

   are equivalences for each object x of $\mathscr{R}$

3. $(\mathscr{R}, \boxtimes, 1_{\mathscr{R}})$ second monoidal structure, distributive over $\boxplus$:

$$\lambda^1_{x,y;z} \colon (x \boxplus y) \boxtimes z \xrightarrow{\sim} (x \boxtimes z) \boxplus (y \boxtimes z)$$

$$\lambda^2_{x;y,z} \colon x \boxtimes (y \boxplus z) \xrightarrow{\sim} (x \boxtimes y) \boxplus (x \boxtimes z)$$

$\mathscr{R}$ is a Picard groupoid (with respect to $\boxplus$)

Categorical rings, informally

A categorical ring $\mathscr{R}$ consists of:

1. A symmetric monoidal structure $(\mathscr{R}, \boxplus, c, 0_{\mathscr{R}})$
2. Group-like:
$$x \boxplus -, - \boxplus x \colon \mathscr{R} \longrightarrow \mathscr{R}$$
are equivalences for each object x of $\mathscr{R}$
3. $(\mathscr{R}, \boxtimes, 1_{\mathscr{R}})$ second monoidal structure, distributive over $\boxplus$:
$$\lambda^1_{x,y;z} \colon (x \boxplus y) \boxtimes z \xrightarrow{\sim} (x \boxtimes z) \boxplus (y \boxtimes z)$$
$$\lambda^2_{x;y,z} \colon x \boxtimes (y \boxplus z) \xrightarrow{\sim} (x \boxtimes y) \boxplus (x \boxtimes z)$$

Distributor isomorphisms must be compatible

Categorical rings, informally

Compatibility

$$
\begin{tikzcd}
 & (x \boxplus y) \boxtimes (z \boxplus t) \arrow[dl, bend right, "\lambda^1_{x,y;z\boxplus t}"'] \arrow[dr, bend left, "\lambda^2_{x\boxplus y;z,t}"] & \\
(x \boxtimes (z \boxplus t)) \boxplus (y \boxtimes (z \boxplus t)) \arrow[d, "\lambda^2_{x;z,t} \boxplus \lambda^2_{y;z,t}"'] & \circlearrowleft & ((x \boxplus y) \boxtimes z) \boxplus ((x \boxplus y) \boxtimes t) \arrow[d, "\lambda^1_{x,y;z} \boxplus \lambda^1_{x,y;t}"] \\
((x \boxtimes z) \boxplus {\color{red}(x \boxtimes t)}) \boxplus ({\color{blue}(y \boxtimes z)} \boxplus (y \boxtimes t)) \arrow[rr, "\hat{c}"'] & & ((x \boxtimes z) \boxplus {\color{blue}(y \boxtimes z)}) \boxplus ({\color{red}(x \boxtimes t)} \boxplus (y \boxtimes t))
\end{tikzcd}
$$

Categorical rings, informally

A categorical ring $\mathscr{R}$ consists of:

1. A symmetric monoidal structure $(\mathscr{R}, \boxplus, c, 0_{\mathscr{R}})$
2. Group-like:
$$x \boxplus -, - \boxplus x \colon \mathscr{R} \longrightarrow \mathscr{R}$$
are equivalences for each object x of $\mathscr{R}$
3. $(\mathscr{R}, \boxtimes, 1_{\mathscr{R}})$ second monoidal structure, distributive over $\boxplus$:
$$\lambda^1_{x,y;z} \colon (x \boxplus y) \boxtimes z \xrightarrow{\sim} (x \boxtimes z) \boxplus (y \boxtimes z)$$
$$\lambda^2_{x;y,z} \colon x \boxtimes (y \boxplus z) \xrightarrow{\sim} (x \boxtimes y) \boxplus (x \boxtimes z)$$

$\boxtimes \colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is bi-additive (with respect to $\boxplus$)

Presentations by stable modules

Definition (Joyal and Street 1993)

A stable crossed module is a crossed module $\partial\colon R_1 \to R_0$ with $\{.,.\}\colon R_0 \times R_0 \to R_1$ such that the groupoid

$$[R_1 \rtimes R_0 \rightrightarrows R_0]$$

is *braided symmetric.*

Definition

A presentation

$$R_1 \xrightarrow{\partial} R_0 \xrightarrow{\pi} \mathscr{R}$$

of $(\mathscr{R}, \boxplus, c, 0_{\mathscr{R}})$ by a stable crossed module $(\partial\colon R_1 \to R_0, c)$ is an equivalence

$$[R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim} \xrightarrow{\sim} \mathscr{R}.$$

Remark

With $A = \pi_1(R) = \operatorname{Ker}\partial$ and $B = \pi_0(R) = \operatorname{Coker}\partial$, stable refers to $k(R) \in H^5(K(B,3), A)$.

Presentations of categorical rings

Given:

- $\boxtimes\colon \mathscr{R} \times \mathscr{R} \longrightarrow \mathscr{R}$ biexact
- A presentation $R_1 \to R_0 \to \mathscr{R}$

Question

Additional structure on $(\partial\colon R_1 \to R_0, c)$ so that

$$\begin{tikzcd}
{[R_1 \rtimes R_0 \rightrightarrows R_0]^\sim \times [R_1 \rtimes R_0 \rightrightarrows R_0]^\sim} \arrow[r] \arrow[d] & {[R_1 \rtimes R_0 \rightrightarrows R_0]^\sim} \arrow[d] \\
\mathscr{R} \times \mathscr{R} \arrow[r, "\boxtimes"] & \mathscr{R}
\end{tikzcd}$$

commutes up to a (coherent) 2-morphism. Top-row is biadditive.

Caveat

Not a degree-wise biexact functor on $[R_1 \rtimes R_0 \rightrightarrows R_0]$!

Presentations of categorical rings

Given:

- $\boxtimes\colon \mathscr{R}\times\mathscr{R} \longrightarrow \mathscr{R}$ biexact
- A presentation $R_1 \to R_0 \to \mathscr{R}$

Question

Additional structure on $(\partial\colon R_1 \to R_0, c)$ so that

$$\begin{tikzcd}
{[R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim} \times [R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim}} \arrow[r] \arrow[d] & {[R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim}} \arrow[d] \\
\mathscr{R}\times\mathscr{R} \arrow[r, "\boxtimes"] & \mathscr{R}
\end{tikzcd}$$

commutes up to a (coherent) 2-morphism. Top-row is biadditive.

Caveat

Not a degree-wise biexact functor on $[R_1 \rtimes R_0 \rightrightarrows R_0]$!

Playground

- Work over site $\mathbf{C}$
- Stable modules are symmetric crossed modules of $\mathbf{T} = \mathrm{Sh}(\mathbf{C})$
- Picard (=symmetric monoidal, group-like) stacks $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{H}, \mathscr{K}, \mathscr{G}, \mathscr{R}, \ldots \to \mathbf{C}$

  objects of a 2-category $\mathfrak{SGrSt}(\mathbf{C})$
- Each object $\mathscr{G}$ admits a presentation $G_1 \to G_0 \to \mathscr{G}$ by stable crossed modules
- Stable crossed modules comprise a *bi*category $\mathfrak{SXMod}(\mathbf{C})$ with butterfly morphisms

$$
\begin{tikzcd}
H_1 \arrow[dd] \arrow[dr] & & G_1 \arrow[dd] \arrow[dl] \\
& E \arrow[dl] \arrow[dr] & \\
H_0 & & G_0
\end{tikzcd}
$$

Theorem (Aldrovandi and Noohi (2009))

There is an equivalence of bicategories $\mathfrak{SXMod}(\mathbf{C}) \xrightarrow{\sim} \mathfrak{SGrSt}(\mathbf{C})$.

Playground

- Work over site $\mathbf{C}$
- Stable modules are symmetric crossed modules of $\mathbf{T} = \mathrm{Sh}(\mathbf{C})$
- Picard (=symmetric monoidal, group-like) stacks $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{H}, \mathscr{K}, \mathscr{G}, \mathscr{R}, \ldots \to \mathbf{C}$

  objects of a 2-category $\mathfrak{SGrSt}(\mathbf{C})$
- Each object $\mathscr{G}$ admits a presentation $G_1 \to G_0 \to \mathscr{G}$ by stable crossed modules
- Stable crossed modules comprise a *bi*category $\mathfrak{SXMod}(\mathbf{C})$ with butterfly morphisms

$$\begin{tikzcd} H_1 \arrow[d] \arrow[dr] & & G_1 \arrow[d] \arrow[dl] \\ & E \arrow[dl] \arrow[dr] & \\ H_0 & & G_0 \end{tikzcd}$$

Theorem (Aldrovandi and Noohi (2009))

There is an equivalence of bicategories $\mathfrak{SXMod}(\mathbf{C}) \xrightarrow{\sim} \mathfrak{SGrSt}(\mathbf{C})$.

(Back to) Biexact functors—in general

A *bifunctor* $F\colon \mathscr{H}\times\mathscr{K}\to\mathscr{G}$ in $\mathfrak{SGrSt}(\mathbf{C})$ is biadditive if:

1. There exist functorial (iso)morphisms

$$\lambda^1_{h,h';k}\colon F(h,k)+F(h',k)\longrightarrow F(h+h',k),\quad \lambda^2_{h;k,k'}\colon F(h,k)+F(h,k')\longrightarrow F(h,k+k')$$

   satisfying the standard associativity conditions and compatibility with the braiding;
2. the two morphisms $F(0_{\mathscr{H}},0_{\mathscr{K}})\to 0_{\mathscr{G}}$ coincide;
3. for all objects h,h' of $\mathscr{H}$ and k,k' of $\mathscr{K}$ there exists a functorial

$$\begin{tikzcd}
(F(h,k)+F(h',k))+(F(h,k')+F(h',k')) \arrow[rr, "\hat{c}"] \arrow[d, "\lambda^1+\lambda^1"'] & & (F(h,k)+F(h,k'))+(F(h',k)+F(h',k')) \arrow[d, "\lambda^2+\lambda^2"] \\
F(h+h',k)+F(h+h',k') \arrow[dr, bend right, "\lambda^2"'] \arrow[rr, phantom, "\circlearrowleft"] & & F(h,k+k')+F(h',k+k') \arrow[dl, bend left, "\lambda^1"] \\
& F(h+h',k+k') &
\end{tikzcd}$$

(Back to) Biexact functors—in general

A *bifunctor* $F\colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ in $\mathfrak{SGrSt}(\mathbf{C})$ is biadditive if:

1. There exist functorial (iso)morphisms
$$\lambda^1_{h,h';k}\colon F(h,k)+F(h',k) \longrightarrow F(h+h',k), \quad \lambda^2_{h;k,k'}\colon F(h,k)+F(h,k') \longrightarrow F(h,k+k')$$
satisfying the standard associativity conditions and compatibility with the braiding;
2. the two morphisms $F(0_{\mathscr{H}},0_{\mathscr{K}}) \to 0_{\mathscr{G}}$ coincide;
3. for all objects h,h' of $\mathscr{H}$ and k,k' of $\mathscr{K}$ there exists a functorial

$$\begin{tikzcd}
(F(h,k)+F(h',k))+(F(h,k')+F(h',k')) \arrow[rr, "\hat{c}"] \arrow[d, "\lambda^1+\lambda^1"'] & & (F(h,k)+F(h,k'))+(F(h',k)+F(h',k')) \arrow[d, "\lambda^2+\lambda^2"] \\
F(h+h',k)+F(h+h',k') \arrow[dr, bend right, "\lambda^2"'] & \circlearrowleft & F(h,k+k')+F(h',k+k') \arrow[dl, bend left, "\lambda^1"] \\
& F(h+h',k+k') &
\end{tikzcd}$$

Biextensions (Grothendieck 1972; Mumford 1969)

Let G, H, K be abelian groups of $\mathbf{T} = \mathrm{Sh}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $G_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial (abelian) group laws

$$\times_1\colon E \times_K E \longrightarrow E, \quad \times_2\colon E \times_H E \longrightarrow E$$

Biextensions (Grothendieck 1972; Mumford 1969)

Let G, H, K be abelian groups of $\mathbf{T} = \mathrm{Sh}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $G_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial (abelian) group laws

$$\times_1 \colon E \times_K E \longrightarrow E, \quad \times_2 \colon E \times_H E \longrightarrow E$$

- $\times_1, \times_2$ are the group laws of central extensions

$$0 \longrightarrow G_K \longrightarrow E \longrightarrow H_K \longrightarrow 0$$
$$0 \longrightarrow G_H \longrightarrow E \longrightarrow K_H \longrightarrow 0$$

Biextensions (Grothendieck 1972; Mumford 1969)

Let G, H, K be abelian groups of $\mathbf{T} = \mathrm{Sh}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $G_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial (abelian) group laws

$$\times_1 : E \times_K E \longrightarrow E, \quad \times_2 : E \times_H E \longrightarrow E$$

- Interchange or compatibility

$$\begin{tikzcd}
E_{h,k} \wedge^G E_{h',k} \wedge^G E_{h,k'} \wedge^G E_{h',k'} \arrow[rr] \arrow[d, "\times_1 \wedge \times_1"'] & & E_{h,k} \wedge^G E_{h,k'} \wedge^G E_{h',k} \wedge^G E_{h',k'} \arrow[d, "\times_2 \wedge \times_2"] \\
E_{h+h',k} \wedge^G E_{h+h',k'} \arrow[dr, bend right, "\times_2"'] & \circlearrowleft & E_{h,k+k'} \wedge^G E_{h',k+k'} \arrow[dl, bend left, "\times_1"] \\
& E_{h+h',k+k'} &
\end{tikzcd}$$

Biextensions, more generally

Let $\mathsf{G} = (\partial\colon G_1 \to G_0, \{\cdot,\cdot\})$ be a stable crossed module of $\mathfrak{SXMod}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $(G_1, G_0)_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial group laws

$$\times_1\colon E \times_K E \longrightarrow E, \quad \times_2\colon E \times_H E \longrightarrow E$$

Biextensions, more generally

Let $\mathsf{G} = (\partial\colon G_1 \to G_0, \{\cdot,\cdot\})$ be a stable crossed module of $\mathfrak{S}\mathfrak{X}\mathfrak{Mod}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $(G_1, G_0)_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial group laws

$$\times_1 : E \times_K E \longrightarrow E\,, \quad \times_2 : E \times_H E \longrightarrow E$$

- $\times_1, \times_2$ are the group laws of extensions

$$\begin{tikzcd} & & G_{1\,K} \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "p"'] \arrow[dr, "\jmath"'] & \\ H_K & & G_{0\,K} \end{tikzcd} \qquad \begin{tikzcd} & & G_{1\,H} \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "p"'] \arrow[dr, "\jmath"'] & \\ K_H & & G_{0\,H} \end{tikzcd}$$

Biextensions, more generally

Let $\mathsf{G} = (\partial\colon G_1 \to G_0, \{\cdot,\cdot\})$ be a stable crossed module of $\mathfrak{SXMod}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $(G_1, G_0)_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial group laws

$$\times_1 : E \times_K E \longrightarrow E\,, \quad \times_2 : E \times_H E \longrightarrow E$$

- Interchange or compatibility

$$\begin{tikzcd}
E_{h,k} \wedge^G E_{h',k} \wedge^G E_{h,k'} \wedge^G E_{h',k'} \arrow[rr, "\hat{c}"] \arrow[d, "\times_1 \wedge \times_1"'] & & E_{h,k} \wedge^G E_{h,k'} \wedge^G E_{h',k} \wedge^G E_{h',k'} \arrow[d, "\times_2 \wedge \times_2"] \\
E_{h+h',k} \wedge^G E_{h+h',k'} \arrow[dr, bend right, "\times_2"'] & \circlearrowleft & E_{h,k+k'} \wedge^G E_{h',k+k'} \arrow[dl, bend left, "\times_1"] \\
& E_{h+h',k+k'} &
\end{tikzcd}$$

Biextensions, more generally

Let $\mathsf{G} = (\partial\colon G_1 \to G_0, \{\cdot,\cdot\})$ be a stable crossed module of $\mathfrak{SXMod}(\mathbf{C})$.

Definition (Biextension of $H \times K$ by G)

- $(G_1, G_0)_{H\times K}$-torsor $p\colon E \to H \times K$
- Partial group laws

$$\times_1\colon E \times_K E \longrightarrow E\,, \quad \times_2\colon E \times_H E \longrightarrow E$$

- The arrow

$$E_{h,k} \wedge^G E_{h',k} \wedge^G E_{h,k'} \wedge^G E_{h',k'} \xrightarrow{\hat{c}} E_{h,k} \wedge^G E_{h,k'} \wedge^G E_{h',k} \wedge^G E_{h',k'}$$

is the braiding of $[G_1 \rtimes G_0 \rightrightarrows G_0]^{\sim} = \textsc{Tors}(G_1, G_0)$, induced by $\{\cdot,\cdot\}$.

Groupoid of biextensions

Denote by $\textsc{Biext}(H, K; \mathsf{G})$ the groupoid of biextensions of $H \times K$ by G

Butterflies

$\mathsf{G} = (\partial\colon G_1 \to G_0)$, $\mathsf{H} = (\partial\colon H_1 \to H_0)$, $\mathsf{K} = (\partial\colon K_1 \to K_0)$ stable crossed modules.

Definition (Butterfly from $\mathsf{H} \times \mathsf{K}$ to G)

- Biextension $E \in \textsc{Biext}(H_0, K_0; \mathsf{G})$
- trivializations $s_1 : H_1 \times K_0 \to E$ and $s_2 : H_0 \times K_1 \to E$:
  - s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id}, \partial)^* s_1 = (\partial, \mathrm{id})^* s_2$.
  - Equivariance: for all $(h, z) \in H_1 \times K_0$, $(y, k) \in H_0 \times K_1$, and $e \in E_{y,z}$

$$s_1(h,z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y,k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi\colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\textsc{Biext}(\mathsf{H}, \mathsf{K}; \mathsf{G})$. The distinguished object is the trivial biextension.

Butterflies

$\mathsf{G} = (\partial\colon G_1 \to G_0)$, $\mathsf{H} = (\partial\colon H_1 \to H_0)$, $\mathsf{K} = (\partial\colon K_1 \to K_0)$ stable crossed modules.

Definition (Butterfly from $\mathsf{H} \times \mathsf{K}$ to G)

- Biextension $E \in \textsc{Biext}(H_0, K_0; \mathsf{G})$
- trivializations $s_1 : H_1 \times K_0 \to E$ and $s_2 : H_0 \times K_1 \to E$:
  - s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id}, \partial)^* s_1 = (\partial, \mathrm{id})^* s_2$.
  - Equivariance: for all $(h, z) \in H_1 \times K_0$, $(y, k) \in H_0 \times K_1$, and $e \in E_{y,z}$

$$s_1(h,z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y,k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi\colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\textsc{Biext}(\mathsf{H}, \mathsf{K}; \mathsf{G})$. The distinguished object is the trivial biextension.

Butterflies

$\mathsf{G} = (\partial\colon G_1 \to G_0)$, $\mathsf{H} = (\partial\colon H_1 \to H_0)$, $\mathsf{K} = (\partial\colon K_1 \to K_0)$ stable crossed modules.

Definition (Butterfly from $\mathsf{H} \times \mathsf{K}$ to G)

- Biextension $E \in \textsc{Biext}(H_0, K_0; \mathsf{G})$
- trivializations $s_1 : H_1 \times K_0 \to E$ and $s_2 : H_0 \times K_1 \to E$:
  - s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id}, \partial)^* s_1 = (\partial, \mathrm{id})^* s_2$.
  - Equivariance: for all $(h, z) \in H_1 \times K_0$, $(y, k) \in H_0 \times K_1$, and $e \in E_{y,z}$

$$s_1(h, z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y, k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi\colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\textsc{Biext}(\mathsf{H}, \mathsf{K}; \mathsf{G})$. The distinguished object is the trivial biextension.

Butterflies

$\mathsf{G} = (\partial\colon G_1 \to G_0)$, $\mathsf{H} = (\partial\colon H_1 \to H_0)$, $\mathsf{K} = (\partial\colon K_1 \to K_0)$ stable crossed modules.

Definition (Butterfly from $\mathsf{H} \times \mathsf{K}$ to G)

- Biextension $E \in \textsc{Biext}(H_0, K_0; \mathsf{G})$
- trivializations $s_1 : H_1 \times K_0 \to E$ and $s_2 : H_0 \times K_1 \to E$:
  - s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id}, \partial)^* s_1 = (\partial, \mathrm{id})^* s_2$.
  - Equivariance: for all $(h,z) \in H_1 \times K_0$, $(y,k) \in H_0 \times K_1$, and $e \in E_{y,z}$

$$s_1(h,z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y,k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi\colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\textsc{Biext}(\mathsf{H}, \mathsf{K}; \mathsf{G})$. The distinguished object is the trivial biextension.

Butterflies

$\mathsf{G} = (\partial\colon G_1 \to G_0)$, $\mathsf{H} = (\partial\colon H_1 \to H_0)$, $\mathsf{K} = (\partial\colon K_1 \to K_0)$ stable crossed modules.

Definition (Butterfly from $\mathsf{H} \times \mathsf{K}$ to G)

- Biextension $E \in \textsc{Biext}(H_0, K_0; \mathsf{G})$
- trivializations $s_1 : H_1 \times K_0 \to E$ and $s_2 : H_0 \times K_1 \to E$:
  - s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id}, \partial)^* s_1 = (\partial, \mathrm{id})^* s_2$.
  - Equivariance: for all $(h,z) \in H_1 \times K_0$, $(y,k) \in H_0 \times K_1$, and $e \in E_{y,z}$

$$s_1(h,z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y,k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi\colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\textsc{Biext}(\mathsf{H}, \mathsf{K}; \mathsf{G})$. *The distinguished object is the trivial biextension.*

Butterflies

$\mathsf{G} = (\partial\colon G_1 \to G_0)$, $\mathsf{H} = (\partial\colon H_1 \to H_0)$, $\mathsf{K} = (\partial\colon K_1 \to K_0)$ stable crossed modules.

Definition (Butterfly from $\mathsf{H} \times \mathsf{K}$ to G)

- Biextension $E \in \textsc{Biext}(H_0, K_0; \mathsf{G})$
- trivializations $s_1 : H_1 \times K_0 \to E$ and $s_2 \colon H_0 \times K_1 \to E$:
  - s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id}, \partial)^* s_1 = (\partial, \mathrm{id})^* s_2$.
  - Equivariance: for all $(h, z) \in H_1 \times K_0$, $(y, k) \in H_0 \times K_1$, and $e \in E_{y,z}$

$$s_1(h, z) \times_1 e = e \times_1 s_1(h^y, z), \quad s_2(y, k) \times_2 e = e \times_2 s_2(y, k^z).$$

- A morphism $\varphi\colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Lemma

Butterflies form a pointed groupoid $\textsc{Biext}(\mathsf{H}, \mathsf{K}; \mathsf{G})$. *The distinguished object is the trivial biextension.*

Question (Name)

Bi-Butterflies?

Butterflies, in diagrams

$$\begin{tikzcd}
H_1 \times K_0 \arrow[dd, "(\partial{,}\mathrm{id})"'] \arrow[dr, "s_1"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"] \arrow[dr, "\jmath"'] & \\
H_0 \times K_0 & & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
H_0 \times K_1 \arrow[dd, "(\mathrm{id}{,}\partial)"'] \arrow[dr, "s_2"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"] \arrow[dr, "\jmath"'] & \\
H_0 \times K_0 & & G_0
\end{tikzcd}$$

$$\begin{tikzcd}
H_1 \times K_1 \arrow[dd, "(\partial{,}\partial)"'] \arrow[dr, "s"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"] \arrow[dr, "\jmath"'] & \\
H_0 \times K_0 & & G_0
\end{tikzcd}$$

Butterflies are biadditive functors

Let

- $\mathscr{G}, \mathscr{H}, \mathscr{K}$ objects of $\mathfrak{SGrSt}(\mathbf{C})$
- $\mathrm{H}\textsc{om}(\mathscr{H}, \mathscr{K}; \mathscr{G})$ groupoid of biadditive functors:

$$F_U \colon \mathscr{H}|_U \times \mathscr{K}|_U \to \mathscr{G}|_U, \quad U \in \mathbf{C}.$$

- presentations by stable crossed modules: $\mathscr{H} \simeq [H_1 \rtimes H_0 \rightrightarrows H_0]^\sim$, $\mathscr{K} \simeq [K_1 \rtimes K_0 \rightrightarrows K_0]^\sim$, and $\mathscr{G} \simeq [G_1 \rtimes G_0 \rightrightarrows G_0]^\sim$

Theorem (E.A. TAC (2017))

There exists a (pointed) equivalence

$$\mathrm{B}\textsc{iext}(\mathsf{H}, \mathsf{K}; \mathsf{G}) \xrightarrow{\sim} \mathrm{H}\textsc{om}(\mathscr{H}, \mathscr{K}; \mathscr{G}).$$

Ideas for the proof.

- Systematically exploit the equivalence $[G_1 \rtimes G_0 \rightrightarrows G_0]^\sim \simeq \mathrm{Tors}(G_1, G_0)$, etc.
- Next...

□

Proof...

From $F\colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ to a butterfly diagram:

$$
\begin{tikzcd}
H_1 \times K_1 \arrow[d, "\partial\times\partial"'] & & G_1 \arrow[d, "\partial"] \\
H_0 \times K_0 \arrow[d, "\pi\times\pi"'] & & G_0 \arrow[d, "\pi"] \\
\mathscr{H} \times \mathscr{K} \arrow[rr, "F"'] & & \mathscr{G}
\end{tikzcd}
$$

Proof...

From $F\colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ to a butterfly diagram:

$$\begin{tikzcd}
H_1 \times K_1 \arrow[d, "\partial\times\partial"'] & & G_1 \arrow[d, "\partial"] \\
H_0 \times K_0 \arrow[d, "\pi\times\pi"'] \arrow[drr] & & G_0 \arrow[d, "\pi"] \\
\mathscr{H} \times \mathscr{K} \arrow[rr, "F"'] & & \mathscr{G}
\end{tikzcd}$$

Proof...

From $F\colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ to a butterfly diagram:

$$
\begin{tikzcd}
H_1 \times K_1 \arrow[dd, "\partial\times\partial"'] & & G_1 \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"'] \arrow[dr, "\jmath"] & \\
H_0 \times K_0 \arrow[d, "\pi\times\pi"'] \arrow[drr] & & G_0 \arrow[d, "\pi"] \\
\mathscr{H} \times \mathscr{K} \arrow[rr, "F"'] & & \mathscr{G}
\end{tikzcd}
$$

Proof...

From $F\colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ to a butterfly diagram:

$$\begin{tikzcd}
H_1 \times K_1 \arrow[dd, "\partial\times\partial"'] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"'] \arrow[dr, "\jmath"] & \\
H_0 \times K_0 \arrow[d, "\pi\times\pi"'] \arrow[drr] & & G_0 \arrow[d, "\pi"] \\
\mathscr{H} \times \mathscr{K} \arrow[rr, "F"'] & & \mathscr{G}
\end{tikzcd}$$

Proof...

From $F\colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ to a butterfly diagram:

$$
\begin{tikzcd}
H_1 \times K_1 \arrow[dd, "\partial\times\partial"'] \arrow[dr, dashed] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"'] \arrow[dr, "\jmath"] & \\
H_0 \times K_0 \arrow[d, "\pi\times\pi"'] \arrow[drr] & & G_0 \arrow[d, "\pi"] \\
\mathscr{H} \times \mathscr{K} \arrow[rr, "F"'] & & \mathscr{G}
\end{tikzcd}
$$

Upgrade to n-variables

Definition (n-Butterflies)

$\mathsf{H}_1, \dots, \mathsf{H}_n, \mathsf{G}$ stable modules.

- $(G_1, G_0)_{H_{1,0}\times\cdots\times H_{n,0}}$-torsor $E \longrightarrow H_{1,0} \times \cdots \times H_{n,0}$
- trivializations $s_i \colon H_{1,0} \times \dots H_{i,1} \cdots \times H_{n,0} \to E$
- "straightforward" generalizations of $n = 2$ data

Theorem

There exists a (pointed) equivalence

$$\underbrace{\mathrm{MExt}(H_{1,\bullet}, \dots, H_{n,\bullet}; G_\bullet)}_{n\text{-Butterflies}} \xrightarrow{\sim} \underbrace{\mathrm{Hom}(\mathscr{H}_1, \dots, \mathscr{H}_n; \mathscr{G})}_{n\text{-Additive functors}},$$

Upgrade to n-variables

Definition (n-Butterflies)

$\mathsf{H}_1, \dots, \mathsf{H}_n, \mathsf{G}$ stable modules.

- $(G_1, G_0)_{H_{1,0} \times \cdots \times H_{n,0}}$-torsor $E \longrightarrow H_{1,0} \times \cdots \times H_{n,0}$
- trivializations $s_i \colon H_{1,0} \times \dots H_{i,1} \cdots \times H_{n,0} \to E$
- "straightforward" generalizations of $n = 2$ data

Theorem

There exists a (pointed) equivalence

$$\underbrace{\mathrm{MExt}(H_{1,\bullet}, \dots, H_{n,\bullet}; G_\bullet)}_{n\text{-Butterflies}} \xrightarrow{\sim} \underbrace{\mathrm{Hom}(\mathscr{H}_1, \dots, \mathscr{H}_n; \mathscr{G})}_{n\text{-Additive functors}},$$

Multi-variable compositions

Butterflies can be composed

- Stable modules G, $\mathsf{H}_1, \dots \mathsf{H}_n$ and $\mathsf{K}_{i,1}, \dots, \mathsf{K}_{i,m_i}$ of $\mathfrak{SXMod}(\mathbf{C})$
- $$\begin{tikzcd}
K_{i,1,1} \times \dots \times K_{i,m_i,1} \arrow[dd, "{(\partial,\dots,\partial)}"'] \arrow[dr, "s"] & & H_{i,1} \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& F_i \arrow[dl, "p"] \arrow[dr, "\jmath"'] & \\
K_{i,1,0} \times \dots \times K_{i,m_i,0} & & H_{i,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
H_{1,1} \times \dots \times H_{n,1} \arrow[dd, "{(\partial,\dots,\partial)}"'] \arrow[dr, "s"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "p"] \arrow[dr, "\jmath"'] & \\
H_{1,0} \times \dots \times H_{n,0} & & G_0
\end{tikzcd}$$

Theorem

$$\begin{tikzcd}
K_{1,1,1} \times \dots \times K_{n,m_n,1} \arrow[dd, "{(\partial,\dots,\partial)}"'] \arrow[dr, "s"] & & H_{i,1} \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& (F_1 \times \dots \times F_n) \bigtimes^{H_{1,1} \times \dots \times H_{n,1}}_{H_{1,0} \times \dots \times H_{n,0}} E \arrow[dl, "p"] \arrow[dr, "\jmath"'] & \\
K_{1,1,0} \times \dots \times K_{n,m_n,0} & & H_{i,0}
\end{tikzcd}$$

Multi-categorical structures

Informal definition

A Multi-Bicategory $\mathsf{M}\mathfrak{C}$ is defined in the same as a multicategory, except the objects

$$\mathrm{Hom}_{\mathsf{M}\mathfrak{C}}(x_1,\dots,x_n;y)$$

are groupoids.

Theorem

- $\mathfrak{SGrSt}(\mathbf{C})$ *and* $\mathfrak{SXMod}(\mathbf{C})$ *are promoted to multi-bicategories with* Hom*-objects* $\mathrm{H}{\scriptstyle\mathrm{OM}}(\mathscr{H}_1,\dots,\mathscr{H}_n;\mathscr{G})$ *and* $\mathrm{ME}{\scriptstyle\mathrm{XT}}(\mathsf{H}_1,\dots,\mathsf{H}_n;\mathsf{G})$*, respectively.*
- *There is an equivalence of multi-bicategories*

$$\mathsf{M}a\colon \mathsf{M}\mathfrak{SXMod} \xrightarrow{\sim} \mathsf{M}\mathfrak{SGrSt}$$

induced by the associated stack functor.

Multi-categorical structures

Informal definition

A Multi-Bicategory $\mathrm{M}\mathfrak{C}$ is defined in the same as a multicategory, except the objects

$$\mathrm{Hom}_{\mathrm{M}\mathfrak{C}}(x_1, \dots, x_n; y)$$

are groupoids.

Theorem

- $\mathfrak{SGrSt}(\mathbf{C})$ *and* $\mathfrak{SXMod}(\mathbf{C})$ *are promoted to multi-bicategories with* Hom*-objects* $\mathrm{H{\scriptstyle OM}}(\mathscr{H}_1, \dots, \mathscr{H}_n; \mathscr{G})$ *and* $\mathrm{ME{\scriptstyle XT}}(\mathsf{H}_1, \dots, \mathsf{H}_n; \mathsf{G})$*, respectively.*
- *There is an equivalence of multi-bicategories*

$$\mathrm{M}a \colon \mathrm{M}\mathfrak{SXMod} \xrightarrow{\sim} \mathrm{M}\mathfrak{SGrSt}$$

induced by the associated stack functor.

Presentations of categorical rings

Given:

- $\boxtimes\colon \mathscr{R}\times\mathscr{R} \longrightarrow \mathscr{R}$ biexact
- A presentation $R_1 \to R_0 \to \mathscr{R}$

Question

Additional structure on $(\partial\colon R_1 \to R_0, c)$ so that

$$\begin{tikzcd} {[R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim} \times [R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim}} \arrow[r] \arrow[d] & {[R_1 \rtimes R_0 \rightrightarrows R_0]^{\sim}} \arrow[d] \\ \mathscr{R}\times\mathscr{R} \arrow[r, "\boxtimes"] & \mathscr{R} \end{tikzcd}$$

commutes up to a (coherent) 2-morphism. Top-row is biadditive.

Caveat

Not a degree-wise biexact functor on $[R_1 \rtimes R_0 \rightrightarrows R_0]$!

Presentations of categorical rings

Given:

- $\boxtimes\colon \mathscr{R}\times\mathscr{R}\longrightarrow\mathscr{R}$ biexact
- A presentation $R_1\to R_0\to\mathscr{R}$

Answer

The biadditive functor $\boxtimes\colon \mathscr{R}\times\mathscr{R}\to\mathscr{R}$ is equivalent to a biextension (=2-butterfly)

$$
\begin{tikzcd}
R_1\times R_1 \arrow[dd, "{(\partial,\partial)}"'] \arrow[dr] & & R_1 \arrow[dl] \arrow[dd, "\partial"] \\
& E^2_{\boxtimes} \arrow[dl] \arrow[dr] & \\
R_0\times R_0 & & R_0
\end{tikzcd}
$$

In fact, due to coherence, we have n-butterflies

$$E^2_{\boxtimes} \bigtimes (E^2_{\boxtimes}\times I) \longleftarrow E^3_{\boxtimes} \longrightarrow E^2_{\boxtimes} \bigtimes (I\times E^2_{\boxtimes}), \quad E^4_{\boxtimes}, \dots$$

Classification by Mac Lane cohomology

Given:

- $\mathscr{R}$ ring-like
- A presentation $R_1 \xrightarrow{\partial} R_0 \xrightarrow{\pi} \mathscr{R}$

Postnikov invariant of the stable crossed module

$$k(R) \in H^5(K(B,3),A),$$

$A = \pi_1 = \operatorname{Ker}\partial$ and $B = \pi_0 = \operatorname{Coker}\partial$
$\mathscr{R}$ ring-like implies B is a ring, and A a B-bimodule of $\mathrm{Sh}(\mathbf{C})$.

Theorem (Jibladze and Pirashvili (2007), Quang (2013), and Aldrovandi (2017))

The data $E^2_{\boxtimes}, E^3_{\boxtimes}, E^4_{\boxtimes}, \dots$ give rise to a cohomology class whose $E^{3,0}$-term (think $K(B,3)$ as a simplicial object of $\mathrm{Sh}(\mathbf{C})$) lives in the standard

$$HML^3(B,A)$$

the third Mac Lane cohomology of B with coefficients in A.

Thank you

Aldrovandi, Ettore (2017). “Biextensions, bimonoidal functors, multilinear functor calculus, and categorical rings”. In: *Theory and Applications of Categories* 32.27, pp. 889–969. arXiv: 1501.04664 [math.CT]. URL: http://www.tac.mta.ca/tac/volumes/32/27/32-27abs.html.

Aldrovandi, Ettore and Behrang Noohi (2009). “Butterflies I: Morphisms of 2-group stacks”. In: *Advances in Mathematics* 221, pp. 687–773. DOI: doi:10.1016/j.aim.2008.12.014. arXiv: 0808.3627 [math.CT].

Jibladze, Mamuka and Teimuraz Pirashvili (2007). “Third Mac Lane cohomology via categorical rings”. In: *J. Homotopy Relat. Struct.* 2.2, pp. 187–216. arXiv: math/0608519.

Joyal, André and Ross Street (1993). “Braided tensor categories”. In: *Adv. Math.* 102.1, pp. 20–78. ISSN: 0001-8708.

Mumford, David (1969). “Bi-extensions of formal groups”. In: *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*. London: Oxford Univ. Press, pp. 307–322.

Quang, Nguyen Tien (2013). “Cohomological classification of Ann-categories”. In: *Math. Commun.* 18.1, pp. 151–169. ISSN: 1331-0623.

Grothendieck, A. (1972). “Biextensions de faisceaux de groupes”. In: *Groupes de monodromie en géométrie algébrique. I*. Vol. 288. Lecture Notes in Mathematics. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Berlin: Springer-Verlag. Chap. VII, pp. 133–217.