Formalizing a sophisticated definition

Patrick Massot (Orsay)

joint work with

Kevin Buzzard (IC London) and Johan Commelin (Freiburg)

Formal Methods in Mathematics – Lean Together
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Theorem

Let $A \subset \mathbb{R}^p$ be a dense subset. Every uniformly continuous function $f : A \to \mathbb{R}^q$ extends to a (uniformly) continuous function $\tilde{f} : \mathbb{R}^p \to \mathbb{R}^q$. 

Example:

$(+) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \subset \mathbb{R}$ extends to $(+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

But multiplication or inversion are not uniformly continuous.
Extending functions

Theorem

Let $A \subset \mathbb{R}^p$ be a dense subset. Every uniformly continuous function $f : A \to \mathbb{R}^q$ extends to a (uniformly) continuous function $\bar{f} : \mathbb{R}^p \to \mathbb{R}^q$.

For every $x \in \mathbb{R}^p$, choose a sequence $a : \mathbb{N} \to A$ converging to $x$. Uniform continuity of $f$ ensures $f \circ a$ is Cauchy, completeness of $\mathbb{R}^q$ gives a limit $y$. Set $\bar{f}(x) = y$. Then prove continuity of $\bar{f}$.

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Theorem

$A \subset \mathbb{R}^p$ dense subset. If $f : A \to \mathbb{R}^q$ is continuous and

$$\forall x \in \mathbb{R}^p, \exists y \in \mathbb{R}^q, \forall u : \mathbb{N} \to A, u_n \to x \Rightarrow f(u_n) \to y$$

then $f$ extends to a continuous function $\tilde{f} : \mathbb{R}^p \to \mathbb{R}^q$.

This applies to multiplication $\mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$. 

A better framework?

In order to handle inversion $\mathbb{Q}^* \rightarrow \mathbb{R}^*$ and more general spaces, we want a version where $\mathbb{R}^p$ and $\mathbb{R}^q$ are replaced by general topological spaces $X$ and $Y$. 
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We can still say that $f(x)$ converges to $y$ when $x$ tends to $x_0$ while remaining in $A$:

$$\forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \forall a \in A \cap V, f(a) \in W.$$
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**Theorem**

Let $X$ be a topological space, $A$ a dense subset of $X$, and $f : A \rightarrow Y$ a continuous mapping of $A$ into a regular space $Y$. If, for each $x_0 \in X$, $f(x)$ tends to a limit in $Y$ when $x$ tends to $x_0$ while remaining in $A$ then $f$ extends to a continuous map $\overline{f} : X \rightarrow Y$. 
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Hint: \( \mathbb{Q} \not\subset \mathbb{R} \).

Better framework:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow i & & \downarrow \exists? \tilde{f} \\
X & & \\
\end{array}
\]
Does this theorem really applies to $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{R} \times \mathbb{R}$?

**Hint:** $\mathbb{Q} \not\subset \mathbb{R}$.

Better framework:

$$
\begin{array}{c}
A \\ i \\ X
\end{array} \xrightarrow{f} \begin{array}{c}
Y \\ \exists \bar{f}
\end{array}
$$

**Issue:** will we need discussions of

$$
\begin{array}{c}
\mathbb{Q} \\ \downarrow \\ \mathbb{R}
\end{array} \xrightarrow{inv} \begin{array}{c}
\mathbb{Q}^* \\ \downarrow \\ \mathbb{R}^*
\end{array} \xrightarrow{inv} \begin{array}{c}
\mathbb{Q} \\ \downarrow \\ \mathbb{R}
\end{array}
$$
Side issue: how to formally refer to \( \tilde{f} \)?

\[ \text{extend } f i \text{ de } h \text{ where } \text{de} \text{ is a proof that } i \text{ is a dense topological embedding, and } h \text{ is a proof that } f \text{ admits a limit...?} \]
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`extend f i de h` where `de` is a proof that $i$ is a dense topological embedding, and `h` is a proof that $f$ admits a limit...?

This looks clunky. Note that $i$ and $f$ can be inferred from the types of `de` and `h`. Should we use `extend de h`?
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$\text{extend } f \ i \ de \ h$ where $\text{de}$ is a proof that $i$ is a dense topological embedding, and $h$ is a proof that $f$ admits a limit...?

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A better solution is to define an extension operator $E_i$ by:

$$E_i(f)(x) = \begin{cases} 
\text{some } y \text{ such that } f(a) \text{ tends to } y \text{ when } a \text{ tends to } x \\
\text{some junk value if no such } y \text{ exists}
\end{cases}$$

Density of image of $i$ is used only to ensure $Y$ is non-empty!
Then use $\text{de} \ . \text{extend } f$
Separation issues

We want to generalize the story going from \( \mathbb{Q} \) to \( \mathbb{R} \), starting with a general topological ring \( R \) (not necessarily metric, or even separated).
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There is a notion of completeness of a topological ring. One can build some $\hat{R}$ which is a (minimal) complete separated space and a natural map $i : R \to \hat{R}$. We want to extend addition and multiplication.
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There is a notion of completeness of a topological ring. One can build some $\widehat{R}$ which is a (minimal) complete separated space and a natural map $i : R \to \widehat{R}$. We want to extend addition and multiplication.

The $i$ map is not injective if $\{0\}$ is not closed in $R$. 
Define $R' = R/\{0\}$ which is separated. Since $\{0\}$ is an ideal, $R'$ inherits addition and multiplication. Continuity is slightly tricky, but ok.
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Then redefine $\hat{R} = \hat{R}'$. 
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Then redefine $\widehat{R} = \widehat{R}'$.

Note: Even in ZFC, if $R$ is already separated, $R' \neq R$. 
Final extension theorem

Return to \[ A \xrightarrow{f} Y \]
Assume \( Y \) is not empty so we can define \( E_i \) without any assumption on \( i \).

**Theorem**

Fix \( x_0 \in X \). If, for every \( x_1 \) in a neighborhood of \( x_0 \), \( f(a) \) tends to a limit in \( Y \) when \( i(a) \) tends to \( x_1 \) then \( E_i f \) is continuous at \( x_0 \).
Final extension theorem

Return to Assume $Y$ is not empty so we can define $E_i$ without any assumption on $i$.

Theorem

Fix $x_0 \in X$. If, for every $x_1$ in a neighborhood of $x_0$, $f(a)$ tends to a limit in $Y$ when $i(a)$ tends to $x_1$ then $E_i f$ is continuous at $x_0$.

If in addition $x_0 = i(a)$, $f$ is continuous at $a$ and $i$ pulls back the topology of $X$ to the topology of $A$ then $E_i[f](i(a)) = f(a)$. 

The assumption in the first part of the previous theorem can be written as

$$\exists U \in \mathcal{N}_{x_0}, \ \forall x \in U, \ \exists y \in Y, \ \forall W \in \mathcal{N}_y, \ \exists V \in \mathcal{N}_x,$$

$$\forall a \in A, i(a) \in V \Rightarrow f(a) \in W.$$
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In mathlib, the assumption is written as

\[ \{ x \mid \exists y, f_\ast i_\ast \mathcal{N}_x \leq \mathcal{N}_y \} \in \mathcal{N}_{x_0}. \]
Perfectoid project assessment

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- Big projects are good. Next one?