Gromov hyperbolic spaces in proof assistants

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**Theorem 1.1.** Let \( a(n, \omega) \) be an integrable and subadditive cocycle relative to the ergodic system \((\Omega, \mu, T)\) as above, with finite asymptotic average \( A \). Then for almost every \( \omega \) there are integers \( n_i := n_i(\omega) \to \infty \) and positive real numbers \( \delta_\ell := \delta_\ell(\omega) \to 0 \) such that for every \( i \) and every \( \ell \leq n_i \),

\[
-\ell \delta_\ell(\omega) \leq a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) - A \ell \leq \ell \delta_\ell(\omega).
\]

**Remark 1.3.** As a test case for the usability of proof assistants for current mathematical research, Theorem 1.1 and its proof given below have been completely formalized and checked in the proof assistant Isabelle/HOL, see the file `Gouezel_Karlsson.thy` in [Go15]. In particular, the correctness of this theorem is certified.

```isar
code
locale conservative_limit =
  conservative M + PS: prob_space P + PZ: real_distribution Z
  for M::"a measure" and P::"a measure" and Z::"real measure" +
  fixes f g::"a ⇒ real" and B::"nat ⇒ real"
  assumes PabsM: "absolutely_continuous M P"
  and Bpos: "∀n. B n > 0"
  and M [measurable]: "f ∈ borel_measurable M" "g ∈ borel_measurable M" "sets P = sets M"
  and non_trivial: "PZ.prob {0} < 1"
  and conv: "weak_conv_m (λn. distr P borel (λx. (g x + birkhoff_sum f n x) / B n)) Z"

datatype

theorem subexponential_growth:
  "(\lambda n. max 0 (ln (B n) /n)) → 0"
Theorem (SG, 2020?)

In a Gromov-hyperbolic group, excursions of length $n$ of a random walk converge in distribution, as metric spaces, towards the continuous random tree.
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The statement involves probability, analysis...
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The statement involves probability, analysis, algebra.
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The statement involves probability, analysis, algebra, geometry.
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The statement involves probability, analysis, algebra, geometry. Additionally, the proof involves complex analysis in Banach spaces, spectral theory of operators, graph theory, potential theory, dynamical systems...
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No hope to formalize the proof in a proof assistant. What about the statement?
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No hope to formalize the proof in a proof assistant. What about the statement? Still very far.
A metric space is Gromov-hyperbolic if there exists $\delta \geq 0$ such that, for all $x, y, z, w$,

$$d(x, y) + d(z, w) \leq \max(d(x, z) + d(y, w), d(x, w) + d(y, z)) + \delta.$$ 

Captures the notion of negative curvature on large scale.
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Captures the notion of negative curvature on large scale.

Geometric intuition when the space is geodesic (i.e., any two points can be joined by a geodesic): triangles are thin.
Theorem (Bonk-Schramm, 2000)

Any $\delta$-hyperbolic metric space embeds isometrically in a $\delta$-hyperbolic geodesic metric space.
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Lemma

Assume that $X$ is $\delta$-hyperbolic. Let $x, y \in X$. If there is no midpoint between $x$ and $y$, one can add one while retaining $\delta$-hyperbolicity.

Proof.

Set $d(m, z) = d(x, y)/2 + \sup_w (d(z, w) - \max(d(a, w), d(b, w)))$.

It works.
Theorem (Bonk-Schramm, 2000)

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It works.

Proof of Bonk-Schramm Theorem.

Enumerate all pairs of points. Add middles, then complete, and do it all over again until it stops by transfinite induction.
instantiation Bonk_Schramm_extension :: (Gromov_hyperbolic_space) Gromov_hyperbolic_space_geodesic
begin
definition deltaG_Bonk_Schramm_extension ::: "('a Bonk_Schramm_extension) itself ⇒ real" where
  "deltaG_Bonk_Schramm_extension _ = deltaG(TYPE('a))"

instance apply standard
unfolding deltaG_Bonk_Schramm_extension_def using Bonk_Schramm_extension_hyperbolic by auto
end (* of instantiation proof *)
Key point: use an inductive type to model both the middle construction and the completion:

```markdown
datatype 'a Bonk_Schramm_extension_unfolded =
  basepoint 'a
| middle "'a Bonk_Schramm_extension_unfolded" "'a Bonk_Schramm_extension_unfolded"
| would_be_Cauchy "nat ⇒ 'a Bonk_Schramm_extension_unfolded"
```
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```plaintext
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Lesson 1

Inductive types are useful (even for mathematicians)
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**Lesson 1**
Inductive types are useful (even for mathematicians)

**Lesson 1’**
Computer scientists are useful (even for mathematicians)

(datatype package in Isabelle/HOL, by Blanchette and al.)
**Definition**

Let $\lambda \geq 1$ and $C \geq 0$. A $(\lambda, C)$-quasigeodesic is a map $f : [a, b] \to X$ such that, for all $s, t \in [a, b]$,

$$\lambda^{-1}|t - s| - C \leq d(f(s), f(t)) \leq \lambda|t - s| + C.$$

**Theorem (Morse Lemma)**

Let $f : [a, b] \to X$ be a $(\lambda, C)$-quasigeodesic, where $X$ is $\delta$-hyperbolic. Then there exists $A = A(\lambda, C, \delta)$ such that $f[a, b]$ and a geodesic from $f(a)$ to $f(b)$ are at distance at most $A$. 

**Theorem (Shchur, 2013)**

One can take $A(\lambda, C, \delta) = 37723\lambda^2(C + \delta)$. Optimal, up to the constant 37723.
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and because the function $e^{-x}$ is decreasing for $X \geq 0$, we can use the estimate

$$\sum_{i=1}^{n} e^{-X_i} (X_{i-1} - X_i) \leq \int_{0}^{\infty} e^{-x} dX = -e^{-x}|_{0}^{\infty} = 1.$$ 

Summarizing all the facts, returning to the initial notation, and recalling that $K = \ln 2/19$, we finally obtain the claimed result

$$H = 4\lambda^2 \left( 78c + \left( 78 + \frac{133}{\ln 2} e^{157\ln 2/38} \right) \delta \right). \quad \Box$$
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*One can take* $A(\lambda, C, \delta) = 92\lambda^2 (C + \delta)$.

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Lesson 2

Mathematicians (as a community) can be wrong, and proof assistants can already help.
Numerical constants are irrelevant in Gromov-hyperbolic geometry. But still, 37723 in Shchur, 92 in Gouëzel-Shchur!

In fact, our constant is $3200 \times \exp\left(-\frac{459}{50} \times \ln 2\right) / \ln 2 + 84$. Sage says it's 91.959195220789730234910660935...
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**by** (approximation 13)
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```markdown
lemma ineq2:

```
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```

by (approximation 98)
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**by** (approximation 98)

**Lesson 2’**

Computer scientists are useful

(approximation package in Isabelle/HOL, by Hölzl, while an undergrad)
Definition

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Definition

Gromov-Hausdorff space: space of all nonempty compact metric spaces up to isometry, with the Gromov-Hausdorff distance.
Theorem

The Gromov-Hausdorff space is a complete second-countable metric space (a.k.a. Polish space).

One can do probability theory on the Gromov-Hausdorff space.
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I formalized the proof of this theorem, but not in Isabelle/HOL because I can not make sense of the sentence “a sequence of compact metric types converges to a compact metric type there”. I formalized it in Lean 3.

```lean
/* The Gromov-Hausdorff space is second countable. */
instance second_countable : second_countable_topology GH_space :=

/* The Gromov-Hausdorff space is complete. */
instance : complete_space (GH_space) :=
```
Lesson 3

Dependent types are useful (especially to mathematicians)
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(Lean 3, developed by de Moura et al.)