Higher Algebra in Homotopy Type Theory

Ulrik Buchholtz

TU Darmstadt

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1. Homotopy Type Theory & Univalent Foundations

2. Higher Groups

3. Higher Algebra

Outline

Homotopy Type Theory & Univalent Foundations

First, recall:

- Homotopy Type Theory (HoTT):
  - A branch of mathematics (& logic/computer science/philosophy) studying
    the connection between homotopy theory & Martin-Löf type theory
  - Specific type theories: Typically MLTT + Univalence
    (+ HITs + Resizing + Optional classicality axioms)
  - Without the classicality axioms: many interesting models
    (higher toposes)
    – one potential reason to case about constructive math
- Univalent Foundations:
  - Using HoTT as a foundation for mathematics.
    Basic idea: mathematical objects are (ordinary) homotopy types.
    (no entity without identity – a notion of identification)
  - Avoid "higher groupoid hell": We can work directly with homotopy types
    and we can form higher quotients.
    (But can we form enough? More on this later.)

Cf.: The HoTT book & list of references on the HoTT wiki

Homotopy levels

Recall Voevodsky's definition of the homotopy levels:

Level	Name	Definition
-2	contractible	$\mathrm{isContr}(A) :\equiv (x : A) \times \big((y : A) \to (x = y)\big)$
-1	proposition	$\mathrm{isProp}(A) :\equiv (x, y : A) \to \mathrm{isContr}(x = y)$
0	set	$\mathrm{isSet}(A) :\equiv (x, y : A) \to \mathrm{isProp}(x = y)$
1	groupoid	$\mathrm{isGpd}(A) :\equiv (x, y : A) \to \mathrm{isSet}(x = y)$
$\vdots$	$\vdots$	$\vdots$
n	n-type	$\cdots$
$\vdots$	$\vdots$	$\vdots$
∞	type	(N/A)

In non-homotopical mathematics, most objects are n-types with $n \leq 1$.

The types of categories and related structures are 2-types.

n-Stuff, Structure, and Properties

In HoTT, any map f is equivalent to a projection $(x : A) \times B(x) \to A$.

If the types $B(x)$ are n-types, we say that f forgets only n-stuff.
We say that f is an equivalence if f forgets only -2-stuff.

-1-stuff is properties.

0-stuff is structure.

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Univalence axiom

For $A, B : \mathrm{Type}$, the map $\mathrm{id\text{-}to\text{-}equiv}_{A,B} : (A =_{\mathrm{Type}} B) \to (A \simeq B)$ is an equivalence.

Synthetic homotopy theory

- In the HoTT book: Whitehead's theorem, $\pi_1(S^1)$, Hopf fibration, etc.
- Quaternionic Hopf fibration (B–Rijke)
- (Generalized) Blakers-Massey theorem (Favonia–Finster–Licata–Lumsdaine, Anel–Biedermann–Finster–Joyal)
- Gysin sequence, Whitehead products and $\pi_4(S^3)$ (Brunerie)
- Homology and cohomology theories, cellular cohomology (B–Favonia)
- Modalities (Rijke–Shulman–Spitters)
- p-localization (Christensen–Opie–Rijke–Scoccola)
- Serre spectral sequence for any cohomology theory (van Doorn *et al.* following outline by Shulman)

Recent Progress on the Meta Theory of HoTT

Last year we made progress on several meta-theoretical problems:

- Coquand–Huber–Sattler proved (arXiv:1902.06572) in *Homotopy canonicity for cubical type theory* that

  > *cubical type theory* [wrt the Dedekind cubes] *is homotopically sound in that we can only derive statements which hold in the interpretation in standard homotopy types.*

- Shulman proved (arXiv:1904.07004) that

  > *any Grothendieck $(\infty, 1)$-topos can be presented by a Quillen model category that interprets homotopy type theory with strict univalent universes.*

- Kapulkin–Sattler proved (slides) Voevodsky's *homotopy canonicity conjecture*:

  > *For any closed term n of natural number type, there is $k \in \mathbb{N}$ with a closed term p of the identity type relating n to the numeral $\mathrm{S}^k\, 0$. Both n and p may make use of the univalence axiom.*

HoTT systems and libraries

Lots of experiments with systems and libraries for HoTT/UF:

- Lean 2
- Lean 3
- Coq (HoTT & UniMath)
- Agda
- Cubical Agda
- RedPRL
- Arend
- ...

Outline

1. Homotopy Type Theory & Univalent Foundations
2. Higher Groups
3. Higher Algebra

Higher Groups

We define $n\text{-Group} :\equiv \mathrm{Type}_{\mathsf{pt}}^{\geq 1, \leq n}$, (pointed, connected, n-types) as the type of n-groups. If G is an n-group, then we write $\mathrm{B}G$ for this pointed, connected type (the classifying type). We allow $n = \infty$.

The type of group elements of G is usually also called G and is defined by $G :\equiv \Omega \mathrm{B}G : \mathrm{Type}_{\mathsf{pt}}^{<n}$. If $n = 1$, then G is a set.

The group operation is path concatenation.

We have an equivalence:

$$n\text{-Group} \simeq (G : \mathrm{Type}_{\mathsf{pt}}^{<n}) \times (\mathrm{B}G : \mathrm{Type}_{\mathsf{pt}}^{\geq 1}) \times (G \simeq_{\mathsf{pt}} \Omega \mathrm{B}G)$$

Automorphism groups

If A is a type and $a : A$, then $\mathrm{Aut}_A(a)$ is a higher group with $\mathrm{BAut}_A(a) :\equiv \mathrm{im}(a : 1 \to A)$.

We have $\mathrm{Aut}_A(a) \simeq (a =_A a)$. The group elements are identifications of a with itself.

Thus, all higher groups are automorphism groups. (Concrete groups.)

Since we have many higher types lying around, such as Set, Group, $R\text{-}\mathrm{Mod}$, Ring, Cat, etc., we get many useful examples.

Some group theory

$\mathrm{B}G \to_{\mathsf{pt}} \mathrm{B}H$	homomorphisms $G \to H$
$\mathrm{B}G \to \mathrm{B}H$	(animated) conjugacy class of homomorphisms
$\mathrm{B}\mathbb{Z} \to_{\mathsf{pt}} \mathrm{B}H$	element of H
$\mathrm{B}\mathbb{Z} \to \mathrm{B}H$	(animated) conjugacy class in H
$\mathrm{B}G \to A$	A-action of G
$\mathrm{B}G \to_{\mathsf{pt}} \mathrm{BAut}(a)$	action of G on $a : A$
$X : \mathrm{B}G \to \mathrm{Type}$	type with an action of G
$(x : BG) \times X(x)$	quotient, (animated) orbit type
$(x : BG) \to X(x)$	fixed points

Symmetric higher groups

Let us introduce the type

$$
\begin{aligned}
(n,k)\text{-Group} &:\equiv \mathrm{Type}_{\mathsf{pt}}^{\geq k,<n+k} \\
&\simeq (G : \mathrm{Type}^{<n}) \times (\mathrm{B}^k G : \mathrm{Type}_{\mathsf{pt}}^{\geq k}) \times (G \simeq_{\mathsf{pt}} \Omega^k \mathrm{B}^k G)
\end{aligned}
$$

for the type of k*-symmetric* n*-groups*.

We can also allow k to be infinite, $k = \omega$, but in this case we can't cancel out the G and we must record all the intermediate delooping steps:

$$
\begin{aligned}
(n,\omega)\text{-Group} := \big(&\mathrm{B}^{-} G : (k : \mathbb{N}) \to \mathrm{Type}_{\mathsf{pt}}^{\geq k,<n+k}\big) \\
&\times \big((k : \mathbb{N}) \to \mathrm{B}^k G \simeq_{\mathsf{pt}} \Omega \mathrm{B}^{k+1} G\big)
\end{aligned}
$$

The periodic table

$k \setminus n$	1	2	$\cdots$	∞
0	pointed set	pointed groupoid	$\cdots$	pointed ∞-groupoid
1	group	2-group	$\cdots$	∞-group
2	abelian group	braided 2-group	$\cdots$	braided ∞-group
3	— " —	symmetric 2-group	$\cdots$	sylleptic ∞-group
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω	— " —	— " —	$\cdots$	connective spectrum

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decategorication	(n, k)-Group $\to (n-1, k)$-Group,
discrete categorification	(n, k)-Group $\to (n+1, k)$-Group,
looping	(n, k)-Group $\to (n-1, k+1)$-Group
delooping	(n, k)-Group $\to (n+1, k-1)$-Group
forgetting	(n, k)-Group $\to (n, k-1)$-Group
stabilization	(n, k)-Group $\to (n, k+1)$-Group

Some more examples

- $\mathrm{B}\mathbb{Z} = \mathbb{S}^1$, other free groups on pointed sets, free abelian groups.
- Fundamental n-group of (A, a), $\pi_1^{(n)}(A, a)$, with corresponding delooping $\mathrm{B}\pi_1^{(n)}(A, a) :\equiv \|\mathrm{BAut}_A\, a\|_n \simeq \mathrm{BAut}_{\|A\|_n}(|a|)$.
- Higher homotopy n-groups of (A, a), $\pi_k^{(n)}(A, a)$, with $\mathrm{B}^k\pi_k^{(n)}(A, a) = \|A\langle k-1\rangle\|_{n+k-1}$.
  The underlying type of elements is $\pi_k^{(n)}(A, a) \simeq_{\mathsf{pt}} \Omega^k(A, a)$.
- $\mathbb{S}^1 \simeq \mathrm{B}\mathbb{Z}$ has delooping $\mathrm{B}^2\mathbb{Z}$, which we can take to be the type of oriented circles.
- For any 1-group G, $\mathrm{B}^2\mathrm{Z}(G)$ is the type of G-banded gerbes (nonabelian cohomology).
- $\vdots$

The stabilization theorem

Theorem (Freudenthal)

If $A : \mathrm{Type}_{\mathsf{pt}}^{>n}$ *with* $n \geq 0$*, then the map* $A \to \Omega\Sigma A$ *is* $2n$*-connected.*

Corollary (Stabilization)

If $k \geq n+1$*, then* $S : (n,k)\text{-Group} \to (n,k+1)\text{-Group}$ *is an equivalence, and any* $G : (n,k)\text{-Group}$ *is an infinite loop space.*

(Formalized in Lean 2)

1-Categorical equivalences

Theorem

We have the following equivalences of 1-categories (*for* $k \geq 2$):

$$(1,0)\text{-Group} \simeq \text{Set}_{\mathsf{pt}};$$
$$(1,1)\text{-Group} \simeq \text{Group};$$
$$(1,k)\text{-Group} \simeq \text{AbGroup}.$$

(Formalized in Lean 2)

Short sequences of higher groups

The following is from: *The long exact sequence of homotopy n-groups* (B–Rijke, arXiv:1912.08696)

Definition

A **short sequence** (or **complex**) of k-symmetric ∞-groups consists of three k-symmetric ∞-groups K, G, H and homomorphisms

$$K \xrightarrow{\psi} G \xrightarrow{\varphi} H,$$

with a identification of $\varphi \circ \psi$ with the trivial homomorphism from K to H *as homomorphisms*. By definition, this means we have a short sequence

$$\mathrm{B}^k K \xrightarrow{\mathrm{B}^k \psi} \mathrm{B}^k G \xrightarrow{\mathrm{B}^k \varphi} \mathrm{B}^k H,$$

of classifying types, i.e., a commutative square with $\mathbb{1}$ in the corner.

Kernels and images

Definition

Given a homomorphism of k-symmetric n-groups $\varphi : G \to H$, we define its **kernel** $\ker(\varphi)$ via the classifying type $\mathrm{B}^k \ker(\varphi) :\equiv \mathrm{fib}(\mathrm{B}^k\varphi)\langle k-1\rangle$.

The type of elements of the kernel is then the fiber of φ.

Definition

Given a homomorphism of k-symmetric n-groups $\varphi : G \to H$, we define the **image** via the classifying type $\mathrm{B}^k \operatorname{im}(\varphi)$ as it appears in the $(n+k-2)$-image factorization of $\mathrm{B}^k\varphi$:

$$\mathrm{B}^k G \longrightarrow \mathrm{B}^k \operatorname{im}(\varphi) \longrightarrow \mathrm{B}^k H,$$

viz., $\mathrm{B}^k \operatorname{im}(\varphi) :\equiv (t : B^k H) \times \|\mathrm{fib}_{\mathrm{B}^k\varphi}(t)\|_{n+k-2}$.

When $n = \infty$, the image is just G again.

Exact sequences of higher groups

Definition

A short sequence of k-symmetric n-groups $K \xrightarrow{\psi} G \xrightarrow{\varphi} H$ is n**-exact** (in the middle) if and only if $\mathrm{im}(\psi) \to \ker(\varphi)$ is an equivalence.

Lemma

Consider a short sequence $K \xrightarrow{\psi} G \xrightarrow{\varphi} H$. The following are equivalent:

1. *The short sequence is n-exact.*
2. *The Ω^j-looped short sequence is n-exact, for $0 \leq j \leq k$.*
3. *The induced map of group elements $K \to \ker(\varphi)$ is $(n-2)$-connected.*
4. *The square of maps of group elements*

$$\begin{array}{ccc} K & \xrightarrow{\psi} & G \\ \downarrow & & \downarrow^{\varphi} \\ \mathbb{1} & \longrightarrow & H \end{array}$$

   is $(n-2)$-cartesian.

Truncating n-exact squares

Theorem

The n-truncation modality preserves k-cartesian squares for any $k < n$.

Corollary

Any fiber sequence $F \hookrightarrow E \twoheadrightarrow B$ induces an n-exact short sequence $\|F\|_{n-1} \to \|E\|_{n-1} \to \|B\|_{n-1}$.

The long n-exact sequence

Corollary

For any fiber sequence $F \hookrightarrow E \twoheadrightarrow B$ *we obtain a long* n*-exact sequence*

$$\cdots \longrightarrow \pi_k^{(n)}(F) \longrightarrow \pi_k^{(n)}(E) \longrightarrow \pi_k^{(n)}(B) \longrightarrow \cdots$$

of homotopy n*-groups, where the morphisms are homomorphisms of* k*-symmetric* n*-groups whenever the codomain is a* k*-symmetric* n*-group.*

Corollary

Given a short n*-exact sequence of* k*-symmetric* n*-groups* $K \xrightarrow{\psi} G \xrightarrow{\varphi} H$, *the resulting* looped sequence $\Omega K \to \Omega G \to \Omega H$ *is a short* $(n-1)$*-exact sequence of* $(k+1)$*-symmetric* $(n-1)$*-groups, and the resulting* decategorified sequence $\mathrm{Decat}(K) \to \mathrm{Decat}(G) \to \mathrm{Decat}(H)$ *is a short* $(n-1)$*-exact sequence of* k*-symmetric* $(n-1)$*-groups.*

The long n-exact sequence

Corollary

For any fiber sequence $F \hookrightarrow E \twoheadrightarrow B$ we obtain a long n-exact sequence

$$\cdots \longrightarrow \pi_k^{(n)}(F) \longrightarrow \pi_k^{(n)}(E) \longrightarrow \pi_k^{(n)}(B) \longrightarrow \cdots$$

of homotopy n-groups, where the morphisms are homomorphisms of k-symmetric n-groups whenever the codomain is a k-symmetric n-group.

Corollary

Given a short n-exact sequence of k-symmetric n-groups $K \xrightarrow{\psi} G \xrightarrow{\varphi} H$, the resulting looped sequence $\Omega K \to \Omega G \to \Omega H$ *is a short $(n-1)$-exact sequence of $(k+1)$-symmetric $(n-1)$-groups, and the resulting* decategorified sequence $\mathrm{Decat}(K) \to \mathrm{Decat}(G) \to \mathrm{Decat}(H)$ *is a short $(n-1)$-exact sequence of k-symmetric $(n-1)$-groups.*

(Formalization TBD)

Outline

Other kinds of higher algebra

There's more to higher algebra than higher groups:

- Higher monoids (A_∞-algebras)
- Higher categories ($(\infty, 1)$-categories)
- Ring spectra (anywhere from E_1- to E_∞-algebras)
- Module spectra for ring spectra other than $\mathbb{S}$ (even just $H\mathbb{Z}\text{-Mod}$)
- $\ldots$

Unfortunately, we don't know how to approach these in HoTT (or cubical type theory).

The problem and attempted solutions

The problem

Even defining the type of *semi-simplicial types* seems to be impossible. Conversely, if we could define this, then we could bootstrap.

Some attempted solutions:

- Work in a two-level type theory with Reedy limits (cf. Annenkov–Capriotti–Kraus–Sattler, arXiv:1705.03307, inspired by Voevodsky's *Homotopy Type System*)
- Work in a simplicial type theory, modeled by simplicial homotopy types (cf. Riehl–Shulman, *A type theory for synthetic ∞-categories*, and B–Weinberger, in progress).
- Work in $\mathbb{S}$-cohesive type theory, modeled by parametrized spectra, for a convenient synthetic approach to ring- and module-spectra (cf. Finster–Morehouse–Licata–Riley, in progress).
- ...?

Perspectives

A combination

We need *all* of these: the specialized type theories as DSLs and a two-level type theory (or something like it) to interpret the DSLs.

Meanwhile

Meanwhile, there's still lots of higher algebra to do in pure HoTT about higher groups, (co)homology, and applications thereof.

Thank you!