National Aeronautics and Space Administration

Applied Nonlinear Arithmetic in PVS

Anthony Narkawicz
NASA Langley Research Center

In collaboration with
César Muñoz
Aaron Dutle

Carnegie Mellon University
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Aeronautics

NASA Langley Research Center:

Approximately 1/3 Space

and 2/3 Aeronautics

Safety

Safety Critical Avionics Systems Branch

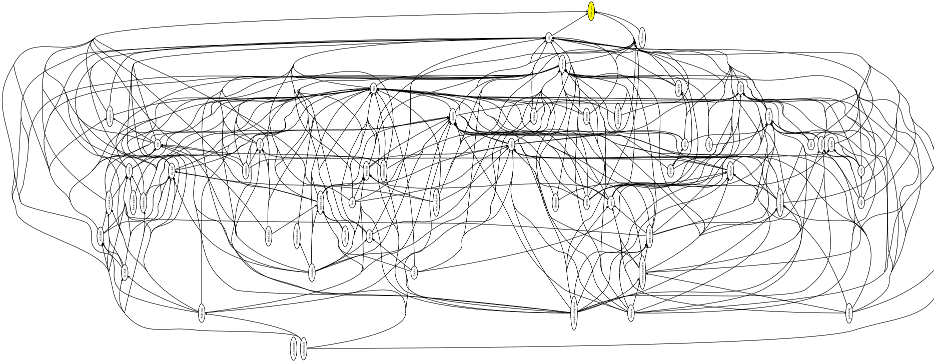
We design **high-integrity** prototype software for research into safety-critical aircraft systems

We don't want incorrect algorithms in safety-critical aircraft systems

Theorem Proving at NASA

The formal methods group at NASA Langley has proved $> 20K$ theorems in PVS from a variety of areas of mathematics

NASA PVS Libraries: `https://github.com/nasa/pvslib`



The NASA PVS Libraries

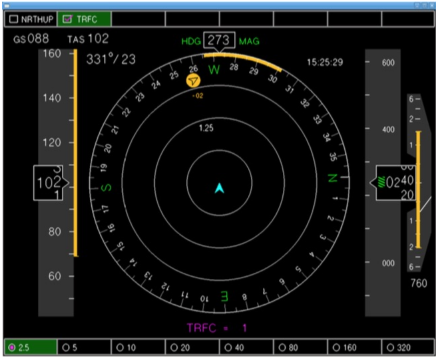
PVS libraries in popular culture

`https://shemesh.larc.nasa.gov/people/cam/TheMartian/`

Example: Daidalus

We have used PVS to verify many of the algorithms in DAIDALUS

Detect and AvoID Alerting Logic for Unmanned Systems



DAIDALUS is the *reference implemenation* for detect and avoid for unmanned aircraft systems in standards document RTCA DO-365

DAIDALUS

$\mathbf{s}_o$, $\mathbf{v}_o$	Horizontal component of ownship's position and velocity
s_{oz}, $\mathbf{v}_{oz}$	Ownship's altitude and vertical speed
$\mathbf{s}_i$, $\mathbf{v}_i$	Horizontal component of intruder's position and velocity
s_{iz}, $\mathbf{v}_{iz}$	Intruder's altitude and vertical speed

Let $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$,

Aircraft have a violation of **well-clear** if they are inside the intersection of horizontal and vertical volumes

$$WCV(\mathbf{s}, \mathbf{s}_z, \mathbf{v}, \mathbf{v}_z) \equiv \texttt{Horizontal\_WCV}(\mathbf{s}, \mathbf{v}) \;\texttt{and}\; \texttt{Vertical\_WCV}(\mathbf{s}_z, \mathbf{v}_z), \qquad (1)$$

Inside this volume the aircraft is **not well clear**.

$$\begin{aligned}
\texttt{Horizontal_WCV}(\mathbf{s}, \mathbf{v}) \equiv\ & \|\mathbf{s}\| \leq \texttt{DMOD} \;\texttt{or} \\
& (d_{\mathrm{cpa}}(\mathbf{s}, \mathbf{v}) \leq \texttt{DMOD} \;\texttt{and}\; 0 \leq \tau_{\mathrm{mod}}(\mathbf{s}, \mathbf{v}) \leq \texttt{TAUMOD}), \\
\texttt{Vertical_WCV}(\mathbf{s}_z, \mathbf{v}_z) \equiv\ & |\mathbf{s}_z| \leq \texttt{ZTHR} \;\texttt{or}\; 0 \leq t_{\mathrm{coa}}(\mathbf{s}_z, \mathbf{v}_z) \leq \texttt{TCOA}.
\end{aligned}$$

Daidalus v2.0

Our current project is Daidalus v2.0

RTCA standards organization requested that we incorporate sensor uncertainty into DAIDALUS

GPS, Radar, and other sensors give information such as variances and covariances for East/North/Up-Down components of positions and velocities

We are trying several different approaches to this problem

Daidalus v2.0

Positions and velocities are random variables

- Inputs $\mathbf{s}, \mathbf{v}$ (reported relative position/velocity) and X,Y (Random Variables)
- Horizontal position is $\mathbf{s} + \begin{pmatrix} X \\ Y \end{pmatrix}$
- Horizontal velocity is $\mathbf{v} + M \cdot \begin{pmatrix} X \\ Y \end{pmatrix}$ (M is a rotation matrix)

We compute random variables, variances, covariances, means, for functions in the well-clear formula

$$[\|\mathbf{s}\| \leq \texttt{DMOD} \ \texttt{or} \ (d_{\mathsf{cpa}}(\mathbf{s},\mathbf{v}) \leq \texttt{DMOD} \ \texttt{and} \ 0 \leq \tau_{\mathsf{mod}}(\mathbf{s},\mathbf{v}) \leq \texttt{TAUMOD})]$$
$$\texttt{and} \ [|\mathbf{s}_z| \leq \texttt{ZTHR} \ \texttt{or} \ 0 \leq t_{\mathsf{coa}}(\mathbf{s}_z,\mathbf{v}_z) \leq \texttt{TCOA}]$$

Daidalus v2.0

The variance of the d_{cpa} function at a future time t is

```
real = 2*(sq(VX)*sq(sq(s`x)*sq(c)+sq(s`y)*sq(a)-2*(s`x*s`y*a*c)-sq(DMOD)*sq(a)-sq(DMOD)*sq(c)))
+2*(sq(VY)*sq(sq(s`x)*sq(d)+sq(s`y)*sq(b)-2*(s`x*s`y*b*d)-sq(DMOD)*sq(b)-sq(DMOD)*sq(d)))
+sq(CXY)*sq(-2*(sq(DMOD)*a*b)-2*(sq(DMOD)*c*d)-2*(s`x*s`y*a*d)-2*(s`x*s`y*b*c)+2*(sq(s`x)*c*d)+2*(sq(s`y)*a*b))
+sq(-2*(s`y*det(s,v)*a)-2*(sq(DMOD)*v`x*a)-2*(sq(DMOD)*v`y*c)+2*(s`x*det(s,v)*c))*VX
+sq(-2*(s`y*det(s,v)*b)-2*(sq(DMOD)*v`x*b)-2*(sq(DMOD)*v`y*d)+2*(s`x*det(s,v)*d))*VY-16*
(VX*s`x*s`y*sq(s`x)*CXY*a*c*c*d)-16*(VX*s`x*s`y*sq(s`y)*CXY*a*a*b*c)-16*(VY*s`x*s`y*sq(s`x)*CXY*b*c*d*d)-16*
(VY*s`x*s`y*sq(s`y)*CXY*a*b*b*d)-8*(VX*sq(s`x)*sq(DMOD)*sq(a)*CXY*c*d)-8*
(VX*sq(s`x)*sq(DMOD)*sq(c)*CXY*a*b)-8*(VX*sq(s`x)*sq(DMOD)*sq(c)*CXY*c*d)-8*
(VX*sq(s`x)*sq(DMOD)*sq(c)*CXY*c*d)-8*(VX*sq(s`y)*sq(DMOD)*sq(a)*CXY*a*b)-8*
(VX*sq(s`y)*sq(DMOD)*sq(a)*CXY*a*b)-8*(VX*sq(s`y)*sq(DMOD)*sq(a)*CXY*c*d)-8*
(VX*sq(s`y)*sq(DMOD)*sq(c)*CXY*a*b)-8*(VY*sq(s`x)*sq(DMOD)*sq(b)*CXY*c*d)-8*
(VY*sq(s`x)*sq(DMOD)*sq(d)*CXY*a*b)-8*(VY*sq(s`x)*sq(DMOD)*sq(d)*CXY*c*d)-8*
(VY*sq(s`x)*sq(DMOD)*sq(d)*CXY*c*d)-8*(VY*sq(s`y)*sq(DMOD)*sq(b)*CXY*a*b)-8*
(VY*sq(s`y)*sq(DMOD)*sq(b)*CXY*a*b)-8*(VY*sq(s`y)*sq(DMOD)*sq(b)*CXY*c*d)-8*
(VY*sq(s`y)*sq(DMOD)*sq(d)*CXY*a*b)-8*(s`x*s`y*sq(s`x)*sq(CXY)*sq(c)*b*d)
-8*(s`x*s`y*sq(s`x)*sq(CXY)*sq(d)*a*c)-8*(s`x*s`y*sq(s`y)*sq(CXY)*sq(a)*b*d)
-8*(s`x*s`y*sq(s`y)*sq(CXY)*sq(b)*a*c)-8*(s`x*s`y*CXY*det(s,v)*det(s,v)*a*d)
-8*(s`x*s`y*CXY*det(s,v)*det(s,v)*b*c)-8*(s`x*sq(DMOD)*v`x*CXY*det(s,v)*a*d)
-8*(s`x*sq(DMOD)*v`x*CXY*det(s,v)*b*c)-8*(s`x*sq(DMOD)*v`y*CXY*det(s,v)*c*d)
-8*(s`x*sq(DMOD)*v`y*CXY*det(s,v)*c*d)-8*(VX*s`x*s`y*sq(s`x)*sq(c)*CXY*a*d)
-8*(VX*s`x*s`y*sq(s`x)*sq(c)*CXY*b*c)-8*(VX*s`x*s`y*sq(s`y)*sq(a)*CXY*a*d)-8*
(VX*s`x*s`y*sq(s`y)*sq(a)*CXY*b*c)-8*(VY*s`x*s`y*sq(s`x)*sq(d)*CXY*a*d)-8*
(VY*s`x*s`y*sq(s`x)*sq(d)*CXY*b*c)-8*(VY*s`x*s`y*sq(s`y)*sq(b)*CXY*a*d)-8*
(VY*s`x*s`y*sq(s`y)*sq(b)*CXY*b*c)-4*(sq(s`x)*sq(CXY)*sq(DMOD)*sq(a)*sq(d))-4*(sq(s`x)*sq(CXY)*sq(DMOD)*sq(b)*sq(c))
-4*(sq(s`x)*sq(CXY)*sq(DMOD)*sq(c)*sq(d))-4*(sq(s`x)*sq(CXY)*sq(DMOD)*sq(c)*sq(d))
-4*(sq(s`y)*sq(CXY)*sq(DMOD)*sq(a)*sq(b))-4*(sq(s`y)*sq(CXY)*sq(DMOD)*sq(a)*sq(b))
-4*(sq(s`y)*sq(CXY)*sq(DMOD)*sq(a)*sq(d))-4*(sq(s`y)*sq(CXY)*sq(DMOD)*sq(b)*sq(c))
+VX*VY*sq(-2*(sq(DMOD)*a*b)-2*(sq(DMOD)*c*d)-2*(s`x*s`y*a*d)-2*(s`x*s`y*b*c)+2*(sq(s`x)*c*d)
+2*(sq(s`y)*a*b))+4*(sq(s`x)*sq(s`x)*sq(CXY)*sq(c)*sq(d))
+4*(sq(s`x)*sq(s`y)*sq(CXY)*sq(a)*sq(d))+4*(sq(s`x)*sq(s`y)*sq(CXY)*sq(b)*sq(c))
+4*(sq(s`y)*sq(s`y)*sq(CXY)*sq(a)*sq(b))+4*(sq(CXY)*sq(DMOD)*sq(DMOD)*sq(a)*sq(b))
+4*(sq(CXY)*sq(DMOD)*sq(DMOD)*sq(a)*sq(d))+4*(sq(CXY)*sq(DMOD)*sq(DMOD)*sq(b)*sq(c))
+4*(sq(CXY)*sq(DMOD)*sq(DMOD)*sq(c)*sq(d))+8*(VX*sq(s`x)*sq(s`x)*sq(c)*CXY*c*d)+
8*(VX*sq(s`x)*sq(s`y)*sq(a)*CXY*c*d)+8*(VX*sq(s`x)*sq(s`y)*sq(c)*CXY*a*b)+8*
(VX*sq(s`y)*sq(s`y)*sq(a)*CXY*a*b)+8*(VX*sq(DMOD)*sq(DMOD)*sq(a)*CXY*a*b)+8*
(VX*sq(DMOD)*sq(DMOD)*sq(a)*CXY*c*d)+8*(VX*sq(DMOD)*sq(DMOD)*sq(c)*CXY*a*b)+8*
(VX*sq(DMOD)*sq(DMOD)*sq(c)*CXY*c*d)+8*(VY*sq(s`x)*sq(s`x)*sq(d)*CXY*c*d)+8*
(VY*sq(s`x)*sq(s`y)*sq(b)*CXY*c*d)+8*(VY*sq(s`x)*sq(s`y)*sq(d)*CXY*a*b)+8*
(VY*sq(s`y)*sq(s`y)*sq(b)*CXY*a*b)+8*(VY*sq(DMOD)*sq(DMOD)*sq(b)*CXY*a*b)+
8*(VY*sq(DMOD)*sq(DMOD)*sq(b)*CXY*c*d)+8*(VY*sq(DMOD)*sq(DMOD)*sq(d)*CXY*a*b)
+8*(VY*sq(DMOD)*sq(DMOD)*sq(d)*CXY*c*d)+8*(sq(s`x)*CXY*det(s,v)*det(s,v)*c*d)
+8*(s`x*s`y*sq(CXY)*sq(DMOD)*sq(a)*b*d)+8*(s`x*s`y*sq(CXY)*sq(DMOD)*sq(b)*a*c)
+8*(s`x*s`y*sq(CXY)*sq(DMOD)*sq(c)*b*d)+8*(s`x*s`y*sq(CXY)*sq(DMOD)*sq(d)*a*c)
+8*(sq(s`y)*CXY*det(s,v)*det(s,v)*a*b)+8*(s`y*sq(DMOD)*v`x*CXY*det(s,v)*a*b)
+8*(s`y*sq(DMOD)*v`x*CXY*det(s,v)*a*b)+8*(s`y*sq(DMOD)*v`y*CXY*det(s,v)*a*d)
+8*(s`y*sq(DMOD)*v`y*CXY*det(s,v)*b*c)+8*(sq(DMOD)*sq(DMOD)*sq(v`x)*CXY*a*b)
+8*(sq(DMOD)*sq(DMOD)*v`x*v`y*CXY*a*d)+8*(sq(DMOD)*sq(DMOD)*v`x*v`y*CXY*b*c)
+8*(sq(DMOD)*sq(DMOD)*sq(v`y)*CXY*c*d)+8*(VX*s`x*s`y*sq(DMOD)*sq(a)*CXY*a*d)+8*
(VX*s`x*s`y*sq(DMOD)*sq(a)*CXY*b*c)+8*(VX*s`x*s`y*sq(DMOD)*sq(c)*CXY*a*d)+8*
(VX*s`x*s`y*sq(DMOD)*sq(c)*CXY*b*c)+8*(VY*s`x*s`y*sq(DMOD)*sq(b)*CXY*a*d)+8*
(VY*s`x*s`y*sq(DMOD)*sq(b)*CXY*b*c)+8*(VY*s`x*s`y*sq(DMOD)*sq(d)*CXY*a*d)+8*
(VY*s`x*s`y*sq(DMOD)*sq(d)*CXY*b*c)+16*(VX*s`x*s`y*sq(DMOD)*CXY*a*a*b*c)+16*
(VX*s`x*s`y*sq(DMOD)*CXY*a*c*c*d)+16*(VY*s`x*s`y*sq(DMOD)*CXY*a*b*b*d)+
16*(VY*s`x*s`y*sq(DMOD)*CXY*b*c*d*d)+16*(sq(s`x)*sq(s`y)*sq(CXY)*a*b*c*d)+
16*(VX*sq(s`x)*sq(s`y)*CXY*a*a*c*d)+16*(VX*sq(s`x)*sq(s`y)*CXY*a*b*c*c)+
16*(VY*sq(s`x)*sq(s`y)*CXY*a*b*d*d)+16*(VY*sq(s`x)*sq(s`y)*CXY*b*b*c*d)
```

This is formally proved in PVS.

The proofs involve reasoning about large and complex expressions with real numbers.

Daidalus v2.0

In the verification, we have to prove

```
Delta_sum: LEMMA sqv(s)>sq(D) AND s*v<0 AND s*nv<0 AND
  Delta[D](s,v)>=0 AND Delta[D](s,nv)>=0 IMPLIES
  Delta[D](s,v+nv)>=0
```

where

```
Delta(s,v) : real =
  sq(D)*sqv(v) - sq(det(s,v))
```

Daidalus v2.0

The proof in PVS reduces to this:

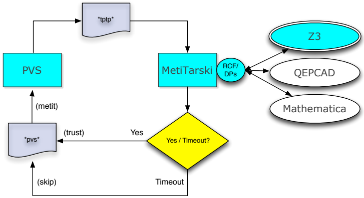
```
[-1]  s`x * s`x + s`y * s`y > D * D
[-2]  s`x * v`x + s`y * v`y < 0
[-3]  s`x * nv`x + s`y * nv`y < 0
[-4]  -1 * (s`x * s`x * v`y * v`y) - s`y * s`y * v`x * v`x +
       2 * (s`x * s`y * v`x * v`y)
       + v`x * v`x * D * D
       + v`y * v`y * D * D
       >= 0
[-5]  -1 * (nv`x * nv`x * s`y * s`y) - nv`y * nv`y * s`x * s`x +
       nv`x * nv`x * D * D
       + 2 * (nv`x * nv`y * s`x * s`y)
       + nv`y * nv`y * D * D
       >= 0
  |-------
{1}   -2 * (nv`x * s`y * s`y * v`x) - 2 * (nv`y * s`x * s`x * v`y) -
       nv`x * nv`x * s`y * s`y
       - nv`y * nv`y * s`x * s`x
       - s`x * s`x * v`y * v`y
       - s`y * s`y * v`x * v`x
       + ((nv`x + v`x) * (nv`x + v`x)) * D * D
       + ((nv`y + v`y) * (nv`y + v`y)) * D * D
       + 2 * (nv`x * nv`y * s`x * s`y)
       + 2 * (nv`x * s`x * s`y * v`y)
       + 2 * (nv`y * s`x * s`y * v`x)
       + 2 * (s`x * s`y * v`x * v`y)
       >= 0

Rule?
```

How do we prove this?

Daidalus v2.0

```
Rule? (metit *)
Substitutions: D -> V1, nv`x -> V2, nv`y -> V3, s`x -> V4, s`y -> V5, v`x -> V6, v`y -> V7
MetiTarski Input =
 fof(pvs2metit,conjecture, (![V1, V2, V3, V4, V5, V6, V7]: ((~ (((V4 * V4) + (V5 * V5)) > (V1 * V1))) | ((~ (((V4 * V6) + (V5
((V4 * V2) + (V5 * V3)) < 0)) | ((~ (((((((-1 / 1) * (((V4 * V4) * V7) * V7)) - (((V5 * V5) * V6) * V6)) + (2 * (((V4 * V5) *
* V6) * V1) * V1)) + (((V7 * V7) * V1) * V1)) >= 0)) | ((~ (((((((-1 / 1) * (((V2 * V2) * V5) * V5)) - (((V3 * V3) * V4) * V4
) * V1)) + (2 * (((V2 * V3) * V4) * V5))) + (((V3 * V3) * V1) * V1)) >= 0)) | (((((((((((((((-2 / 1) * (((V2 * V5) * V5) * V6)
* V4) * V7))) - (((V2 * V2) * V5) * V5)) - (((V3 * V3) * V4) * V4)) - (((V4 * V4) * V7) * V7)) - (((V5 * V5) * V6) * V6)) + ((
6)) * V1) * V1)) + ((((V3 + V7) * (V3 + V7)) * V1) * V1)) + (2 * (((V2 * V3) * V4) * V5))) + (2 * (((V2 * V4) * V5) * V7))) +
5) * V6))) + (2 * (((V4 * V5) * V6) * V7))) >= 0))))))))).

Result = Generated Include List
Axioms/trans.ax
Axioms/general.ax
Axioms/minmax.ax
----------------------------
SZS status Theorem for /Users/anarkawi/Documents/SVN_Software/DAIDALUS/SUM/optimal_vectors/pvsbin/Delta_sum.tptp
Processor time: 0.782 = 0.058 (Metis) + 0.724 (RCF)
Maximum weight in proof search: 17
MetiTarski succesfully proved.
Trusted oracle: MetiTarski.

Proving formula(s) * with MetiTarski,
Q.E.D.



Run time  = 1.93 secs.
Real time = 183.42 secs.

nil
pvs(18):
```

Integration of MetiTarski into PVS



This calls MetiTarski/Z3, which uses the algorithm from the book

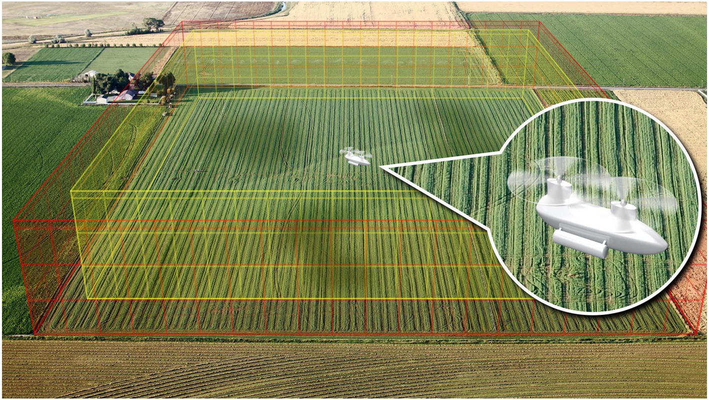
Algorithms in Real Algebraic Geometry by Basu, Pollack, and Roy

In Z3: Dejan Jovanovic and Leonardo de Moura. Solving non-linear arithmetic. In Automated Reasoning - 6th International Joint Conference, IJCAR 2012, Manchester, UK, June 26-29, 2012. Proceedings, volume 7364 of Lecture Notes in Computer Science, pages 339-354. Springer, 2012.

Another Application of MetiTarski

NASA Langley has developed multiple systems with geofencing capabilities.

Geofencing: Ensuring that an aircraft obeys stay-in and stay-out regions.



Another Application of MetiTarski

ICAROUS

Integrated Configurable Algorithms for Reliable Operations of Unmanned Systems

A recent problem: Restrict the controller input so that the component of acceleration (control input) in the direction of each edge guarantees future separation from that edge.

```
position_accel_time_scal_nnreal.2.1.2.2 :

[-1]  Acc < 0
[-2]  0 <= Acc * t1 * t1 + 2 * (v * t1)
[-3]  v > 0
[-4]  v * Acc < 0
[-5]  t1 <= t2
[-6]  t2 <= -v / Acc
  |-------
[1]   Acc * t1 * t1 + 2 * (v * t1) <= Acc * t2 * t2 + 2 * (v * t2)
```

Another Application of MetiTarski

```
position_accel_time_scal_nnreal.2.1.2.2 :

[-1]  Acc < 0
[-2]  0 <= Acc * t1 * t1 + 2 * (v * t1)
[-3]  v > 0
[-4]  v * Acc < 0
[-5]  t1 <= t2
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  |-------
[1]   Acc * t1 * t1 + 2 * (v * t1) <= Acc * t2 * t2 + 2 * (v * t2)

Rule? (metit *)
```

Another Application of MetiTarski

```
position_accel_time_scal_nnreal.2.1.2.2 :

[-1]  Acc < 0
[-2]  0 <= Acc * t1 * t1 + 2 * (v * t1)
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[-4]  v * Acc < 0
[-5]  t1 <= t2
[-6]  t2 <= -v / Acc
  |-------
[1]   Acc * t1 * t1 + 2 * (v * t1) <= Acc * t2 * t2 + 2 * (v * t2)

Rule? (metit *)
Substitutions: t1 -> V1, Acc -> V2, v -> V3, t2 -> V4
MetiTarski Input =
 fof(pvs2metit,conjecture, (![V1, V2, V3, V4]: ((~ (V2 < 0)) | ((~ (0 <= (((V2 * V1)
* V1) + (2 * (V3 * V1))))) | ((~ (V3 > 0)) | ((~ ((V3 * V2) < 0)) | ((~ (V1 <= V4)) |
 ((~ (V4 <= (-V3 / V2))) | ((((V2 * V1) * V1) + (2 * (V3 * V1))) <= (((V2 * V4) * V4)
 + (2 * (V3 * V4)))))))))))).

Result = Generated Include List
Axioms/trans.ax
Axioms/general.ax
Axioms/minmax.ax
----------------------------
SZS status Theorem for /Users/anarkawi/Documents/Papers/ICAS2018/pvs/pvsbin/position_
accel_time_scal_nnreal.2.1.2.2.tptp
Processor time: 0.079 = 0.057 (Metis) + 0.022 (RCF)
Maximum weight in proof search: 310
MetiTarski succesfully proved.
Trusted oracle: MetiTarski.

Proving formula(s) * with MetiTarski,

This completes the proof of position_accel_time_scal_nnreal.2.1.2.2.
```

Is it absolutely necessary to prove this automatically? No.

Is it easier than proving it manually? **Yes!**

Integration of MetiTarski into PVS

Formally proving the algorithm in PVS:

- Metit is used as an oracle in PVS (an addition to the kernel)
- Ultimate Goal: Build and formally prove a version of metit on *top* of the kernel of PVS
- We have two different implementations for the univariate case
- ... and **zero** implementations for the multivariate case

Exact Real Algebraic Geometry

- Sturm Sequences
- Tarski Queries

Tarski queries use Sturm sequences

Tarski queries determine if any polynomial system $p_1(x_1, \ldots, x_n) > 0 \wedge \cdots \wedge p_m(x_1, \ldots, x_m) \leq 0$ has a solution.

Tarski's Theorem

Let p and g be polynomials. Compute the remainder sequence

$$p_0(x),\ p_1(x),\ \ldots,\ p_m(x),$$

where $p_0 = p$, $p_1 = g \cdot p'$, $p_{j+2} = -rem(p_j, p_{j+1})$, and $p_m \equiv 0$

$\mathtt{TQ}(p, q)$ - A Tarski Query - a computable function based on the sequence $p_0, \ldots, p_m$.

<u>Tarski's Basic Theorem</u>

$$\mathtt{card}\{x \mid p(x) = 0 \wedge g(x) > 0\} - \mathtt{card}\{x \mid p(x) = 0 \wedge g(x) < 0\} = \mathtt{TQ}(p, g)$$

Book: *Algorithms in Real Algebraic Geometry* by Saugata Basu, Richard Pollack, Marie-Franoise Roy

Tarski's (Basic) Matrix Theorem

$$\begin{bmatrix} \mathrm{TQ}(p,1) \\ \mathrm{TQ}(p,g) \\ \mathrm{TQ}(p,g^2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)=0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)>0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)<0\}) \end{bmatrix}$$

Corollary

$$\begin{bmatrix} \mathrm{TQ}(p,1) \\ \mathrm{TQ}(p,g) \\ \mathrm{TQ}(p,g^2) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)=0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)>0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)<0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)\neq 0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)\geq 0\}) \\ \mathrm{card}(\{x \,:\, p(x)=0 \,\wedge\, g(x)\leq 0\}) \end{bmatrix}$$

Call this matrix $\mathtt{M}$

Multiple Polynomials

This is the base case of an induction proof for multiple polys

Tarski

$$\mathtt{TQ}(p, g_0, \ldots, g_k) = \mathtt{M}^{\otimes(k+1)} \cdot \mathtt{N}(p, g_0, \ldots, g_k)$$

Corollary

$$\mathtt{N}(p, g_0, \ldots, g_k) = (\mathtt{M}^{\otimes(k+1)})^{-1} \mathtt{TQ}(p, g_0, \ldots, g_k)$$

We want to compute entries of $\mathtt{N}(p, g_0, \ldots, g_k)$ (cards of solution sets)

Just need to compute $(\mathtt{M}^{\otimes(k+1)})^{-1} = (\mathtt{M}^{-1})^{\otimes(k+1)} =$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}^{\otimes(k+1)}$$

Sturm's Theorem

We can therefore compute cardinalities

$$\texttt{card}\{x \in \mathbb{R} : \; p(x) = 0 \;\wedge\; g_0(x)\; R_0\; 0 \;\wedge\; \ldots \;\wedge\; g_k(x)\; R_k\; 0\}$$

We can also determine if

$$\texttt{card}\{x \in \mathbb{R} : \; g_0(x)\; R_0\; 0 \;\wedge\; \ldots \;\wedge\; g_k(x)\; R_k\; 0\} \;=\; 0$$

(i.e. If the system $g_0(x)\; R_0\; 0 \;\wedge\; \ldots \;\wedge\; g_k(x)\; R_k\; 0$ has a solution)

```
example_1: LEMMA
  FORALL(y,x,z:real): (x-2)^2*(-x+4)>0 AND x^2*(x-3)^2>=0 AND x-1>=0 AND -(x-3)^2+1>0
  IMPLIES
  -(x-11/12)^3*(x-41/10)^3>=0
```

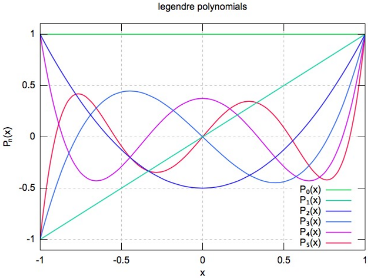
This proves now by simply typing "(tarski)" in the PVS prover. It's all automated.

Implementation of Tarski

- There is a deep embedding of the algorithm in PVS...
- ... and an theorem stating its correctness
- A property is proved by invoking the theorem in the prover...
- ... and the *evaluating* the resulting call to the algorithm...
- through reflection... The evaluation is done in PVS's underlying LISP language.

Hutch

A simpler method for proving unsatisfiability of a univariate system $p_1(x) < 0 \wedge \cdots \wedge p_5(x) < 0$



- Subdivide until every every interval has at most one root of any polynomial.
- Check this by counting roots of $\sum_i p_i(x)^2$ or $\prod_i p_i(x)$...
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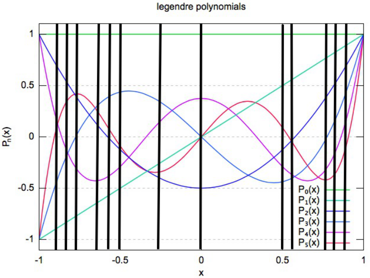

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This new command is called Hutch.

This proves now by simply typing "(tarski)" or "(hutch)" in the PVS prover. It's all automated.

The implemenations are similar and both use reflection.

Narkawicz, A. J.; Munoz, C. A.; and Dutle, A. M.: A Decision Procedure for Univariate Polynomial Systems Based on Root Counting and Interval Subdivision, Journal of Formalized Reasoning, 2018 (To appear)

Comparison of Tarski and Hutch

Problem	tarski	hutch	hutch :sos? nil	metit
Ex1	1.78 (0.12)	2.68 (0.02)	1.73 (0.02)	0.05
Ex2	4.80 (2.16)	2.91 (0.07)	3.19 (0.38)	0.05
Ex3	5.07 (0.26)	30.19 (27.22)	6.41 (1.74)	N/A
Ex4	12.58 (9.69)	4.07 (0.01)	4.09 (0.05)	0.04
Ex5	225.45 (237.96)	5.55 (0.01)	4.44 (0.17)	0.05
Ex6	–	70.02 (3.02)	–	N/A
Ex7	–	76.76 (51.85)	–	0.05
quad2	1.82 (0.01)	1.85 (0.00)	1.85 (0.00)	0.05
quad3	2.28 (0.05)	1.05 (0.00)	2.27 (0.00)	0.05
quad4	1.83 (0.44)	2.74 (0.00)	2.75 (0.01)	0.05
quad5	5.57 (2.88)	3.20 (0.01)	3.25 (0.03)	0.05
quad6	22.16 (21.61)	3.75 (0.01)	3.82 (0.08)	0.05
quad7	154.43 (175.47)	4.26 (0.01)	4.48 (0.24)	0.05
quad8	–	8.73 (0.01)	4.11 (0.53)	0.05
quad9	–	11.90 (0.01)	4.46 (1.09)	0.05
quad10	–	14.19 (0.02)	7.98 (2.07)	0.05

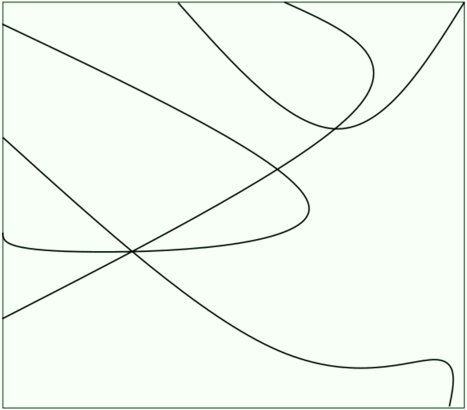
Multivariate Subdivision Algorithm

Is there a multivariate version of Hutch, and would that be the same as standard CAD?

- This would be much easier to prove in PVS
  - ... and add on top of the kernel (in a sound way)
- Subdivide n-dim box until every sub-box is small enough so that
  - For every nonempty subvariety of the form $\{x \mid \sum_i p_i(x)^2 = 0\}$ taken over a subset of the polynomials, either
    - it has positive dimension and is sliced by the edge of some box, *or*
    - it has zero dimension and is contained in a sub-box that contains no other zero dimensional subvariety
- Check satisfiability on the boundary of each sub-box **and** at every zero dimensional subvariety

Multivariate Subdivision Algorithm

2-dimensional example: plot the zero sets of polynomials $\mathbb{R}^2 \to \mathbb{R}$:



- Such a subdivision probably exists.
- But can we compute whether we are finished subdividing without doing a full CAD projection? (maybe not)

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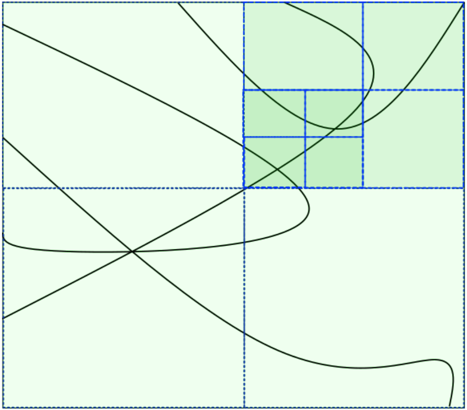

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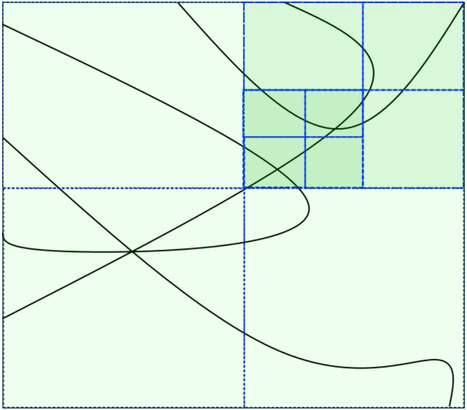

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Nonlinear Arithmetic

We have implemented/formalized/proved several other algorithms for determining satisfiability of a system

$$p_1(x_1, \ldots, x_n) > 0 \land \cdots \land p_m(x_1, \ldots, x_m) \leq 0$$

This includes some approximation methods:

- Interval Arithmetic
- Bernstein Polynomials
- Affine Arithmetic

Numerical Methods

Numerical Approximation Methods

- Interval Arithmetic
- Bernstein Polynomials

Both of these methods give a *crude* estimate of a range of $p_i(x_1, \ldots, x_n)$. These estimates get *better* for small boxes

Solution: Keep subdividing a big box into smaller boxes until you can prove the result.

A Generic Branch and Bound Algorithm

Using the branch and bound algorithm in PVS is *automated*

```
Heart(x1,x2,x3,x4,x5,x6,x7,x8): MACRO real =
  -x1*x6^3+3*x1*x6*x7^2-x3*x7^3+3*x3*x7*x6^2-x2*x5^3+3*x2*x5*x8^2-x4*x8^3+3*x4*x8*x5^2-0.9563453

Heart_forall_: LEMMA
  -0.1 <= x1 AND x1 <= 0.4 AND
  0.4 <= x2  AND x2 <= 1 AND
  -0.7 <= x3 AND x3 <= -0.4 AND
  -0.7 <= x4 AND x4 <= 0.4 AND
  0.1 <= x5  AND x5 <= 0.2 AND
  -0.1 <= x6 AND x6 <= 0.2 AND
  -0.3 <= x7 AND x7 <= 1.1 AND
  -1.1 <= x8 AND x8 <= -0.3 IMPLIES
  Heart(x1,x2,x3,x4,x5,x6,x7,x8) >= -1.7435

Heart_exists_: LEMMA
  EXISTS (x1,x2,x3,x4,x5,x6,x7,x8:real):
    -0.1 <= x1 AND x1 <= 0.4 AND
    0.4 <= x2  AND x2 <= 1 AND
    -0.7 <= x3 AND x3 <= -0.4 AND
    -0.7 <= x4 AND x4 <= 0.4 AND
    0.1 <= x5  AND x5 <= 0.2 AND
    -0.1 <= x6 AND x6 <= 0.2 AND
    -0.3 <= x7 AND x7 <= 1.1 AND
    -1.1 <= x8 AND x8 <= -0.3 AND
    Heart(x1,x2,x3,x4,x5,x6,x7,x8) <= -1.7434
```

These theorems can both be proved by simply typing "(bernstein)" or "(interval)" in PVS

Conclusion

- We depend heavily on built-in decision procedures for real arithmetic
- We have built our own in PVS
  - Bernstein polynomials
  - Interval arithmetic
  - Sturm and Tarski theorems
  - Sturm's theorem and subdivision
  - Affine arithmetic
  - Exact real arithmetic
- We would like a fast, verified multivariate CAD in PVS (Integrated in a sound way).
- Matt Damon used the PVS libraries

Questions

Questions?