

# **Sheaf semantics and nonstandard intuitionistic arithmetic**

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## **Outline**

Why nonstandard intuitionistic arithmetic?

Why sheaf semantics?

Conservation results

## Nonstandard analysis

External (Robinson):

- Start with a classical structure (e.g.  $\mathbb{N}$ ,  $\mathbb{R}$ ,  $V_\omega(\mathbb{R})$ , a model of set theory)
- Use compactness, or an ultrapower construction, to find an elementary extension with saturation properties
- Reason about what is true in the extension
- Transfer to the original structure

Internal (Nelson):

- Start with a classical theory
- Add a predicate  $\text{st}(x)$ , for “ $x$  is standard”, and appropriate axioms
- Show the new theory is a conservative extension of the old one

## Nonstandard arithmetic

Why nonstandard *arithmetic*?

- (Chauqui, Suppes, Sommer) Can carry out parts of real analysis
- (Nelson) Can carry out probability theory
- (Wilkie, Ajtai, Woods) Can carry out combinatorial arguments

Why nonstandard *intuitionistic* arithmetic?

In the nonstandard setting, many arguments have a constructive flavor.

**Thesis:** Nonstandard theories of intuitionistic arithmetic provides a natural framework for formalizing a number of interesting mathematical arguments.

Palmgren (BSL 99) develops intuitionistic nonstandard analysis.

## Sheaf semantics

### Background:

- Grothendieck: spaces of sheaves (topoi) are useful in algebraic topology, algebraic geometry, etc.
- Lawvere, Tierney: topoi provide an algebraic semantics for (higher type) intuitionistic logic
- Joyal: this semantics can be seen as a generalization of Kripke semantics and Beth semantics

We will actually use a slight generalization of sheaf semantics, due to Palmgren 97.

## Kripke semantics

Start with a first-order language  $L$ . A Kripke model consists of

- A poset
- A “universe” at each node of the poset
- An interpretation of the function symbols at each node
- An interpretation of the relation symbols at each node

where the universes are increasing, the interpretation of the function symbols agree between nodes, and the interpretation of the relations is monotone.

## Kripke semantics (continued)

Truth at each node is determined by a forcing relation:

- $p \Vdash (\theta \wedge \eta)[\vec{a}]$  if and only if  $p \Vdash \theta[\vec{a}]$  and  $p \Vdash \eta[\vec{a}]$
- $p \Vdash (\theta \vee \eta)[\vec{a}]$  if and only if  $p \Vdash \theta[\vec{a}]$  or  $p \Vdash \eta[\vec{a}]$
- $p \Vdash (\theta \rightarrow \eta)[\vec{a}]$  if and only if for every  $q \leq p$ , if  $q \Vdash \theta[\vec{a}]$ , then  $q \Vdash \eta[\vec{a}]$ .
- $p \Vdash (\forall x \theta(x))[\vec{a}]$  if and only if for every  $q \leq p$  and  $u \in D(q)$ ,  $p \Vdash \theta(z)[\vec{a}, b]$
- $p \Vdash (\exists x \theta(x))[\vec{a}]$  if and only if there is a  $b$  in  $D(p)$  such that  $p \Vdash \theta(z)[\vec{a}, b]$

## Beth semantics

For Beth semantics:

- Make the poset a tree
- Say “ $q_1, \dots, q_k$  covers  $p$ ” if every maximal path passing through  $p$  passes through one of the  $q_i$  as well
- Make the interpretation of the relations satisfy a covering condition.

The forcing definition is analogous, except for  $\vee$  and  $\exists$ :

- $p \Vdash (\theta \vee \eta)[\vec{a}]$  if and only if there is a cover  $\{q_1, \dots, q_k\}$  of  $p$  such that for each  $i$  either  $q_i \Vdash \theta[\vec{t}]$  or  $q_i \Vdash \eta[\vec{a}]$
- $p \Vdash (\exists x \theta(x))[\vec{a}]$  if and only if there are a cover  $\{q_1, \dots, q_k\}$  of  $p$  and a sequence of elements  $b_1 \in D(q_1), \dots, b_l \in D(q_l)$ , such that for each  $i$ ,  $q_i \Vdash \theta(z)[\vec{a}, b_i]$

## Sheaf semantics

For sheaf semantics:

- Use an arbitrary category  $\mathcal{C}$
- Use a basis for a Grothendieck topology (that is, a generalized notion of covering)

For standard sheaf semantics, one uses a sheaf to interpret the universe. Palmgren notes that for first-order logic, one only needs a presheaf (but the interpretation of the relations must still obey the covering condition).



## Completeness

The completeness of Kripke, Beth, and sheaf semantics (and so, a fortiori, Palmgren's semantics) is well known.

For Palmgren's semantics, however, the construction is almost trivial.

Given a theory  $T$ ,

- Let the objects of  $\mathcal{C}$  be formulas
- An arrow  $\varphi \xrightarrow{f} \psi$  is a renaming  $f$  of the variables of  $\psi$  such that  $\varphi \vdash_T \psi^f$
- For the notion of covering, take the smallest basis for a Grothendieck topology satisfying:
  1. If  $\varphi \vdash_T \theta \vee \eta$ , then  $\{\varphi \wedge \theta \rightarrow \varphi, \varphi \wedge \eta \rightarrow \varphi\}$  covers  $\varphi$ .
  2. If  $\varphi \vdash_T \exists x \theta(x)$ , and  $y$  is not a free variable of  $\varphi$  or  $\exists x \theta(x)$ , then  $\{\varphi \wedge \theta(y) \rightarrow \varphi\}$  covers  $\varphi$ .
- Take the universe at  $\varphi$  to be the set of terms with free variables among those of  $\varphi$ .
- Interpret function symbols in the obvious way.
- Interpret  $R$  at  $\varphi$  by  $\varphi \vdash_T R(t_1, \dots, t_k)$ .

**Theorem.** For every  $\theta$ ,  $\Vdash \theta$  iff  $\vdash_T \theta$ .

## Back to nonstandard arithmetic

Let  $L$  be the language of arithmetic, and let  $L^{\text{st}}$  be  $L$  with a new predicate symbol,  $\text{st}(x)$ .

Nonstandard Peano arithmetic consists of the following axioms:

- All the axioms of Peano arithmetic.
- $\exists x \neg \text{st}(x)$
- $\text{st}(x_1) \wedge \dots \wedge \text{st}(x_n) \rightarrow \text{st}(f(x_1, \dots, x_n))$ , for each function symbol  $f$
- *External induction:* For each formula  $\varphi(x)$  in  $L^{\text{st}}$  (possibly with other free variables),

$$\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall^{\text{st}} x \varphi(x)$$

- *Transfer:* For each formula  $\varphi$  in  $L$  with free variables  $x_1, \dots, x_n$ ,

$$\text{st}(x_1) \wedge \dots \wedge \text{st}(x_n) \rightarrow (\varphi \leftrightarrow \varphi^{\text{st}})$$

**Theorem:** (Friedman, unpublished, c. 1967) Nonstandard  $PA$  is a conservative extension of  $PA$ .

## Nonstandard Heyting arithmetic

Take nonstandard Heyting arithmetic,  $HAI$ , to be given by the following axioms:

- All the axioms of  $HA$
- $\text{st}(x_1) \wedge \dots \wedge \text{st}(x_n) \rightarrow \text{st}(f(x_1, \dots, x_n))$  for each function symbol  $f$
- $\neg\neg\text{st}(x) \rightarrow \text{st}(x)$
- *External induction:* for each formula  $\varphi(x)$  of  $L^{\text{st}}$ ,

$$\varphi(0) \wedge \forall^{\text{st}}x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall^{\text{st}}x \varphi(x).$$

- *Overspill:* for each formula  $\varphi(x)$  of  $L$ ,

$$\forall^{\text{st}}x \varphi(x) \rightarrow \exists x (\neg\text{st}(x) \wedge \varphi(x)).$$

- *Underspill:* for each formula  $\varphi(x)$  of  $L$ ,

$$\forall x (\neg\text{st}(x) \rightarrow \varphi(x)) \rightarrow \exists^{\text{st}}x \varphi(x).$$

**Theorem** (Moerdijk and Palmgren 97)  $HAI$  is a conservative extension of  $HA$ . Also, the transfer principles imply the law of the excluded middle.

## History

- Palmgren (95), and also Coquand and Smith (96), obtain conservation results for weaker nonstandard versions of  $HA$
- Moerdijk (95): Presents a nonstandard model of arithmetic, using a sheaf construction over a category of filters
- Moerdijk and Palmgren (97): Obtains the conservation result, using a category of provable filter bases
- Avigad and Helzner:
  - A slight modification of the completeness proof above yields the same result
  - Internalizing the argument yields an additional transfer rule
  - This transfer rule is optimal
- Butz independently presents another proof of the M-P conservation result.

## A nonstandard model

Modifying the completeness proof:

- Use types  $\Gamma$  instead of formulas  $\varphi$
- For the notion of covering, take the smallest basis for a Grothendieck topology satisfying the following:
  1. If  $\Gamma \vdash \theta \vee \eta$ , then  $\{\Gamma \cup \{\theta\} \rightarrow \Gamma, \Gamma \cup \{\eta\} \rightarrow \Gamma\}$  covers  $\Gamma$ .
  2. If  $\Gamma \vdash \exists x \theta(x)$ , and  $y$  is not free in  $\Gamma$  or  $\exists x \theta(x)$ , then

$$\Gamma \cup \{\theta(y)\} \cup \{y \geq \bar{n} \mid \Gamma \vdash_T \theta(\bar{n})\}$$

covers  $\Gamma$ .

- Interpret  $st$  at the node  $\Gamma$  by the relation

$$\exists n (\Gamma \vdash_T t \leq \bar{n}).$$

**Theorem.** For  $\theta$  in  $L$ ,  $\Vdash \theta$  iff  $\vdash_{HA} \theta$ .

**Theorem.** Each axiom of  $HAI$  is forced.

## Transfer principles

Positive results:

- If  $HAI$  proves  $\varphi$ ,  $\varphi$  in  $L$ , then  $HAI$  proves  $\varphi^{\text{st}}$
- If  $HAI$  proves  $\varphi^{\text{st}}$ ,  $\varphi$  in  $L$  negative, then  $HA$  proves  $\varphi$
- If  $HAI$  proves  $\forall^{\text{st}}x \varphi$ ,  $\varphi$  in  $L$ , then  $HA$  proves  $\varphi$ .
- If  $HAI$  proves  $\varphi^{\text{st}}$ ,  $\varphi$  in  $L$  and  $\Pi_2$ , then  $HA$  proves  $\varphi$ .

Negative results: there are primitive recursive  $A(x)$ ,  $B(x)$ ,  $C(x)$ ,  $D(x)$ , and  $E(x, y)$  such that

- $HAI + \exists x A(x) \rightarrow \exists^{\text{st}}x A(x)$  is not conservative over  $HA$
- $HAI + \forall^{\text{st}}x B(x) \rightarrow \forall x B(x)$  is not conservative over  $HA$
- $HAI$  proves  $\forall^{\text{st}}x C(x) \vee \forall^{\text{st}}x D(x)$ , but  $HA$  does not prove  $\forall x C(x) \vee \forall x D(x)$
- $HAI$  proves  $\exists^{\text{st}}x \forall y E(x, y)$  but  $HA$  does not prove  $\exists x \forall y E(x, y)$

Corollary:  $HAI$  has neither the existence property nor the disjunction property.

## Future work

Things to do:

- Find nicer translations for classical theories.
- Obtain the right conservation results for weak theories.
- See what kinds of mathematics can be developed in these theories.