A realizability interpretation for classical arithmetic

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Two flavors of arithmetic

First-order arithmetic comes in two flavors: classical and intuitionistic.

Though the two theories prove the same Π^0_2 ("computational") assertions,

- intuitionistic arithmetic has a nice constructive interpretation;
- classical arithmetic does not.

Classical (Peano) arithmetic

Language: $A, \bar{A}, \wedge, \vee, \forall, \exists$

$\neg\varphi$ is defined using DeMorgan equivalences

Prove sequents $\{\varphi_1, \ldots, \varphi_k\}$

$$\Gamma, A, \bar{A}$$

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi} \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi(x)}{\Gamma, \forall x\ \varphi(x)} \qquad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x\ \varphi(x)}$$

$$\frac{\Gamma, \varphi \quad \Gamma, \neg\varphi}{\Gamma}$$

QF axioms $\qquad \dfrac{\Gamma, \varphi(0) \quad \Gamma, \neg\varphi(x), \varphi(x')}{\Gamma, \forall x\ \varphi(x)}$

Intuitionistic (Heyting) arithmetic

Language: $\wedge, \vee, \rightarrow, \forall, \exists, \perp$

$\sim\varphi$ is defined as $\varphi \rightarrow \perp$

Prove sequents $\{\varphi_1, \ldots, \varphi_k\} \vdash \psi$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \qquad \frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

...

$$\frac{\Gamma \vdash \varphi(0) \quad \Gamma, \varphi(x) \vdash \varphi(x')}{\Gamma \vdash \forall x\, \varphi(x)}$$

Normalization vs. cut-elimination

On the intuitionistic side:

- HA has a constructive interpretation ("propositions as types," "realizability")
- HA comes with a natural set of "simplifying" reductions
- Strong normalization: arbitrary normalization strategies are guaranteed to terminate
- Church-Rosser: various normalization procedures all yield the same result

In contrast, cut-elimination procedures seem less canonical; it is not always clear that the transformations "simplify" the proof.

Maybe the situation isn't so bad

In an associated paper, I present:

- A realizability interpretation for classical arithmetic
- An new translation of classical arithmetic into intuitionistic arithmetic
- A set of reductions for classical arithmetic

I show:

- Under the translation, my realizability is just intuitionistic realizability plus the Friedman-Dragalin translation
- Under the translation, the reductions are compatible with intuitionistic normalization
- "Typical" finitary and infinitary cut-elimination procedures use the reductions
- With a reasonable restriction, the reductions are strongly normalizing

Conclusions

- It is easy to extract skolem terms from proofs of Π_2 theorems of classical arithmetic
- Classical arithmetic has a nice set of reductions
- A wide class of cut-elimination procedures all yield the same result
- The Friedman-Dragalin translation is "implicit" in these cut-elimination procedures

The "one-and-a-half negation" translation

Intuitionistically, take $\sim\varphi$ to be $\varphi \to \bot$.

Define the following translation from "classical" formulas to "intuitionistic" ones:

$$
\begin{array}{rcl}
A^M & = & A \\
\bar{A}^M & = & \sim A \\
(\varphi \lor \psi)^M & = & \varphi^M \lor \psi^M \\
(\varphi \land \psi)^M & = & \sim(\neg\varphi \lor \neg\psi)^M \\
(\exists x\ \varphi)^M & = & \exists x\ \varphi^M \\
(\forall x\ \varphi)^M & = & \sim(\exists x\ \neg\varphi)^M.
\end{array}
$$

Theorem. Intuitionistically, we have $\sim\varphi^M \equiv \sim\varphi^N$.

Corollary. If $\{\varphi_1, \ldots, \varphi_k\}$ is provable classically, then

$$(\neg\varphi_1)^M, \ldots, (\neg\varphi_k)^M \vdash \bot$$

intuitionistically (in fact, in minimal logic).

The theorem and corollary still hold true if we define

$$(\varphi \land \psi)^M \equiv \varphi^M \land \psi^M.$$

Translating proofs

Cut,

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}$$

translates to

$$\frac{\dfrac{(\neg\Gamma)^M, (\neg\varphi)^M \vdash \bot}{(\neg\Gamma)^M \vdash {\sim}(\neg\varphi)^M} \qquad \dfrac{(\neg\Gamma)^M, \varphi^M \vdash \bot}{(\neg\Gamma)^M \vdash {\sim}\varphi^M}}{(\neg\Gamma)^M \vdash \bot}$$

The $\wedge$ rule,

$$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

translates to

$$\frac{(\neg\Gamma)^M, (\neg\varphi)^M \vdash \bot \qquad (\neg\Gamma)^M, (\neg\psi)^M \vdash \bot}{(\neg\Gamma)^M, (\neg\varphi)^M \vee (\neg\psi)^M \vdash \bot}$$

The $\vee$ rule,

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}$$

translates to

$$\dfrac{\dfrac{(\neg\Gamma)^M, (\neg\varphi)^M \vdash \bot}{(\neg\Gamma)^M \vdash \sim(\neg\varphi)^M} \quad \sim(\varphi^M \vee \psi^M) \vdash \sim\varphi^M}{(\neg\Gamma)^M, \sim(\varphi^M \vee \psi^M) \vdash \bot}$$

Applying the Friedman-Dragalin translation

Given a proof of $\exists x\ A(x)$ in classical arithmetic, obtain a proof of $\perp$ from $\forall x \sim A(x)$ in arithmetic over minimal logic.

Now, replace $\perp$ everywhere by $\exists x\ A(x)$. This yields a proof of $\exists x\ A(x)$ from

$$\forall x\ (A(x) \to \exists x\ A(x)),$$

and hence a proof of $\exists x\ A(x)$.

Corollary. If classical arithmetic proves $\forall y\ \exists x\ A(x, y)$ then intuitionistic arithmetic proves it as well.

Some reductions

A principal cut:

$$\dfrac{\dfrac{\begin{matrix} d_0 \\ \Gamma, \varphi \vee \psi, \varphi \end{matrix}}{\Gamma, \varphi \vee \psi} \qquad \begin{matrix} d_1 \\ \Gamma, \neg\varphi \wedge \neg\psi \end{matrix}}{\Gamma}$$

reduces to

$$\dfrac{\dfrac{\begin{matrix} d_0 \\ \Gamma, \varphi \vee \psi, \varphi \end{matrix} \qquad \begin{matrix} d_1 \\ \Gamma, \neg\varphi \wedge \neg\psi \end{matrix}}{\Gamma, \varphi} \qquad \dfrac{\begin{matrix} d_1 \\ \Gamma, \neg\varphi \wedge \neg\psi \end{matrix}}{\Gamma, \neg\varphi}\,\text{(invert)}}{\Gamma}$$

A principal inversion:

$$\dfrac{\dfrac{\begin{matrix} d_0 \\ \Gamma, \varphi \wedge \psi, \varphi \end{matrix} \qquad \begin{matrix} d_1 \\ \Gamma, \varphi \wedge \psi, \psi \end{matrix}}{\Gamma, \varphi \wedge \psi}}{\Gamma, \varphi}$$

reduces to

$$\dfrac{\begin{matrix} d_0 \\ \Gamma, \varphi \wedge \psi, \varphi \end{matrix}}{\Gamma, \varphi}$$

A taxonomy of reductions

Add inversion rules: $\dfrac{\Gamma, \varphi \wedge \psi}{\Gamma, \varphi}$, $\dfrac{\Gamma, \forall x\, \varphi(x)}{\Gamma, \varphi(n)}$, ...

Five kinds of reductions:

1. principal inversions
2. nonprincipal inversion
3. principal cut
4. nonprincipal cut
5. unnecessary free variables

The results

- These reductions are compatible with the normalization of the corresponding intuitionistic proof
- They be used in a Gentzen-style finitary cut elimination procedure
- They are also implicit in infinitary cut elimination procedures
- The Friedman-Dragalin translation corresponds to extracting a witness from a cut-free proof
- The witness extracted is independent of the order in which reductions are applied
- You can eliminate cuts from proofs of Σ_1 sentences, even without "permutative" reductions
- (Buchholz) If you restrict the permutative reductions, you have strong normalization

Comments

1. Gentzen's original cut-elimination procedure used a more symmetric cut reduction:

$$\dfrac{\dfrac{\begin{array}{c} d_0 \\ \Gamma, \forall x\ \varphi(x), \varphi(y) \end{array}}{\Gamma, \forall x\ \varphi(x)} \qquad \dfrac{\begin{array}{c} d_1 \\ \Gamma, \exists x\ \neg\varphi(x), \neg\varphi(t) \end{array}}{\Gamma, \exists x\ \neg\varphi(x)}}{\Gamma}$$

reduces to

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} d_0 \\ \Gamma, \forall x\ \varphi, \varphi(y) \end{array}}{\Gamma, \forall x\ \varphi} \qquad \begin{array}{c} d_1 \\ \Gamma, \exists x\ \neg\varphi, \neg\varphi(t) \end{array}}{\Gamma, \neg\varphi(t)} \qquad \dfrac{\begin{array}{c} d_0[t/y] \\ \Gamma, \forall x\ \varphi, \varphi(t) \end{array} \qquad \dfrac{\begin{array}{c} d_1 \\ \Gamma, \exists x\ \neg\varphi, \neg\varphi(t) \end{array}}{\Gamma, \exists x\ \neg\varphi}}{\Gamma, \varphi(t)}}{\Gamma}$$

These are *not* compatible with normalization, under the translation above.

2. The translation isn't sharp on fragments of arithmetic; for example, $I\Sigma_1$ doesn't translate to $I\Sigma_1^i$. For one that is (due to Coquand), see

Interpreting classical theories

in constructive ones

on my home page.