Weak theories of nonstandard arithmetic and analysis

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Contents

Two traditions

Developing mathematics in weak theories:

- In the tradition of Weyl, Hilbert, Bernays, Kreisel, Feferman, Takeuti, Friedman, Simpson, . . .
- Recent interest in conservative extensions of primitive recursive arithmetic, elementary arithmetic, feasible arithmetic
- Goals:
  - Minimizing ontological commitments
  - Understanding mathematics in concrete computational terms

Nonstandard analysis:

- Semantic approach (Robinson): reason about saturated models
- Syntactic approach (Kreisel, Nelson): reason axiomatically
- Obtain enriched universes for doing mathematics

A mixed marriage

Why combine the two traditions?

Weak theories of nonstandard arithmetic and analysis may provide a natural setting for:

- Developing real analysis (Chauqui, Suppes, Sommer)
  - studying complexity issues (à la Ko, Ferreira)
  - extracting numeric bounds (à la Kohlenbach)
- Formalizing combinatorial arguments (like those of Ajtai, Wilkie, Woods)
- Formalizing Nelson's *Radically elementary probability theory*

Weak theories of arithmetic

The set of primitive recursive functions is the smallest set of functions from $\mathbb{N}$ to $\mathbb{N}$ (of various arities)

- containing 0, $S(x) = x + 1$, $p_i^n(x_1, \ldots, x_n) = x_i$
- closed under composition
- closed under primitive recursion:

$$f(0, \vec{z}) = g(\vec{z}), \quad f(x + 1, \vec{z}) = h(f(x, \vec{z}), x, \vec{z})$$

Primitive recursive arithmetic is an axiomatic theory, with

- defining equations for the primitive recursive functions
- quantifier-free induction

PRA can be presented either as a first-order theory or as a quantifier-free calculus.

Similarly, ERA axiomatizes the elementary functions, and PV axiomatizes the polynomial time computable functions.

A nonstandard version

Add to the language of PRA:

- a predicate, $st(x)$ ("x is standard")
- a constant, ω

Let $NPRA$ consist of PRA plus the following axioms:

- $\neg st(\omega)$
- $st(x) \wedge y < x \to st(y)$
- $st(x_1) \wedge \ldots \wedge st(x_k) \to st(f(x_1, \ldots, x_k))$, for each function symbol f
- $\forall$-transfer without parameters: $\forall^{st} \vec{x}\ \psi(\vec{x}) \to \forall \vec{x}\ \psi(\vec{x})$, for ψ quantifier-free with the free variables shown.

A short model-theoretic argument shows the following:

Theorem 1 *Suppose NPRA proves* $\forall^{st} x\ \exists y\ \varphi(x, y)$*, with* φ *quantifier-free in the language of PRA. Then PRA proves* $\forall x\ \exists y\ \varphi(x, y)$.

In particular, the conclusion holds if $NPRA$ proves either $\forall x\ \exists y\ \varphi(x, y)$ or $\forall^{st} x\ \exists^{st} y\ \varphi(x, y)$.

Higher type versions

The *finite types* are defined as follows:

- N is a finite type
- If σ and τ are finite types, so are $\sigma \times \tau$ and $\sigma \to \tau$

The *primitive recursive* functionals *of finite type* allow:

- λ abstraction, and application
- Restricted higher-type primitive recursion:

$$F(0, \vec{z}) = G(\vec{z}), \quad F(n+1, \vec{z}) = H(F(n), n, \vec{z})$$

  where $F(n, \vec{z})$ has type N.

The theory PRA^ω axiomatizes these functionals, and is a conservative extension of PRA.

Define $NPRA^\omega$ in analogy to $NPRA$.

Theorem 2 *Suppose* $NPRA^\omega$ *proves* $\forall^{st} x\ \exists y\ \varphi(x, y)$*, for* φ *a quantifier-free formula in the language of* PRA^ω*. Then* PRA^ω *proves* $\forall x\ \exists y\ \varphi(x, y)$.

A direct interpretation

Why go beyond the model theoretic proof?

- obtain an explicit translation
- obtain bounds on lengths of proofs, additional information

Ideas:

- Use a forcing relation to describe nonstandard extension.
- Add a "generic" nonstandard element, ω.
- Work internally, in the language of PRA^ω.
- If $NPRA^\omega$ proves φ, PRA^ω proves "φ is forced."

The forcing interpretation

Names:

- Replace the constant ω by a variable.
- Replace each variable x_i by a term $\tilde{x}_i(\omega)$.
- Replace terms $t[\omega, x_1, \ldots, x_k]$ by $t[\omega, \tilde{x}_1(\omega), \ldots, \tilde{x}_k(\omega)]$. (Call this $\widehat{t}$.)

Conditions: A condition is a 3-ary relation $p(u, v, \omega)$. Intuitively, this represents the assertion $\forall^{st} u \; \forall v \; p(u, v, \omega)$, or the set

$$\{\forall v \; p(0, v, \omega), \forall v \; p(1, v, \omega), \forall v \; p(2, v, \omega), \ldots\}$$

A condition p is *stronger than* q, written $p \preceq q$, if $\forall u, v, \omega \; (p(u, v, \omega) \to q(u, v, \omega))$.

The atomic case: Say $p \Vdash t_1 = t_2$ if and only if

$$\exists z \; \forall \omega \; (\forall u < z \; \forall v \; p(u, v, \omega) \to \widehat{t_1} = \widehat{t_2})$$

In other words, $p \Vdash t_1 = t_2$ if and only if $t_1 = t_2$ follows from a finite subset of the set above.

The forcing interpretation (continued)

The full forcing relation is defined inductively, as follows:

1. $p \Vdash \bot \equiv \exists z\, \forall \omega\, \neg \forall u < z\, \forall v\, p(u, v, \omega)$.
2. $p \Vdash t_1 = t_2 \equiv \exists z\, \forall \omega\, (\forall u < z\, \forall v\, p(u, v, \omega) \to \widehat{t_1} = \widehat{t_2})$.
3. $p \Vdash t_1 < t_2 \equiv \exists z\, \forall \omega\, (\forall u < z\, \forall v\, p(u, v, \omega) \to \widehat{t_1} < \widehat{t_2})$.
4. $p \Vdash st(t) \equiv \exists z\, \forall \omega\, (\forall u < z\, \forall v\, p(u, v, \omega) \to \widehat{t} < z)$.
5. $p \Vdash \varphi \to \psi \equiv \forall q \preceq p\, (q \Vdash \varphi \to q \Vdash \psi)$.
6. $p \Vdash \varphi \wedge \psi \equiv (p \Vdash \varphi) \wedge (p \Vdash \psi)$.
7. $p \Vdash \forall x\, \varphi \equiv \forall \tilde{x}\, (p \Vdash \varphi)$

Define $\neg \varphi$, $\varphi \vee \psi$, $\exists x\, \varphi$ from these connectives in the usual way.

Theorem 3 *If $NPRA^\omega$ proves φ, PRA^ω proves $\forall^{st} u\, (\omega > u) \Vdash \varphi$.*

The conservation theorem follows from this.

Interlude

Remember the table of contents:

Developing real analysis

Definitions in $NPRA^\omega$:

- $\mathbb{N}^*$: the nonstandard natural numbers (type N)
- $\mathbb{N}$: the standard numbers (i.e. satisfying $st(x^{\mathrm{N}})$)
- $\mathbb{Z}^*, \mathbb{Z}$: the nonstandard / standard integers
- $\mathbb{Q}^*, \mathbb{Q}$: the nonstandard / standard rationals
- $q \in \mathbb{Q}^*$ is *bounded* if $\ulcorner q \urcorner$ is standard
- q is *infinitesimal* if it is zero or $1/q$ is unbounded
- $q \sim r$ if $q - r$ is infinitesimal
- $x \in \mathbb{R}$ means that $x \in \mathbb{Q}^*$ and x is bounded
- $x =_{\mathbb{R}} y$ means $x \sim y$

In other words, we are taking $\mathbb{R}$ to be $(\mathbb{Q}^*)^{bdd} / \sim$, and dispensing with $\mathbb{R}^*$ entirely.

The advantage: reals are type 0 objects.

A *function* $f : \mathbb{R} \to \mathbb{R}$ is a function $\mathbb{Q}^* \to \mathbb{Q}^*$ satisfying

$$\forall r \in \mathbb{R}\ (f(r) \in \mathbb{R}) \land \forall r, s \in \mathbb{R}\ (r =_{\mathbb{R}} s \to f(r) =_{\mathbb{R}} f(s)).$$

A surprise

Theorem 4 **($NERA^\omega$)** *Every function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

What is going on? Variables range over *internal* functions.

The function $f \in \mathbb{Q}^* \to \mathbb{Q}^*$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq_{\mathbb{Q}^*} 0 \\ 1 & \text{otherwise,} \end{cases}$$

is not a function from $\mathbb{R}$ to $\mathbb{R}$: for example, $1/\omega =_{\mathbb{R}} 0$ but $f(1/\omega) \neq_{\mathbb{R}} f(0)$.

On the other hand, the function $g \in \mathbb{Q}^* \to \mathbb{Q}^*$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq_{\mathbb{R}} 0 \\ 1 & \text{otherwise} \end{cases}$$

is not represented by a term of $NERA^\omega$, since $x \leq_{\mathbb{R}} 0$ is external.

The intermediate value theorem

Theorem 5 *Suppose $f \in [0,1] \to \mathbb{R}$, $f(0) = -1$, and $f(1) = 1$. Then there is an $x \in [0,1]$ such that $f(x) = 0$.*

Proof. Considering f as a function on $\mathbb{Q}^*$, let

$$j = \max\{i < \omega \mid f(i/\omega) <_{\mathbb{Q}^*} 0\}$$

and let $x = j/\omega$. Since $j/\omega \sim (j+1)/\omega$, we have

$$f((j+1)/\omega) =_{\mathbb{R}} f(j/\omega) \leq_{\mathbb{R}} 0 \leq_{\mathbb{R}} f((j+1)/\omega)$$

and so $f(x) =_{\mathbb{R}} 0$.

The extreme value theorem

Theorem 6 *If $f \in [0,1] \to \mathbb{R}$, then f attains a maximum value.*

Proof. Again considering f as a function on $\mathbb{Q}^*$, let

$$y = \max_{0 \leq i \leq \omega} f(i/\omega),$$

let $x = j/\omega$ satisfy $f(x) =_{\mathbb{Q}^*} y$. That y is a maximum is guaranteed by the fact that for any $x' \in [0,1]$, there is an i such that $x' \sim i/\omega$.

Notes

References:

1. Jeremy Avigad, "Weak theories of nonstandard arithmetic and analysis," to appear in Stephen Simpson, ed., *Reverse Mathematics 2001*.
2. Jeremy Avigad and Jeffrey Helzner, "Transfer principles in intuitionistic nonstandard arithmetic," to appear in the *Archive for Mathematical Logic*.

Notes:

1. One can also study intuitionistic theories.
2. In the case of $NPRA$, we can allow Σ_1 standard induction.
3. Using nonstandard numbers, one can interpret weak König's lemma.
4. We can show that many of the results are optimal.

Questions

1. Can some of the results be strengthened?
2. Can these methods be used to to extract bounds from proofs in analysis?
3. What does it take to formalize nonstandard arguments in combinatorics and proof complexity?
4. What does it take to formalize measure-theoretic probability (following Nelson)?
5. Can one develop a nonstandard feasible analysis?