Proof Theory and Proof Mining: Recaps

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Recap of Lecture 1

- computable reals
- computable functions $\mathbb{R} \to \mathbb{R}$
- computable $f : [0,1] \to \mathbb{R}$ has a computable modulus of uniform continuity
- These are "morally the same":
  - Every open cover of $[0,1]$ has a finite subcover.
  - Every open cover of $\{0,1\}^\omega$ has a finite subcover.
  - Every binary tree with no path is finite.
- They are constructively iffy; the converses are computationally false.

Recap of Lecture 1

- computable metric spaces (complete, separable)
- computable elements and functions of computable metric spaces
- computable Banach spaces, Hilbert spaces, measure spaces
- there is a bounded, increasing sequences of rationals, whose limit is not computable.

Recap of Lecture 2

- computability in ergodic theory
  - in general, there is no computable bound on the rate of convergence in the mean ergodic theorem
  - a rate of convergence can be computed from the norm of the limit of the ergodic averages
  - computability vs. rate
- axiomatic theories
  - primitive recursive arithmetic (PRA)
  - PRA is more powerful than it seems
  - quantifier PRA Π_2 conservative over quantifier-free version
  - $I\Sigma_1 = PRA + \Sigma_1$ induction
  - $I\Sigma_1$ is Π_2 conservative over PRA
  - $PA = PRA +$ full induction

Recap of Lecture 3

- *PRA*, $I\Sigma_1$, *PA*
- *HA* and the double-negation interpretation
- second-order arithmetic
- subsystems: RCA_0, WKL_0, ACA_0, ATR_0, $\Pi^1_1-CA_0$
  - natural models
  - reversals
  - conservation results
- analysis in subsystems
  - complete separable metric spaces
  - compactness

What we have discussed

I have provided an impressionistic look at:

- computable analysis
- proof theory and conservation results
- reverse mathematics
- proof mining

Note: all the slides available on the web page, together with reading suggestions.

What I would have covered in Part IV

Nonstandard analysis and ultraproducts:

- nonstandard analysis
- weak theories of nonstandard analysis
- conservation results for ultraproducts
- ultraproducts and metastability

Slides are available on the web page.

What's left out

- reverse math of stronger theories (ATR_0, $\Pi^1_1-CA_0$, ...)
- constructive mathematics and constructive analysis
- reverse mathematics and combinatorics
- proof mining and combinatorics
- ultraproducts and nonstandard methods in combinatorics

Recap of Lecture 4

- reverse mathematics and analysis (e.g. ergodic theory)
- higher-type systems (PRA^{ω}, ...)
- Kreuzer's conservation result for Lebesgue measure
- the Dialectica interpretation

Semantic methods in proof theory

Observations:

- Conservation results involving classical first- or second-order theories can generally be done semantically or syntactically (using compactness, forcing arguments, definable ultrapowers, Skolem hull constructions, . . .).
- Reductions from classical theories to constructive theories are generally easier to do with a translation, but can usually be done semantically (you need model theory for inuitionistic systems — Kripke models, sheaf models, etc.).

Semantic methods in proof theory

- I know of no good model theory for systems of finite type functionals.
    - Some of the models are straightforwardly described (though often by recursion on the types)
    - What does it mean to "expand" a model, adding something conservatively?
    - I know of no semantic proof of Gödel's Dialectica result.
- Topic 4 includes the use of ultraproducts to obtain uniformity results that were originally obtained using the Dialectica interpretation.