Eliminating definitions and Skolem functions in first-order logic

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Definitions in propositional proofs

Start with a standard axiomatic proof system for propositional logic, with modus ponens the only rule of inference.

Add definitions: iteratively introduce new variables P_φ and axioms $P_\varphi \leftrightarrow \varphi$.

Naive elimination of definitions can be exponential. Can one do better? In other words:

> Are extended Frege systems p-equivalent to Frege systems?

This is a major open question.

First-order logic

Let Γ be a set of first-order sentences in a language L, and let $R_0, R_1, R_2, \ldots$ denote new relation symbols.

Definition 0.1 *Say that* Γ *has an* efficient elimination of definitions *if there is a polynomial* $p(x)$ *such that if* d *is a proof of a formula* ψ *in* L *from*

$$\Gamma \cup \{\forall \vec{x}_0 \; (R_0(\vec{x}_0) \leftrightarrow \varphi_0(\vec{x}_0)), \ldots, \\ \forall \vec{x}_k \; (R_k(\vec{x}_k) \leftrightarrow \varphi_k(\vec{x}_k))\},$$

where each φ_i *involves at most* $R_0, \ldots, R_{i-1}$*, then there is a proof* d' *of* ψ *from* Γ *using only formulae in* L*, with* $|d'| \leq p(|d|)$*.*

This definition is monotone in Γ: if Γ has an efficient elimination of definitions and $\Gamma' \supseteq \Gamma$ then so does Γ'.

Eliminating definitions

Theorem 0.2 $\{\exists x, y \; (x \neq y)\}$ *has an efficient elimination of definitions.*

Notes:

- Proof is not difficult (and may be folklore)
- Relies on equality
- Similar tricks have been used elsewhere

Corollary 0.3 *First-order logic (with equality) has efficient elimination of definitions if and only if propositional logic does as well.*

Corollary 0.4 *One can eliminate "$\leftrightarrow$" efficiently from standard first-order proof systems.*

The proof

Add constants a, b, with $a \neq b$. Code each natural number i as a sequence of values $a, a, \ldots, a, b, a, \ldots, a, a$ with b in the ith position.

Recursively define a sequence of formulae $\hat{\varphi}_i(\vec{z}, \vec{x})$ such that

- for each $j < i$, $\hat{\varphi}_i(\bar{j}, \vec{x})$ is equivalent to $\varphi_j(\vec{x})$, and
- $\hat{\varphi}_{i+1}$ is used only once in the definition of $\hat{\varphi}_i$.

For example, suppose φ_{i+1} is the formula

$$R_i(\vec{t}) \wedge \neg R_i(\vec{s})$$

Use a and b as truth values. Let $\theta(v, v')$ be

$$\forall \vec{x}, y \; ((R(\vec{x}) \leftrightarrow y = a) \rightarrow \\ (\vec{x} = \vec{t} \rightarrow y = v) \wedge (\vec{x} = \vec{s} \rightarrow y = v')).$$

Then φ_{i+1} is equivalent to

$$\forall v, v' \; (\theta(v, v') \rightarrow (v = a \wedge v' \neq a)).$$

More generally:

- Put formulae in prenex form.
- If $\leftrightarrow$ is not in the language, use positive and negative representations of each definition.

Skolem functions

A *Skolem axiom* has the form

$$\forall \vec{x}, y \; (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, f(\vec{x}))),$$

"if anything satisfies $\exists y \; \varphi(\vec{x}, y)$, $f(\vec{x})$ does."

These can be eliminated from first-order proofs.

- Model-theoretic argument is easy.
- Syntactic arguments are harder, and worse than exponential.

Pudlák: Is there an example of a single Skolem axiom that cannot be eliminated efficiently?

Eliminating Skolem functions

Let Γ be a set of first-order sentences in a language L.

Definition 0.5 *Say that* Γ *has an* efficient elimination of Skolem functions *if there is a polynomial* $p(x)$ *such that if* d *is a proof of a formula* ψ *in* L *from*

$$\Gamma \cup \{\forall \vec{x}_0, y\ (\varphi_0(\vec{x}_0, y) \rightarrow \varphi_0(\vec{x}_0, f_0(\vec{x}_0))), \ldots,$$
$$\forall \vec{x}_k, y\ (\varphi_k(\vec{x}_k, y) \rightarrow \varphi_k(\vec{x}_k, f_k(\vec{x}_k)))\},$$

where each φ_i *involves at most* $f_0, \ldots, f_{i-1}$*, then there is a proof* d' *of* ψ *from* Γ *using only formulae in* L*, with* $|d'| \leq p(|d|)$.

By internalizing the model-theoretic argument, e.g. Zermelo-Fraenkel set theory has efficient an elimination of Skolem functions.

How little can we get away with?

Coding finite functions

Definition 0.6 *Say a set of sentences* Γ codes finite functions (efficiently) *if for each* n *there are*

- *a definable element, "*$\emptyset_n$*";*
- *a definable relation, "*$x_0, \ldots, x_{n-1} \in dom_n(p)$*";*
- *a definable function, "*$eval_n(p, x_0, \ldots, x_{n-1})$*"; and*
- *a definable function, "*$p \oplus_n (x_0, \ldots, x_{n-1} \mapsto y)$*"*

such that, for each n*,* Γ *proves*

- $\vec{x} \notin dom_n(\emptyset_n)$
- $\vec{w} \in dom_n(p \oplus (\vec{x} \mapsto y)) \leftrightarrow (\vec{w} \in dom_n(p) \vee \vec{w} = \vec{x})$
- $eval_n(p \oplus_n (\vec{x} \mapsto y), \vec{x}) = y$
- $\vec{w} \neq \vec{x} \rightarrow eval_n(p \oplus_n (\vec{x} \mapsto y), \vec{w}) = eval_n(p, \vec{w})$,

and such that the lengths of all the definitions and proofs are bounded by a polynomial in n.

Intuition: $eval_n(p, x_0, \ldots, x_{n-1})$ means $p(x_0, \ldots, x_{n-1})$.

Any "sequential" theory meets these criteria.

The main theorem

Theorem 0.7 *Suppose Γ codes finite functions. Then Γ has an efficient elimination of Skolem functions.*

Notes:

- Use forcing to describe a generic extension of the universe with a new Skolem function.
- Conditions are finite partial functions approximating the Skolem function being added.
- This is familiar to set theorists, but a novel application to weak theories.
- Need to express the forcing relation in the underlying language.
- Only the iterated version needs definitions.

Outline of the argument:

- If Γ plus the Skolem axiom proves φ, Γ proves "φ is forced."
- If φ does not mention the Skolem function, then Γ proves φ.

The forcing definition

Let us deal with a single Skolem axiom. $Cond(p)$ says p is a condition:

$$\forall \vec{x} \in dom(p)\ \forall y\ (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, p(x))).$$

For terms t involving f, define t^p inductively as follows:

- $x^p \equiv x$, for each variable x (other than p),
- $(g(t_0, \ldots, t_m))^p \equiv g(t_0^p, \ldots, t_m^p)$, for each function symbol g of L, and
- $(f(t_0, \ldots, t_n))^p \equiv p(t_0^p, \ldots, t_n^p)$.

Define "t^p is defined" inductively as follows:

- "x^p is defined" is always true.
- "$(g(t_0, \ldots, t_m))^p$ is defined," where g is a function symbol of L, is true if and only if $t_0^p, \ldots, t_m^p$ are all defined.
- "$(f(t_0, \ldots, t_n))^p$ is defined" is true if and only if $t_0^p, \ldots, t_n^p$ are all defined and $t_0^p, \ldots, t_n^p \in dom(p)$.

The forcing definition (cont'd)

If p and q are conditions, say $p \preceq q$, "p is stronger than or equal to q," if p extends q as a function:

$$\forall \vec{x}\ (\vec{x} \in dom(q) \rightarrow \vec{x} \in dom(p) \wedge p(\vec{x}) = q(\vec{x})).$$

Define the relation $p \Vdash \theta$ inductively:

1. $p \Vdash R(t_0, \ldots, t_m) \equiv \forall q \preceq p\ \exists r \preceq q\ \ (t_0^r, \ldots, t_m^r$ are all defined and $R(t_0^r, \ldots, t_m^r))$.
2. $p \Vdash \theta \wedge \eta \equiv p \Vdash \theta$ and $p \Vdash \eta$.
3. $p \Vdash \theta \rightarrow \eta \equiv \forall q \preceq p\ (q \Vdash \theta \rightarrow q \Vdash \eta)$.
4. $p \Vdash \neg\theta \equiv \forall q \preceq p\ q \nVdash \theta$.
5. $p \Vdash \forall x\ \theta \equiv \forall x\ p \Vdash \theta$.

The quantifiers involving q and r range over conditions.

"θ is forced", written $\Vdash \theta$, means $\forall p\ (p \Vdash \theta)$,

The main lemmata

Lemma 0.8 (monotonicity) *For each formula θ of L_f, Γ proves*

$$p \Vdash \theta \wedge q \preceq p \rightarrow q \Vdash \theta.$$

Lemma 0.9 *For each formula θ of L_f, Γ proves*

$$p \Vdash \theta \leftrightarrow \forall q \preceq p\ \exists r \preceq q\ r \Vdash \theta.$$

Corollary 0.10 *For each formula θ of L_f, Γ proves*

$$\Vdash (\theta \leftrightarrow \neg\neg\theta).$$

Lemma 0.11 *For any term t of L_f, Γ proves*

$$\forall q\ \exists r \preceq q\ (t^r \text{ is defined}).$$

Lemma 0.12 *For each formula θ of L_f, if θ is provable in classical first-order logic, then Γ proves $\Vdash \theta$.*

Lemma 0.13 Γ *proves* $\Vdash \forall \vec{x}, y\ (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, f(\vec{x})))$.

Lemma 0.14 *For each formula θ of L, Γ proves* $(p \Vdash \theta) \leftrightarrow \theta$.

For nested Skolem axioms, use an iteration, with definitions.

Questions

1. Can one eliminate definitions efficiently in the propositional case?
2. Can one eliminate Skolem functions efficiently in pure first order logic?
3. What can one say about first-order definitions in the absence of equality?
4. What can one say about eliminating "$\leftrightarrow$" in the absence of equality?
5. What can one say about intuitionistic theories?
6. Are there other interesting applications of forcing arguments "low down"?