The *Lean* Theorem Prover

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(Lean’s principal developer is Leonardo de Moura,
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Automated theorem proving and formal verification

- Automated theorem proving: want powerful, fast methods
- Formal verification: want secure guarantees

These pull in different directions.
Automated theorem proving

Domain general
- fast satisfiability methods
- equational theorem proving
- first-order theorem proving (resolution, tableau)

Domain specific
- integer / linear arithmetic
- nonlinear real arithmetic
- numerical methods
- algebraic methods

Combination methods aim to get the best of both worlds.
Interactive theorem proving

Some systems: Mizar, HOL, Isabelle, Coq, HOL-light, ACL2, PVS, Agda, ...

The user works interactively with the system to construct a formal proof.

Design space:
- Logic: first-order, simple types, dependent types
- Classical vs. constructive
- Interaction with computation (internal vs. external)
Lean

Aims to bring the two worlds together:

- An interactive theorem prover with powerful automation.
- An automated reasoning tool that
  - produces proofs,
  - has a rich language,
  - can be used interactively, and
  - is built on a verified mathematical library.
Lean

The Lean theorem prover is being developed by
  - Leonardo de Moura (Microsoft Research)
  - Soonho Kong (CMU, a student of Ed’s!)

The Lean standard library is being developed by
  - Jeremy Avigad (CMU)
  - Floris van Doorn (CMU)
  - Leonardo de Moura (Microsoft Research)

Contributors
  - Cody Roux (Draper)
  - Robert Lewis (CMU)
  - Parikshit Khanna (Indian Institute of Technology, Kanpur)
The logical framework

Lean’s default logical framework is a version of the Calculus of Constructions with:

- an impredicative, proof-irrelevant type \( \text{Prop} \) of propositions
- a non-cumulative hierarchy of universes, \( \text{Type 1, Type 2, ...} \) above \( \text{Prop} \)
- universe polymorphism
- inductively defined types

Features:

- The core is constructive.
- Can comfortably import classical logic.
- Can work in homotopy type theory.
inductive nat : Type :=
  zero : nat,
  succ : nat → nat

namespace nat

notation `ℕ` := nat

theorem zero_or_succ_pred (n : ℕ) :
  n = 0 ∨ n = succ (pred n) :=
induction_on n
  (or.inl rfl)
  (take m IH, or.inr
    (show succ m = succ (pred (succ m)),
     from congr_arg succ pred_succ⁻¹)))
**inductive** decidable (p : Prop) : Type :=
inl : p → decidable p,
inr : ¬p → decidable p

**theorem** em (p : Prop) {H : decidable p} :
p ∨ ¬p :=
induction_on H
  (∀ Hp, or.inl Hp)
  (∀ Hnp, or.inr Hnp)

**theorem** and_decidable [instance] {a b : Prop} (Ha : decidable a) (Hb : decidable b) :
decidable (a ∧ b)

**theorem** has_decidable_eq [instance] [protected] :
decidable_eq N
The elaborator

Can handle:
- Dependent type theory
- Implicit arguments, higher-order unification
- Overloading
- Coercions
- Type classes

Features:
- No other proof system handles all of these.
- The elaborator uses nonchronological backtracking.
- It is really fast.
The implementation

- Written in C++, for performance.
- Functional data structures for backtracking, parallelization.
- Small kernel (7,000 lines of C++ code).
- Lua bindings, for user-defined tactics and parser extensions.
The user interface

Features:

- text files, with unicode symbols
- emacs
- a “Lean server” tracking changes and answering queries
- flycheck checks in the background, highlights errors
- the ninja build system maintains dependencies
- the Lean server provides type information, goals
- robust autocompletion

For a demo, see: https://asciinema.org/a/12277
A release in early 2015, with:

- A stable kernel and elaborator.
- A stable user interface.
- A basic tactic language.
- Some automation (e.g. the term simplifier)
- The beginnings of a standard library:
  - basic data types: nat, int, lists, ...
  - algebraic structures: orderings, equivalence relations, groups, rings, categories, ...

See: https://github.com/leanprover/lean
Long terms goals

A powerful system for

- reasoning about complex systems,
- reasoning about mathematics, and
- verifying claims about both.