Semantic approaches to ordinal analysis

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Overview

Ordinal analysis typically proceeds by "unwinding proofs."

Can we use ordinals, instead, to "build models"?

Motivation:

- Use ideas and methods from model theory, set theory, recursion theory
- Constructions may suggest combinatorial independences

Semantic approaches

- Hilbert and Ackermann: epsilon substitution
- Friedman: models of Σ^1_1-AC and ATR_0
- Paris-Kirby, Sommer, Avigad: α-large intervals
- Kripke, Quinsey: fulfillment
- Carlson: ranked partial structures

The α-large approach:

- Use ordinals to define large intervals in $\mathbb{N}$
- Carve out models from those

This two-step process becomes difficult for stronger theories.

Another approach

To analyze a theory T:

- Use Skolem functions to embed T in a universal theory
- Herbrand's theorem: it suffices to assign values to finitely many terms, consistent with axioms
- Use ordinals to do this
- Gradually eliminate nonconstructive principles

Advantage: seems to be as flexible as cut elimination

Disadvantage: starts to look less like model theory, and more like cut elimination

Ordinal recursive functions

Fix a system of ordinal notations.

A *$\prec\alpha$-iterative algorithm* is given by a notation $\beta \prec \alpha$ and elementary functions

- $start(\vec{x})$
- $next(q)$
- $norm(q)$
- $result(q)$

These data define a function $F(\vec{x})$:

```
clock ← β
state ← start(x⃗)
while norm(state) ≺ clock do
    clock ← norm(state)
    state ← next(state)
return result(state)
```


Ordinal recursive functionals

The previous definition relativizes well.

A *relativized $\prec\alpha$-iterative algorithm* is given by a notation $\beta \prec \alpha$ and elementary functions

- $start(\vec{x})$
- $query(q)$
- $next(q, u)$
- $norm(q)$
- $result(q)$

These data define a functional $F(\vec{x}, f)$:

```
clock ← β
state ← start(x⃗)
while norm(state) ≺ clock do
    clock ← norm(state)
    state ← next(state, f(query(state)))
return result(state)
```

The ordinal analysis of arithmetic

Theorem. Suppose $PA(f)$ proves $\forall x \; \exists y \; \varphi(x, y, f)$ for some Δ_0 formula φ. Then there is a $\prec\varepsilon_0$-recursive functional $F(x, f)$ such that PRA proves

$$\forall x, y \; (F(x, f) \downarrow = y \rightarrow \varphi(x, y, f)).$$

This is essentially due to Gentzen, and implies all the usual results of an ordinal analysis.

In the new approach, use "least element" functions to make Peano arithmetic quantifier free:

$$f(x, \vec{z}) = 0 \rightarrow f(\mu_f(\vec{z}), \vec{z}) = 0 \wedge \mu_f(\vec{z}) \leq x.$$

Nesting corresponds to complexity of induction.

Goal: given a finite set of μ axioms, assign consistent values to μ terms.

The general idea

Suppose $F(x, \mu_0, \mu_1, \ldots, \mu_n)$ is $\prec\alpha$-recursive, and each μ_i has depth i.

Replace this by a $\prec\omega^\alpha$-recursive function $G(x, \mu_0, \ldots, \mu_{n-1})$ which simultaneously computes F and a finite approximation to μ_n that is consistent with the values used in the computation.

Argument has the flavor of a finite injury priority argument. Start with $\mu_n = \emptyset$. Then:

1. Carry out computation of F.
2. If you find a value inconsistent with axiom for the μ_n, correct this value, and repeat.

Assign ordinals to computations, so that the ordinal drops with each step.

The Howard-Bachman ordinal

Let Ω denote the first uncountable cardinal, and let $\varepsilon_{\Omega+1}$ denote the $\Omega+1$st ε-number, i.e. the limit of the sequence

$$\Omega, \Omega^{\Omega}, \Omega^{(\Omega^{\Omega})}, \ldots$$

Any ordinal $\alpha < \varepsilon_{\Omega+1}$ can be written in Cantor normal form to the base Ω,

$$\alpha = \Omega^{\alpha_1}\beta_1 + \ldots \Omega^{\alpha_k}\beta_k$$

where

- $\alpha > \alpha_1 > \ldots > \alpha_k$
- each β_k is an element of Ω.

The β's occuring in the expansion (as well as in those of the α_i) are called the *components* of α.

The Howard-Bachman ordinal (cont'd)

For $\alpha \leq \varepsilon_{\Omega+1}$, define

- $C_\alpha : \Omega \to P(\Omega)$
- $\theta_\alpha : \Omega \to \Omega$

by transfinite recursion, as follows:

$$\begin{aligned} C_\alpha(\beta) &= \text{the closure of } \{0,1\} \cup \beta \text{ under } + \text{ and} \\ &\quad \text{the functions } \theta_\gamma\text{, where } \gamma < \alpha \text{ and the} \\ &\quad \text{components of } \gamma \text{ are in } C_\alpha(\beta) \\ \theta_\alpha &= \text{the enumerating function of} \\ &\quad \{\delta \mid \delta \notin C_\alpha(\delta) \wedge \alpha \in C_\alpha(\delta)\}. \end{aligned}$$

One has $\theta_\alpha(\beta) < \theta_\gamma(\delta)$ if and only if one of the following holds:

- $\alpha < \gamma$, $\beta < \theta_\gamma(\delta)$, and all the components of α are less than $\theta_\gamma(\delta)$
- $\alpha = \gamma$ and $\beta < \delta$
- $\gamma \leq \alpha$ but either δ or some component of γ is greater than or equal to $\theta_\alpha(\beta)$.

The Howard-Bachmann ordinal is $\theta_{\varepsilon_{\Omega+1}}(0)$.

Admissible set theory

The axioms of $KP\omega$ are as follows:

1. Extensionality: $x = y \rightarrow (x \in w \rightarrow y \in w)$
2. Pair: $\exists x\ (x = \{y, z\})$
3. Union: $\exists x\ (x = \bigcup y)$
4. Δ_0 separation: $\exists x\ \forall z\ (z \in x \leftrightarrow z \in y \wedge \varphi(z))$ where φ is Δ_0 and x does not occur in φ
5. Δ_0 collection: $\forall x \in z\ \exists y\ \varphi(x, y) \rightarrow \exists w\ \forall x \in z\ \exists y \in w\ \varphi(x, y)$, where φ is Δ_0
6. Foundation: $\forall x\ (\forall y \in x\ \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x\ \varphi(x)$, for arbitrary φ
7. Infinity: $\exists x\ (\emptyset \in x \wedge \forall y \in x\ (y \cup \{y\} \in x))$

In the absence of infinity, this is inter-interpretable with PA.

Theorem 0.1 *Suppose $KP\omega$ proves $\forall x\ \exists y\ \varphi(x, y)$, where φ is Σ_1. Then there is an ordinal $\alpha < \varepsilon_{\Omega+1}$ such that for every β, we have $\forall x \in L_\beta\ \exists y \in L_{\theta_\alpha(\beta)}\ \varphi(x, y)$.*

Primitive recursive set functions

To (re)obtain this result, let us first lift the definition of $\prec\alpha$-recursion to functions on sets.

In analogy to the elementary functions on the natural numbers, we need a collection of set functions that is robust, but does not grow too fast.

Use the *primitive recursive set functions* arising from work of Takeuti, Kino, Jensen, Karp, and Gandy.

Let φ_ω $(= \theta_\omega)$ be the ωth Veblen function.

Lemma 0.2 *For each α, $L_{\varphi_\omega(\alpha)}$ is closed under the primitive recursive set functions.*

Recursion on notations

Now think of Ω as the order type of the universe. We can define notations for $\varepsilon_{\Omega+1}$ in the class of sets, just as we can define notations for ε_0 in $\mathbb{N}$:

$$\hat{\alpha} = \Omega^{\hat{\alpha}_1}\beta_1 + \ldots \Omega^{\hat{\alpha}_k}\beta_k$$

where $\hat{\alpha}_1, \ldots, \hat{\alpha}_k$ are notations, and $\beta_1, \ldots, \ldots \beta_k$ are ordinals.

A $\prec\varepsilon_{\Omega+1}$-recursive functional $F(\vec{x}, f)$ is given by a notation $\hat{\beta}\prec\varepsilon_{\Omega+1}$ and primitive recursive *set* functions

- $start(\vec{x})$
- $query(q)$
- $next(q, u)$
- $norm(q)$
- $result(q)$

Lifting Gentzen's result

Let $PRS\omega$ be an axiomatization of the primitive recursive set functions (with ω as a constant).

Theorem 0.3 *Suppose*

$$PRS\omega + (Foundation) \vdash \forall x \; \exists y \; \varphi(x, y, \vec{f}),$$

where φ is quantifier-free. Then there is a $\prec\hat{\varepsilon}_{\Omega+1}$-recursive set function $F(x, \vec{f})$ such that

$$PRS\omega \vdash \forall x, y \; (F(x, \vec{f}) \downarrow= y \rightarrow \varphi(x, y, \vec{f})).$$

Compare to Genzten's result for PA:

- Foundation replaces induction
- $\varepsilon_{\Omega+1}$ replaces ε_0

We have not said anything about collection yet.

Skolemizing collection

Remember that an instance of Δ_0 collection is of the form

$$\forall v, z\ (\forall x \in v\ \exists y\ \theta(x,y,z) \rightarrow \exists w\ \forall x \in v\ \exists y \in w\ \theta(x,y,z))$$

Rewrite this as

$$\forall v, z\ (\exists x\ (x \in v \wedge \forall y\ \neg\theta(x,y,z)) \vee \\ \exists w\ \forall x \in w\ \exists y \in v\ \theta(x,y,z)).$$

Pair v and z, bring quantifiers to the front, and Skolemize:

$$\forall u, y\ ((coll(u) \in (u)_0 \wedge \neg\theta(coll(u), y, (u)_1)) \vee \\ \forall x \in u\ \exists y \in coll(u)\ \theta(x, y, (u)_1)).$$

In short, $coll(\langle v, z\rangle)$ is supposed to return either

- a value x satisfying $x \in v \wedge \neg\theta(x,y,z)$, or
- a value w satisfying $\forall x \in u\ \exists x \in w\ \theta(x,y,z)$.

Skolemizing collection

Let $Coll'(u,y,c)$ denote the primitive recursive relation

$$(c \in (u)_0 \wedge \neg\theta((u)_0, y, (u)_1)) \vee \forall x \in u\ \exists y \in c\ \theta(x, y, (u)_1).$$

This says "c is a sound interpretation of $coll(u)$ at y."

Collection is then equivalent to the universal axiom

$$\forall u, y\ Coll'(u, y, coll(u)) \qquad (Coll)$$

$KP\omega$ is contained in $PRS\omega + (Coll) + Foundation$.

Lemma 0.4 *Suppose $PRS\omega + (Coll) + Foundation$ proves*

$$\forall x\ \exists y\ \varphi(x,y),$$

where φ is Δ_0. Then there is a $\prec\varepsilon_{\Omega+1}$-recursive functional F such that $PRS\omega$ proves

$$\forall x, y\ (F(x, coll) \downarrow= y \wedge Coll'((y)_0, (y)_1, coll((y)_0)) \rightarrow \varphi(x,y)).$$

To finish it off, we only need to show that for some $\alpha \prec \varepsilon_{\Omega+1}$, whenever x is in L_γ, there is an approximation to the $coll$ function and a computation of F in $L_{\theta_\alpha(\gamma)}$ robust enough to answer the queries and satisfy the final test.

A combinatorial lemma

Lemma 0.5 *Suppose $F(x, f)$ is $\hat{\alpha}$-recursive, and $x \in L_\gamma$. Then there is a pair $\langle s, m\rangle \in L_{\theta_{\omega+\hat{\alpha}}(\gamma)}$ such that*

- *m is a function,*
- *s is a computation sequence for F at x, m, and*
- *if the result of s is y, then $Coll'((y)_0, (y)_1, m((y)_0))$.*

Proof: use transfinite induction on $\theta_{\omega+\hat{\alpha}}(\gamma)$ and a slightly stronger induction hypothesis.

This is analogous to a proof-theoretic "collapsing" lemma.

Conclusion

References:

- "Ordinal analysis without proofs": from fragments of arithmetic to predicative analysis
- "An ordinal analysis of admissible set theory using recursion on ordinal notations": admissible set theory
- "Update procedures and the 1-consistency of arithmetic": a more combinatorial packaging of the ordinal analysis of arithmetic

Further work:

- *Rewrite old results:* Cut elimination arguments can probably be translated to the new framework. Is there any advantage to doing so?
- *Polish the methods:* Can one make them seem even more combinatorial, more semantic, and easier to understand?
- *Prove new results:* Can one use the methods to extract interesting combinatorial principles for ordinals, sets, and numbers?