Proof as Method: A New Case for Proof in Mathematics Curricula

M.S. Thesis

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Abstract

In recent years there has been a call to reform mathematics education to produce what the NCTM calls "mathematical literacy" for all students. One of the NCTM's Standards involves the use of problem solving as a method of learning mathematics. In this thesis I put forward the hypothesis that proof is valuable in the school curriculum because it is instrumental in the cognitive processes required for successful problem solving. My view of proof does not supersede, but rather supplements, the traditional arguments for teaching proof. The evidence I present here draws on those traditional arguments as well as evidence from cognitive psychology concerning the role of metacognition in learning. The picture of proof that emerges emphasizes a role in mathematical discovery which mathematicians have noted but which is overlooked in educational literature.

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Introduction

For a little over a decade now, there has been much talk of reform in mathematics education. In particular, the National Council of Teachers of Mathematics has published several documents containing "standards" at which reforms should aim. These Standards aim to create and promote what is termed "mathematical literacy" for all students by the use of a developmentally and cognitively appropriate curriculum. Literacy, as understood by the NCTM, involves much more than performing desired calculations and solving exercises by working with symbols or even doing the routine proofs for which high school geometry is notorious. Mathematical literacy is a vague concept, involving "developing a mathematical viewpoint," "making connections" in mathematics, "mathematical reasoning," "communicating mathematically," and building an appropriate picture of the discipline, among other things. The Standards themselves help to flesh out what these terms mean, but in the end they still leave much to be desired when it comes to measuring and testing whether these goals have been achieved. Still, the NCTM's Standards have been highly influential, so that a new mathematics curriculum with any hope of being adopted by a school district advertises itself as "conforming to the Standards."

A conspicuous feature of some of these curricula is the de-emphasis of proof. Indeed, a turn away from proof is specifically (and disappointingly) advocated in the 1989 presentation of the *Standards* [13], where proof is explicitly de-emphasized. There, the NCTM cites the difficulty of teaching and learning to do proofs; the amount of time proof takes up, which is out of proportion to its benefit; the fact that proof is really unnecessary for the majority of students, including many of those who are college-bound; and proof's tendency to convey a picture of a static subject in which the students simply re-hash geometrical facts which have been known for 2500 years. At the time, perhaps, this picture of proof was a useful wake-up call for the mathematics community. The NCTM was aiming to effect a revolution in the way mathematics was viewed and taught, and undermining the importance of institutionalized, traditional elements of the curriculum was a bold way to achieve that objective. To give the council credit, it is important to recognize that the 1989 Standards did not call for a complete elimination of proof, though it may have been seen that way by many.

This de-emphasis on proof in the 1989 version of the *Standards* created a tension within the document. On the one hand, it called for a move away from proof, and on the other it stated goals of treating and teaching mathematics "as reasoning" and "as communication." The 2000 version of the *Standards* [14] ameliorates this tension. Among the standards enumerated is one titled "Reasoning and Proof," which recognizes that the traditional use of proof in a single, isolated section of the mathematics curriculum is a major reason why proof is perceived as immaterial to learning mathematics, and that rather than eliminating it, what is called for is a widening of its use:

> Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena. People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification.
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> Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and… at all grade levels, students should see and expect that mathematics makes sense. By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments.
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> Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. … Perhaps students at the postsecondary level find proof so difficult because their only experience in writing proofs has been in a high school geometry course, so they have a limited perspective. … Reasoning

> mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts.

Furthermore, in the "Communication" standard, the NCTM notes the importance of proof as a way of consolidating and organizing mathematical thinking in order to communicate it to the mathematical community. Asking and answering why something happens the way it does promotes the mathematical "understanding" which is the goal of mathematics education as enumerated by the *Standards*.

This revised outlook on proof is, no doubt, a response to a great deal of literature generated in the decade after the publication of the first *Standards* document. This body of work sprang up partly in explicit defense of proof (Hanna [6], Epp [4], Schoenfeld [22], [23]), and partly to support and fill in the NCTM's overall picture of what mathematics education should be. In light of that research, the NCTM was able to revise and hone its aims. But even the 2000 *Standards* are rather vague about just what the value of proof is.

In this thesis I will argue that proof, understood in the right way as part of an entire problem-solving process and not just as a paragraph in which the truth of some proposition or theorem is demonstrated, is the very thing which must be emphasized in order to achieve the NCTM's goals. I claim that solving problems by aiming to prove that the solution is correct develops the metacognitive skill required to create the "understanding" which the NCTM, and certainly any dedicated teacher, desires. In addition, to underline the reasons why this sort of understanding is desirable, I will supplement this discussion with some discussion of the role of proof in mathematics as a discipline. My purpose is to give a philosophical account of classroom-appropriate proof which, combined with the discussion of metacognitive skill, will provide a firm and explicit basis for the argument that proof is a valuable and necessary part of a mathematics curriculum. The discussion will develop in parts:

1. A review of literature on cognitive and metacognitive models of problem solving;

2. A discussion of the conventional view and use of proof in the classroom;

3. The main argument: that proof and problem solving are in a sense largely the same process and that teaching this process leads to the "literacy" or "understanding" the NCTM desires;

4. Conclusions: a prescription for teaching proof, and directions for future empirical study.

Chapter 1

Problem Solving and Metacognition

To begin building my case that proof should be taught as a part of the problem solving process, I open with a discussion of problem solving. I take it for granted that students' ability to solve problems is one of the desired outcomes of mathematical instruction and leave the arguments for this premise aside as separate from the current project.

Here I will present the general outlines along which problem solving performance is judged in much of the literature, relying heavily on Alan Schoenfeld's model in particular. The research on problem solving has led in recent years to a concentration on thinking skills which investigators term "metacognition," and after the general presentation I spend some time examining this notion. At the end of the chapter I describe and discuss research which indicates that metacognition is a vital element in successful problem solving. This last discussion will be a key point on which I base my argument in later chapters.

1.1 Models of Problem Solving: Factors in navigating through a space

Newell and Simon's [12] now classic presentation of problem solving presents the process of solving a problem as a search through a "problem space." This space is characterized by:

- a set of elements which represent a state of knowledge about a task,
- a set of operators on the elements, which produce new states of knowledge,
- an initial state of knowledge,
- the problem, as specified by a set of final states to be reached by applying operators to knowledge states,
- the total knowledge available to the problem solver in a given knowledge state.

This basic idea is reflected implicitly or explicitly in many models which have been developed since the publishing of *Human Problem Solving*. (See, for example, Dean [3], Hayes [7], Sternberg [24], and Schoenfeld [20].) In other words, Newell and Simon laid out the framework within which most later elaborations of problem solving have been articulated.

Thus, solving a problem involves having a problem to solve (i.e. a goal to reach), having background information and a starting point from which to approach it, and familiarity with the rules of the domain (objects to use and moves to make). Searching through this space, i.e. solving a problem, involves moving from one knowledge state to another through the use of the operators. The solver must be able to back up to previous states when he realizes that the current path is not working. Researchers since Newell and Simon have described the ideal search as occurring in roughly the following stages (see, for example, [20], [21], [3]):

- open, which includes both reading the problem and analyzing it: the solver reads through the problem to make sure terms are understood and the goal is clear, writing out definitions, recalling related examples, listing unknowns and givens and the things that follow immediately from them;
- explore, which involves looking for patterns, making guesses, conjectures and hypotheses, which may lead back to more analysis;
- plan, which is where a promising avenue is chosen and an appropriate representation set up for the next step;

- implement, which should be straightforward if the previous steps have been done carefully;

- verify, which includes not only checking results but also extending them by considering corollaries, as well as reviewing the process by which the problem was solved. This stage may lead to new, further proofs of the same theorem.

Researchers do exhibit some variation in their presentations, but in general, this is the basic outline of how a problem solver navigates (or should navigate) through the problem space. Notice that the "actual solving" of the problem, the implementation stage, is only one step of many; and further, that implementation is the step which requires the least amount of thought. The other steps involve recalling or inventing a great deal of auxiliary apparatus based on a store of previous knowledge, or thinking beyond the problem's requirements to its applications and extensions. Thus to focus on problem solving as a way of teaching mathematics is (or should be) to focus on how certain mathematical facts are related to and built from others.

Success in problem solving, according to Schoenfeld [20], is the product of four broad factors: resources, heuristics, control, and belief systems.

- Resources: A solver's resources include not only the facts and relationships she has at her disposal, but also the organization of that knowledge and its accessibility for the solver. Novices have different kinds of knowledge from experts, as research on chess players demonstrates [7]. Information is organized into chunks, and those chunks evolve as a person becomes more and more familiar with a domain. Thus expert chess players see the board as the configuration of the pieces, whereas novices see individual pieces. Similarly, professional mathematicians will quickly characterize problems under certain types which tend to reflect the deeper structure of the problem, whereas students tend to characterize problems by more superficial characteristics [22]. Resources Schoenfeld lists include informal or intuitive knowledge, facts and definitions, routine procedures, and knowledge about the rules of discourse in a domain. All of these things are important factors in problem solving ability.

- Heuristics: A heuristic is a general schema which outlines a procedure from which to work to solve a problem. Pólya's famous work *How* *to Solve It* describes heuristics which accurately reflect the behavior of expert mathematicians working through problems, but it turns out that those heuristics are ineffective in teaching students. Schoenfeld suggests that this is because heuristics are not sufficiently prescriptive and are described too broadly; they require more specific instantiation in order to be useful to non-experts. Using heuristics already involves a significant amount of sophistication on the part of the problem solver. He must first be able to choose an appropriate strategy, and then be familiar with specific forms of the strategy; and on top of this, he must be able to break the problem down and relate its parts to familiar or accessible problems, solve those problems, and apply those solutions to his present task. All of these things take good use of available resources, and require practice.

- Control: Since control is related very closely to metacognition, I defer its discussion to the next section.

- Belief systems: What a problem solver believes about her capabilities, the nature of the domain, and the requirements of the task at hand all influence how she deals with a problem. Schoenfeld notes that sophisticated problem solvers rely much more on argumentation and planning than do beginners, because they have learned how to use it and they believe in its efficacy. Beginners tend to rely more on empirical evidence and specific cases when approaching and exploring problems. Proof, particularly, is not often employed unless demanded by the task. Students tend to think of proof as confirming or verifying rather than discovering, and are often less convinced by proofs than by empirical evidence or cases. The study of students' proof conceptions by Healy and Hoyles [8] supports this finding. Furthermore, Shoenfeld observes that the context in which a task arises influences the way students think about the task. Classroom tasks can seem artificial, which may cause students to make silly mistakes they would never make when confronted with a similar problem in real life. One example he gives [21, p. 197] is a division problem in which students were asked to figure out how many buses the army needs to transport 1128 soldiers, if each bus holds 36 soldiers. Nearly one-third of the students gave the answer "31 remainder 12." Most students on a playground asked to find how many cars it takes to transport them somewhere would not

> make the same mistake, Schoenfeld claims—so the answer to the math problem is related to the context of the classroom and the students' beliefs about the nature of mathematics. They do not make the connection between the situations; one is a Math Problem done in the classroom with certain parameters and expectations, and the other is just a practical question. Students make the mistake in the classroom because of what they believe the nature of the task to be: they are to extract numbers and an operation, carry out the operation on the numbers, and that's all. They don't think about the practicality that buses come in whole numbers.

These four factors (resources, heuristics, control, and beliefs) are factors which influence mostly the problem-solving steps of opening, exploring, planning and verifying. Having extensive mathematical knowledge (resources) is critical in the opening and exploratory stages, so that students can discern patterns and draw analogies between their current tasks and the things they already know. Heuristics are useful in the planning stage in particular, and control (as we shall see) is vital in the exploratory and planning stages. Beliefs play a role at the beginning of a problem, when the solver is first working to set the problem into her space of knowledge, and at the end when the student must make sense of the solution she has come up with. Implementation is the step which draws on the least amount of a solver's background, and thus when teaching problem solving teachers should spend their time developing students' facility with the other aspects of the problem solving process.

From the discussion thus far it should be clear that problem solving requires a great deal of attention and care on the part of the solver. I remind you that true problems are non-routine; that is, the solver does not know immediately after reading the problem how it should be solved. Furthermore, successful problem solving requires much more than simply knowing many mathematical facts. It requires using these facts strategically. In the next section I turn to this strategic aspect of problem solving.

1.2 Clarification of metacognition and control

The term "metacognition" has become something of a buzzword all over the educational and psychological community, and for that reason has lost some of its meaning. In this section I try to get a handle on what metacognition is in the context of mathematics education, particularly problem solving, again relying largely on Alan Schoenfeld's explication of the term.

In the vaguest terms, metacognition is thinking about thinking. It is sometimes broken up into branches: there is metacognitive *knowledge* and metacognitive *skill* [9]. Metacognitive knowledge is awareness of thought processes and beliefs, and metacognitive skill is the use one makes of one's metacognitive knowledge to regulate further thought processes. Schoenfeld [21] breaks metacognition into three aspects:

1. Knowledge of one's own thought processes: how well a student can describe her own thinking;

2. Control or self-regulation: how well a student keeps track of what he's doing and to what extent he uses the knowledge gained from that activity;

3. Beliefs and intuitions: the ideas the student brings to mathematics and how those ideas shape the student's learning.

Item 1 forms something of a basis for items 2 and 3. Children's perception of their thought processes develops gradually, and it is not until middle school or later that they are able to reflect accurately on how they think. Once they are able to do so, however, they can begin to be self-regulating. The process of self-regulation is what interests us here, because successful problem solving depends not only on knowing what you know, but using it effectively. Thus, following Schoenfeld, I now discuss items 2 and 3 in more detail.

1.2.1 Control and self-regulation

This aspect of metacognition is important to problem solving for straightforward reasons: problem solvers who spend no time reflecting on what they are doing often jump hastily into a problem and therefore spend disproportionate amounts of time on wild-goose chases. At best, this is an inefficient use of time and resources; at worst, it results in failure to reach any solution at all (particularly in timed situations). Schoenfeld notes that in his problem-solving courses, students have a tendency at the beginning of the year to spend little time on understanding the problem. When he poses a problem to the class, he asks for suggestions and often there is one forthcoming almost immediately. He teaches caution by asking, "Before we explore this suggestion, does everyone understand the problem?" [21, p. 201]. This question requires the class to keep track of what they are collectively doing; typically, Schoenfeld reports, a few students say they do not understand, so the class spends time exploring problem conditions, drawing diagrams, working examples, or anything else which seems necessary. By the time all students are satisfied that they do understand the problem, the first suggestion has usually been withdrawn or revised because of what they have learned by pausing to be sure they understand the problem completely.

Control is, of course, also important in the rest of the problem solving session. Analysis of the problem (including its logical structure), which is tightly bound to the initial understanding of the problem, uncovers important conceptual aspects which can be utilized later in the problem and helps to generate ideas upon which to build plans. Planning is especially valuable to arriving at a solution. Students who spend little time planning are in danger of following a wrong solution path for an inordinate amount of time; it is usually advantageous to step back and ask what the result of the current exploration or calculation or construction activity is expected to contribute to the ultimate solution. Schoenfeld contrasts the performance of students on a given problem to that of an expert; after reading the problem, the students spent the rest of their allotted time on one unhelpful pursuit, whereas the expert analyzed, planned, explored and rejected different ideas, consequently solving the problem.

1.2.2 Beliefs and intuitions

In the problem-solving context, Schoenfeld cites two examples which, he suggests, show that students "believe that school mathematics consists of mastering formal procedures that are completely divorced from real life, from discovery, and from problem solving" [21, p. 197]. One example is the division problem mentioned above in which the student is asked to figure out how many buses the army needs to transport soldiers. The other example involved a disconnect between a geometry proof and a construction the students were asked to do. The proof task was as follows :

> *Given two intersecting lines and a circle tangent to both of them, show that the center of the circle lies on the intersection of the angle bisector of the two lines and the perpendiculars from the*

points of tangency.

And the construction task:

> *Given two intersecting lines and a point P on one of them, construct the circle tangent to both lines and passing through P.*

Most students had little difficulty completing the proof, but were unable to perform the construction afterward, even though the proof contains the key to the construction. The students apparently saw no relation between the problems; they have discrete schemas, one for confirmation and one for discovery, and the two don't seem to intersect. This, according to Schoenfeld, goes to show that "'[k]nowing' a lot of mathematics may not do the students much good if their beliefs keep them from using it" [21, p. 198]. So bringing students' beliefs to the surface should allow them to work with those beliefs and build bridges between them.

The point of all this is that even when students have sufficient mathematical knowledge (in the form of propositional facts and theorems) at their disposal and could make a list of the things they know, poor control and mistaken beliefs about mathematics result in poor use of that knowledge. This can make math seem pointless and irrelevant to the students, and by the NCTM's criteria they do not "understand" mathematics. In the next section, I review research designed to show that explicitly promoting metacognition increases students' success in solving problems. This will also help us focus on what "metacognition" is in this context.

1.3 Metacognitive training enhances problem solving performance

1.3.1 IMPROVE

Kramarski, Mevarech and Lieberman [9] conduct a study designed to investigate the effects of metacognitive training on seventh-grade students' mathematical achievement. The metacognitive aspect of the study, a method called IMPROVE, focuses on metacognitive *knowledge* in the form of understanding a problem, and metacognitive *skill* in the forms of using strategies and linking aspects of the problem to previous knowledge. The achievement aspect of the study measured 1) what we might call "ordinary" mathematical skill in basic facts and calculation, 2) mathematical reasoning such as predicting outcomes, generalizing, and choosing the mathematical laws appropriate for evaluating expressions, and 3) transfer of these skills from the situation of math class to a "real-life" problem. In general, the study found that students who were given metacognitive training both in reading English (a foreign language for these Israeli students) *and* in mathematics did better than students whose metacognitive training was only math-related, and the latter in turn did better than a control group.

In an similar, earlier study in which they were testing the IMPROVE method, Mevarech and Kramarski [11] state that their study is rooted in constructivist theories of learning. Citing Lauren Resnik, they describe such theories as holding that "learning occurs not by recoding information but by interpreting it" [11]. This "interpretation" involves making ties between previously existing and new knowledge, either by incorporating new knowledge into existing schemata or revising a schema in light of the new knowledge. For example, consider a student who knows that adding two positive numbers $a + b$ requires moving b steps to the right from point a on a number line. When she comes to learn about negative numbers and is exposed to problems such as $a + (-b)$, this adding schema will need to be revised to incorporate the *leftward* shift denoted by the negative number. She could have two different adding schemata, one moving right for positive and one moving left for negative numbers, or she could revise the existing schema so that adding is simply motion on a number line and the sign of the second addend provides information as to which direction to move. Thus, under constructivist theories, learning essentially involves "intentions, self-monitoring, elaborations, and representational constructions of the individual learners" [11, citing Resnik]. Based on these theories and the observations of previous researchers on the problem-solving behavior of experts, they designed their metacognitive questions to make the construction of links between new and old knowledge explicit. They hypothesized that making this process explicit would "elicit elaborate explanations and enhance mathematical reasoning," with the expectation that the effects of the IMPROVE method would be stronger on reasoning than basic skills.

In the IMPROVE method, a teacher would introduce new material in several ways, to take advantage of the different backgrounds students brought to the lesson. Among the techniques the teacher employed were the set of metacognitive questions (which I will discuss in more detail below). The students also had the questions on prompt cards for use in individual and group work. They were told directly that the questions were designed to help them in their mathematical performance. After the introduction of the material, students were given assignments and worked in groups of four. Each student took a turn reading a problem aloud and guiding the group through the answers to the metacognitive questions as they applied to the problem; then the group solved the problem. When disagreements arose, the students were expected to resolve the difficulties through discussion.

Mevarech and Kramarski claim that in the course of asking and answering these questions, students learn to uncover not only the surface structure, but the deeper structure of the problems they encounter. They cite an example of a protocol in which the students were doing a train problem. The students immediately identify the problem as being about distance, time, and speed. One student connects the new problem to a previous one, saying it's just like the one before. Other members of the group point out that it's a little different because the two trains in the new problem do not leave at the same time. The students then continue to work out a representation of the new problem, and again discover the similarities and differences between the new problem and the old one. The new one has the same "surface structure" as the old one— both are about distance, time and speed— but the new situation has a different "deep structure" because the techniques used to set up and solve the new one are different from those used to solve the old one.

Strictly speaking, the terms "surface structure" and "deep structure" are misleading. What Mevarech and Kramarski call "surface structure" isn't *structure* at all, but identification of the important, potentially useful features of the problem, such as what quantities are involved. "Deep structure" involves more than just the techniques used to set up and solve the problem— unless we count the logical underpinnings of the problem as part of the setting up and solving process. This is not entirely inappropriate; the metacognitive questions the students in the treatment groups for both studies were taught to ask themselves are essentially asking for the students to think about the logic involved with the problem. The questions included the following: "What is the problem about?" (surface features); "How is this problem different from ones I have already solved?" (surface and deeper features); "Which strategies and principles are appropriate for solving the problem?" (deep features). These questions require the students to relate their current task to the knowledge they already have, and to consider how they ought to think about the task. That is, the questions demand that students think about the overall patterns of thought involved in solving the problem, patterns which are very likely to have clear representations in formal logic, and hence fairly standard moves to use in solving the problem. In a move related to the advice Polya gives in *How to Solve It* [15, p. 206–7], the students in the study were told to use the above questions when working together in small groups, as well as in written explanations of problem solutions. The teacher also asked the questions of the whole class as they began a new topic or explored a problem together.

In the 2001 study, all students were given a pretest and a posttest to assess their mathematical achievement. The pretest and a major component of the posttest consisted of multiple choice questions concerning basic facts, or open-ended calculation problems. These were scored 0 or 1 depending on whether the final answer was correct or incorrect. On the posttest, however, were additional items in which not only were the students scored for the answer, but for their explanations of the answers as well. A score of 0 was given for incorrect or irrelevant explanations, 1 for relevant but incomplete explanations, and 2 for full answers showing "clear, unambiguous explanation of one's mathematical reasoning" [9].

In addition to these two tests, students were given a "pizza task" in which they were to make a proposal to the class treasurer on the best way to use a limited budget to provide the class with a pizza party (that is, they were to figure out the maximum amount of pizza they could get with their money). Students' answers were scored according to 1) whether they referenced all the data they were given, 2) their organization of the information, e.g. in a table, 3) their processing of the information, i.e. the calculations and explicit descriptions of how they used the information, and 4) the conclusions: presenting a solution to the problem and justifying that solution to the treasurer.

Notice that organization is particularly valuable in solving problems, according to the researchers. The highest scoring solutions had to include not only a choice of pizza company and the justification for that choice, but how well the information presented was used. This fits with what was said in previous sections, namely that the kind of mathematical knowledge critically important for solving problems (and thus for understanding mathematics) has to do with the *use* of information.

Recall that the roughest definition of metacognition is that it is "thinking about thinking." Right now, we are interested in the thought processes the researchers brought out in the students participating in the study: we want to know what metacognition is in the context of learning mathematics and how it promotes mathematical "understanding" as understood in this particular study.

As mentioned above, the questions in the metacognitive training of the experimental groups concern the following: "(a) nature of the problem (e.g., What is the problem all about?); (b) use of strategies appropriate for solving the problem (e.g., What are the strategies/tactics/principles that are appropriate to solve the problem, and why?); and (c) construction of relationships between previous and new knowledge (e.g., What are the similarities/differences between the problem at hand and the problems solved in the past?)" [9].

The salient feature of these questions is that they make the information students routinely use in problem solving explicit and conscious. The strategies question makes a strategy one more schema (or chunk) in the inventory from which a student can draw when encountering new problems. That is, students accustomed to stepping back and assessing their problem-solving process will probably recognize patterns which they can later apply in new contexts. The construction of relationships is a reinforcer; it works from the assumption that ties between pieces of knowledge are stronger when they are made explicitly rather than implicitly. Asking what the problem is about encourages the students to make observations about deeper structure and, thus, again make connections between the new situation and old ones. So the term "metacognition" as used here involves the conscious construction of relationships between pieces of knowledge.

In their concluding remarks on the 1997 study, Mevarech and Kramarski explain that their research shows a difference in cognitive responses as well as achievement among the students. Those who employed the IMPROVE method could discuss problems from multiple perspectives and gave verbal explanations supported by evidence and mathematical principles, whereas those in the control group rarely went beyond providing a final answer to a problem [11]. While providing a correct final answer is certainly evidence of some grasp of the material, it is not *necessarily* evidence of the kind of understanding sought by the researchers and by the NCTM, because just an answer with no explanation does not directly demonstrate that the student has relied on a broad knowledge of principles and their application in order to produce that answer. Students who underwent metacognitive training could demonstrate their understanding more directly.

1.3.2 How IMPROVE improves understanding

In this section, I discuss the 2001 study on a finer level in order to get at just what good metacognition can do for students' understanding of mathematics. The focus here shifts slightly from a discussion of problem solving to a discussion of understanding, but in light of the fact that solving problems is supposed to be a way to demonstrate understanding according to the NCTM, I think the shift is acceptable.

The pretest and posttest in the study relied heavily on calculational, "drill-type" problems. The results indicate that the metacognitive training in both the treatment groups led to improvement in these groups' scores. On the assumption that the metacognitive groups differed from the control group only in the metacognitive training (and not, for example, by getting extra drill practice), surely this is an indication that the students in the metacognitive groups have some more "understanding" which enables them to perform better than their colleagues. But what sort of "understanding" can metacognition bring out in routine, algorithmically solvable problems?

The first metacognitive question asked the students what the problem was all about. Now, it makes a difference here whether the problems are presented as "story problems" or simply as calculations. If they are story problems, "What is this problem about?" might ask whether the problem is a subtraction problem or a division problem (that is, it might ask the student to sort the situational details about Margaret and Susie and stickers from the mathematical ones). Or it might ask something deeper about what the concept of subtraction or division is. Given the possibility that the problems were presented without a covering story, I think we must assume that "What is the problem about?" is asking something about the mathematical procedure rather than the story.

But if we assume that, what kind of answer are we expecting the student to give? If the given problem is to evaluate $5-(-7)$, the problem is "about" subtraction. But perhaps there is something even deeper than that. What is subtraction about? One answer is that subtracting one number from another determines their difference, or the distance between them, i.e. how far apart they are on a number line. If a student gives this answer or one like it, at the seventh-grade level, I believe we would say that she understands subtraction. But can we expect all students to come to this kind of understanding just by getting them to ask routinely "What is going on here?" A problem such as $5-(-7)$ isn't easy to do unless you have the distance conception of subtraction already under your belt.

Perhaps, however, it's not that simple. For example, could the metacognitive question "What is this problem about?" require a student to take a problem such as $5 - (-7)$ and invent the scenario which helps him understand what the problem is about? We could imagine an inner monologue such as this: "Ok, five minus negative seven. Suppose Brian has five dollars. That's the 5. Now, what are negative dollars? Owing money. Ok, Laura owes someone seven dollars. Now, how does subtraction work in this case? Subtraction is difference. I guess I want to know how much money Brian has compared to Laura." What is the problem about— what is it asking? To find a difference between quantities. Now that he has understood what the problem is asking, the student asks himself the next metacognitive question, "What strategy is appropriate to solve the problem?" His answer might be, "All right, the difference between them is how much I would have to give Laura so that she had the same amount as Brian. She needs seven to pay her debt, then five more to get what Brian has. That's twelve dollars. That's the difference between them." Now that the student has solved the problem, the teacher can ask him to relate it to other subtraction problems he has done. Depending on how he has previously learned mathematics, there may be many different ways of doing this.

Note, however, that the money/debt scenario isn't a necessary feature of the explanation required by the question "What's going on here?" The student may very well have been capable of thinking the problem out entirely in terms of numbers and the number line: "Ok, five minus negative seven. I'm at negative seven, and five is five units on the other side of zero from here. Now, I need to find how big 5 is compared to -7." His strategy might look like this: "Well, to get to zero from where I am I go up seven. From zero to five I go up five more, which is a total of twelve. So the difference between them is twelve units, and therefore $5 - (-7)$ is 12."

This is very interesting, of course, but how is it helping us uncover the researchers' notion of understanding? Consider two students who are seeing the subtraction of negative numbers for the first time. Assume they have done subtraction problems in which the smaller positive number is subtracted from the larger, as well as larger positive numbers subtracted from smaller ones (so they already know about negative numbers). Assume also that they know what it means to add a negative number to another number. Now, when confronted with $5 - (-7)$, the metacognitively trained student will begin by asking "What is this problem about?", and he might proceed as above.

How will a student in the study's control group approach the problem? Perhaps she will need an entirely new schema for subtracting negative numbers, because she has not been used to seeing the subtraction operation as a single concept of difference or distance, but instead has memorized what you do (in terms of moving along the number line) when you see certain signs in the problem. Unless she is particularly clever, she might look at $5-(-7)$ and ask for help because this type of problem doesn't match any of her previous experience.

If this scenario is correct, then the "understanding" tested for in the study is not only operational or computational (as it would appear from the types of test items), but approach- and application-based. A problem's identified "aboutness" can involve the relevant operations and cognitive schemata a student should use to solve it, or it can involve translating to or from a general concept (such as subtraction) and a representative instance (such as owing money). To understand a mathematical problem is to locate where it lies on the mathematical map as understood thus far— how the concepts (such as negative numbers and the operation of subtraction, in our example) involved have been used before in familiar situations and how they can be extended. Having such understanding allows a student to break down novel problems in terms of things she already knows, thus giving her an inroad to begin a solution. Having a general understanding of subtraction as difference allows her to apply that operation to negative numbers as well as positive numbers, rather than getting stuck because what she sees before her doesn't fit any recognized schema. Something similar could be said about Schoenfeld's proof and construction problem pair concerning intersecting lines and the circle tangent to them: students who have been trained to make connections would surely be more likely to see how the proof they just completed would help them with the construction task.

How well does this analysis apply to the other aspect of mathematical achievement the researchers investigated, namely mathematical explanations?

On the posttest there were eight items in which the students were asked not only to give an answer, but to explain their reasoning in writing. The problem example in the study is as follows: "In the following item, $2^3 \ldots (-2)^3$, write the sign $>$, $<$, or $=$ so that a correct statement will be received. Explain your answer." A score of 0 was given for an incorrect or irrelevant explanation, e.g. "$2^3 = (-2)^3$ because when there is a minus in brackets in powers, the minus becomes +." A score of 1 was given for partial explanations, e.g. "$2^3 > (-2)^3$ because when the exponent of the power is odd, the result will always be negative." This explanation is called incomplete because nothing is mentioned about 2^3 or the reason for the sign to be $>$. A score of 2 would be given for an explanation such as "$2^3 > (-2)^3$ because when the exponent of the power is odd and the base of the power is positive, the result is positive. When the exponent of the power is odd and the base of the power is negative, the result will be negative even with brackets. A positive number is always bigger than a negative number." So a student's explanation scores a 2 when all aspects of the problem are included in the (correct) explanation [9].

Let us manufacture 0, 1, and 2-level answers for the subtraction problem discussed above. The question the student sees on the test is "What is $5 - (-7)$? Explain your answer." Perhaps a score of 0 would look like this: "$5 - (-7) = -2$ because 5 is 2 less than 7." The next level might be an answer such as "$5 - (-7) = 12$ because two negatives make a plus and $5 + 7 = 12$." This is incomplete if we are testing for an understanding of subtraction because it makes no mention of the difference or distance concept. Now, it could very well be that the student understands subtraction as adding a negative quantity and this rule is the basis for his answer; but without further evidence, we cannot be certain that the student is not simply reciting a rule he does not know the origin of. A 2-level answer would be, "$5 - (-7) = 12$ because the distance from -7 to 5 on a number line is 12."

Now, above I concocted an example in which the student relied on general principles by instantiating them in an example using money and debt. Would we accept as a level 2 answer an explanation such as "$5 - (-7) = 12$ because removing a debt of seven dollars is the same as receiving seven dollars, and in order to get from having a debt of seven dollars to having five, you need to receive twelve dollars"? We want to say no; relying on the instantiation is dangerous because there may be other problems involving different surface concepts which may not lead the student to recognize what's going on in the problem. But is this all there is to it? The student began with a plain, abstract equation and knew a proper way to concretize it, which certainly counts for some understanding. Still, without the student relying on a general mathematical principle that makes no reference to everyday objects, we cannot be certain that the student can instantiate the equation properly in any situation, and so I think we would give that answer a score of 1.

Translating between general cases and appropriate instances is certainly a feature of understanding, but the generality is the most important feature of understanding in this context.

In my version of a level 2 answer, we can see how the metacognitive questions would aid the student in arriving at this explanation. A student who has asked himself what the problem is about and how it is similar to others he has encountered will not only be able to figure out what the answer is, but also that it is a reasonable answer. His explanation displays that thinking. In the level 1 answer, however, we have evidence only of a cognitive schema used to figure out the answer but which may not be applicable in new situations. The student with the level 2 answer has been able to relate the particular problem at hand to others like it, and knows that he has done so by employing the generalization about the meaning of subtraction. Thus my analysis of "understanding" tested for in the computational items of the posttest is appropriate for the explanation-oriented items as well.

And what about the pizza task? Is my analysis of what the researchers were looking for as "understanding" applicable there too? I said that to understand in the context of this study was to locate where a math problem lies in the mathematical landscape, and how the concepts involved in the problem have been used before and how they can be extended. In the pizza task, students were awarded points along four axes: 1) referencing all data, 2) organizing the information, 3) processing the information, and 4) drawing conclusions. Presumably, a student with the most understanding would have the best score along all four axes. The task for us here is to figure out what each of these criteria contributes to a student's showing understanding.

Referencing all data. Why must a student who understands mathematics make reference to all (relevant!) data in the problem? Well, first, referencing relevant data shows that the student can distinguish what is relevant in light of the problem's demands— and by figuring out what the problem is about, she can figure out what data is relevant. In order to solve a problem, the student goes through her inventory of mathematical facts and operations to decide what she can use and how to use it to arrive at a solution. In the case of the pizza task, the diameter of the pizza and the price are relevant because they provide a way to compare different companies' pizzas (price per area), and the budget is relevant as a limiting factor. Notice that the information necessary to solve the problem goes beyond what is given: from the diameter and the price the students must calculate a new quantity which is the crucial one to solve the problem. The key relationship is not given in the problem, so the student must determine just what the information she *is* given tells her— i.e. she must locate the problem task in her space of mathematical knowledge of diameters, circle areas, prices, and so on. So by evaluating students' solutions along the axis of data reference, researchers are assuming that understanding mathematics relies on discerning what data are important and how they are useful.

Organizing the information. A student who understands mathematics knows that one strategy for solving a problem is to organize the data given in the problem, for instance in a table, because that will display the relevant relationships among various quantities (relevance determined by what the problem is seen to be about). This organization is a part of the approach to the problem. It is also a strategy for arriving at a solution, because a table helps display explicitly the steps necessary to solve the problem. For the pizza task, the table will need to include the initial data on pizza diameter and price for the various companies, as well as the areas of the pizzas and the total amount of pizza it is possible for the class to order from each company given the budget constraint. Making a table is a tool a student should have, and using it in the particular task at hand shows that a student understands not only the mathematical use of tables but also that the structure of the problem is one that calls for a table.

Processing information. Certainly, understanding mathematics involves arriving at correct solutions, but not only that. The researchers also judged students' ability to provide an explicit description of the solution process. That is, they want a student to show that he has arrived at the answer in a principled way. A student must demonstrate that he proceeded by drawing on his previous knowledge in clear and conscious ways— in short, that he has a mathematical map and can navigate with it.

Drawing conclusions. This is the final presentation of the solution to the treasurer, which must be clear and give justifications so that the treasurer, who has presumably not worked out the problem for herself, can see that the proposed plan is the best use of the class's budget. Presumably the researchers used this criterion on the principle that if the student presenting the solution does not fully understand it, neither will the treasurer. Metaphorically speaking, someone who is truly familiar with a place can not only get somewhere himself, but can also direct others— he not only locates it on his own map, but can describe the map to someone else. Therefore, being able to explain and justify the solution is also a demonstration of the kind of understanding I have depicted the researchers as operating with.

To summarize the last few sections: Mevarech, Kramarski and Lieberman's study suggests that metacognitive training in problem solving, i.e. training students to make themselves aware of the strategies they use to solve problems and requiring them to make connections between facts, improves their problem-solving ability. Problem solving ability is linked to mathematical understanding in that good problem solvers can locate their immediate task in the space of the mathematical knowledge they have and can navigate within that space by exploiting connections among facts to achieve new states of knowledge and thus reach a solution to the problem. Or, in Schoenfeld's terms, good problem solvers draw effectively on their mathematical resources; and the effective use of resources requires metacognitive control and the appropriate beliefs regarding the nature of mathematical justification, the structure of mathematical knowledge, and the relationship between abstract mathematical knowledge and real-world applications of that knowledge.

1.3.3 Pausing to think globally: Schoenfeld's work

In his problem solving classes, Alan Schoenfeld also uses questions to get his students thinking about their approach to problems. His questions are:

- What are you doing?
- Why are you doing it?
- How does it help you?

He uses these questions as he circulates among small groups as they work on problems in class, because he has found (as discussed above) that novice problem solvers have a propensity to dive into a problem whenever they come up with an idea. They usually don't step back and consider whether that idea is a good one, and if it isn't, they waste their time on unfruitful pursuits.

The broad story Schoenfeld wants to tell looks like this: students who are able to explain what they're doing and why, when they're solving a problem, can do so because they're aware of the process and how steps along the way are supposed to relate to the final solution. This is metacognition. Awareness facilitates control decisions, since a verbal/propositional grasp of their activities gives them something concrete to reflect on. It's clear that this is the case because Schoenfeld's use of his three questions works, in his experience, to get the students to break their old problem-solving habits and become better mathematicians. Good control decisions certainly facilitate success at problem solving, which Schoenfeld illustrates by the comparison of students and experts and noting that the major difference between the two is the control. He further supports the observation by comparing the problem-solving procedures of his own students at the beginning and end of his course.

The heart of the "Control" chapter of his book [20] is a general problem solving strategy which Schoenfeld taught in his problem-solving courses at Berkeley and at Hamilton College, and it is this which is of the most interest for the current project. He presents excerpts from his handouts which introduce the strategy (to be elaborated below), emphasizing that it is not to be taken as a "program" or algorithm the students are supposed to implement mechanically. "Rather," he reports, "it was intended as a default strategy—a guide to use when the student did not know what to do next and could use guidance [and reminders] in selecting from among the heuristic techniques that might be appropriate" [20, p. 107]. Students who knew what to do should be guided first by their knowledge, without any prescribed strategy interfering with their thought processes.

The handout begins by presenting a high-level description of the strategy and an explanation of the term "heuristics." Schoenfeld then proceeds to describe each phase of the strategy in more detail, giving heuristics to use at each level. The strategy's "steps" are roughly the ones by which Schoenfeld analyzes the protocols of novices and experts and which were described in general terms earlier in this chapter: analysis, design or planning, exploration, implementation and verification. They are presented in a flow chart to show the development of a solution.

One begins with analysis, making sure one understands the problem by drawing diagrams, looking at special cases, and simplifying where possible. Restating the problem in one's own words is also part of analysis. Schoenfeld does not mention logical analysis in his discussion, but Susanna Epp [5] advocates the use of some basic formal logic in approaching problems, and her point is worth noting.[1] When students begin analyzing a problem, very often it will take a form such as "Prove that for all…" or "Show that for any…, there is a…". For these fairly common and fairly simple sorts of statements, there are standard ways of dealing with them, such as using proof by contradiction on a hypothetical counterexample. Students who are able to identify the logical structure of a problem will then have standard logical tools available as strategies for working with the problem. This sort of logical analysis is a straightforward answer to a metacognitive "What is going on here?" question.

Once a student is satisfied that he has understood the problem, he proceeds to planning and design. "Most generally," Schoenfeld explains, "it means keeping a global perspective on what you are doing and proceeding hierarchically" [20, p.108]. Students should not proceed down a path without a clear idea of the reasons for choosing it, given that they have proposed several alternatives and proceeding further in the problem requires following it. The plan should be supplemented with exploration, that is, investigating analogous or equivalent problems, slightly modified problems, and perhaps also broadly modified problems. Schoenfeld outlines some suggestions for coming up with such problems. The exploration step might show her that she can proceed to implement the tentative plan she outlined in the previous step. Or it might turn out that during exploration, the student realizes that she did not understand the problem thoroughly in the first place or the design she came up with was faulty, in which case she may return to the analysis phase of the strategy. For example, in the tangent circle construction problem mentioned above (p. 14), the student may have guessed at the outset that by finding the diameter of the desired circle she would be able to draw the circle, so she sets up a subgoal of finding the diameter of the circle. She may realize that the center of the circle will lie on the bisector of the angle between the two lines, and decide to relax the condition of finding a circle through point P, seeking instead a circle tangent to both lines at an arbitrary point on the bisector. If she can do this, she may construct the diameter of that "generic" circle and note that it does not bear any obvious relation to the point of tangency or the angle bisector, and she may give up her subgoal of looking for the diameter of the circle. Instead, she may make use of what she learned under the relaxed condition to complete the desired construction.

[1] I thank Wilfried Sieg for pointing this out.

Implementation should be fairly straightforward at this point, and it is the last step in the problem solution itself. After the implementation has been executed, Schoenfeld stresses, the student should look back and verify both locally and globally that the solution is reasonable. This recap is crucial not only to catch small mistakes, but to cement the process. In this step the student may discover connections to other problems or reveal other methods of solving the same problem, or note the generality of the solution technique for use in future problems.

The next part of the chapter is a more detailed discussion of the effect of control decisions (or lack thereof) on arriving at a solution. Schoenfeld examines four cases: Type A, where bad control decisions guarantee failure because the solver fails to take a global perspective on the problem and jumps into an entirely unhelpful wild goose chase; Type B, in which control avoids disaster but does not serve as a strong guide toward the solution; Type C, where control shapes the process in positive ways, making efficient use of resources; and Type D in which there is practically no control behavior because the solution is easily apparent—that is, the "problem" is really an exercise.

It is Type C which Schoenfeld wishes to hold up as the model of a good problem solver, and he implicitly suggests that this, rather than Type D, is really the model which should be the aim of a good mathematics education. Even students who do not advance beyond high school mathematics ought to be able to work through problems with the understanding demonstrated by the professor whose protocol models Type C control. This professor frequently makes comments such as "Isn't that what I want? Right!" and "mmm, wait a minute..." and "That's dumb!" Schoenfeld remarks that these exclamations serve as checkpoints in the professor's strongly goal-directed behavior; the professor has a plan (though it is not explicitly stated) and he continually reviews whether he is on track to achieve it. He is continually generating ideas, evaluating their promise and discarding ones which could grow into wild goose chases. His ability to make such judgments shows a global understanding which is exactly what mathematics education is (or should be) aiming for.

Note also that in the outline of the strategy and in his discussion of the protocols, Schoenfeld implicitly de-emphasizes the actual solution (the construction, the numerical answer, etc.). The implementation step gets one sentence in an entire three pages of discussion. The process which brings the student to the point of implementation, and the review of that process, are always the focus. I take this as evidence that Schoenfeld sees "learning mathematics" as crucially involving a kind of thought process, as much or more than it involves the memorization of mathematical facts. Of course, implementation is not trivial when one does not have a working knowledge of, for instance, geometry. Certainly there are no relationships to exploit if there are no facts to be related to one another. But what Schoenfeld and the NCTM are looking for goes beyond those facts.

Control is not, in and of itself, the desired end. But students who are trained to keep control of their problem solving behavior will be successful at solving problems, an activity in which they must exploit known relationships and along the way they will discover new ones. There is a great deal of psychological evidence that learning is more effective when students "construct"—or discover—information for themselves. They will also see how useful and important all their facts really are, when they are expected to use them in meaningful activity. Problem solving provides a forum for meaningful mathematical activity.

Chapter 2

Proof in Schools: Current Practice

In this chapter I set problem solving aside and turn to a discussion of proof as it is currently taught and viewed in schools. I present and discuss some of the conventional justifications of and roles for proof in mathematics education, in order to set off my argument for proof from what has already been said on its behalf. I will also survey some of the common objections to these conventional arguments, which will serve later to motivate my own argument. This discussion will be supplemented by an empirical study of students' perceptions of proof in the United Kingdom. In the next chapter I present an argument that there is another justification for proof which involves its tendency to promote metacognition, which as we saw above is instrumental to problem solving.

2.1 The "obvious" point: Truth and Justification

No doubt one of the first things that comes to mind when we think of proof in any context is its role in assuring us of the truth of some fact. That idea certainly fits our notion of what lawyers and scientists do: they show that their clients are or are not guilty, or that their hypotheses concerning subatomic particles or genetic processes are true. Surely mathematicians do much the same thing; proofs demonstrate to us the truth of intuitively correct facts such as the congruence of two triangles whose side lengths are congruent ("SSS"), or less obvious facts such as the Pythagorean theorem, or outright surprising theorems such as the fact that the cardinalities of the sets of integers ($\mathbb{Z}$), rationals ($\mathbb{Q}$), and natural numbers ($\mathbb{N}$) are all equal. Most of us would readily assent to the proposition that two triangles with congruent sides are congruent, but in case there was any doubt, the proof will show us that it must be so by using accepted rules of reasoning to build the proposition from facts we already accept. In the case of the Pythagorean theorem, someone unacquainted with it might raise his eyebrows at it, but any number of proofs will demonstrate its truth. My last example, because it is so counterintuitive, is usually not accepted by novices except when the proof is given and cannot be denied; after all, it seems obvious that there ought to be twice as many integers as there are natural numbers. Yet the proof that the cardinalities of $\mathbb{Z}$ and $\mathbb{N}$ are equal is fairly simple, and because of the deductive nature of proof, one cannot deny the truth of the proposition once it has been proven.

Assurance of truth is only one of proof's roles in mathematics, in the classroom or in professional practice. We should note that presenting justification as the sole reason to do proof has a few weaknesses, and some argue that because of these weaknesses, there is no need to teach proof in schools. For example: as it turns out, most students are not as convinced by a rigorous proof as they are by a number of examples or "empirical" evidence ([25], [8]). They may be happy to take a teacher's—or Euclid's—word for the truth of the Pythagorean theorem or by their own investigation of examples. The reply to this objection is twofold: first, of course it doesn't matter that they're convinced by empirical arguments; they shouldn't be, and part of the point of education is to teach them not to be. They should value rationality over authority. Second, students can be led astray from intuition and perceived patterns. In the discussion following Susanna Epp's explication of the role of proof in problem solving [4], Schoenfeld mentions a problem he uses to get students to realize why proof is necessary:

> Suppose you place n points on the circumference of a circle so that when you draw the line segments joining them pairwise, no three of the line segments intersect in a common point. What is the number of regions into which the line segments partition the circle?

For $n = 0, 1, 2, 3$, and 4, the number of regions is 2^n, but when $n = 5$ the pattern breaks because the number of regions is 31. "[A]fter the students have checked their diagrams half a dozen times," Schoenfeld remarks, "...we can begin to talk about why proof is necessary." That is, proof can be a tool to *dis*confirm conjectures or correct mistakes, in the sense that the process of finding a proof for a hypothesis may reveal that the hypothesis was wrong or incomplete.

Nevertheless, another weakness is that if students are presented with a picture of proof which makes it seem as though proof's value lies in confirming facts, they may easily be turned off. After all, Euclid's *Elements* is over two thousand years old; if students are rehearsing the same proofs Euclid did, i.e. proving things we already know to be true, they are likely to see no point in the exercise of proving ([13], [22]). Unless educators address this issue directly by explaining (for instance) that the best way to learn to prove unknown things is to prove known things and gain experience, students may end up with answers to questions they never had [2], because many geometrical facts seem straightforward and therefore in no need of proof. Still, this weakness does not seem insurmountable when some care is taken to provide relevant motivation for proving.

2.2 Other conventional arguments for proof: communication and tradition

Very few teachers would cite assurance of truth as the sole reason for teaching proof in the classroom, of course. Proof is also a method of communicating results to others in a clear and fairly conventional form. This purpose of proof is relatively straightforward; a good proof will show how the theorem to be proved follows from other already-known facts by a chain of good reasoning. Communication is tied to assurance of truth in that it is not generally enough to present a new mathematical fact to someone else and say "I've proved it." Most people will not be entirely convinced, particularly with counterintuitive results, unless they have been shown the reasoning which supports the fact.

The NCTM also claims that teaching students to communicate in mathematical language will help them clarify their thinking and thus increase their understanding [14].

> The communication process...helps build meaning and permanence for ideas and makes them public. When students are challenged to think and reason about mathematics and to communi-

> cate the results of their thinking to others orally or in writing, the learn to be clear and convincing. ... Students who are involved in discussions in which they justify solutions—especially in the face of disagreement—will gain better mathematical understanding as they work to convince their peers about differing points of view. [p. 60]

In the section quoted above, the NCTM is not referring explicitly to proof, since mathematical communication can happen in many forms, but proof certainly falls under their discussion. On top of the reasons just given, the council also notes that "In order for a mathematical result to be recognized as correct, the proposed proof must be accepted by the community of professional mathematicians" [p. 61]. It is not likely, of course, that students in schools will be proving results of professional caliber. But as Schoenfeld explains [23], a classroom climate can be created so that the students are the mathematical authority. They can argue about problems and solutions, bringing reasons to bear on the problem, and accepting a proof only when they themselves are convinced by it. When proof is used in such a fashion, its value in the classroom is quite apparent: students learn to rely on arguments and reasoning rather than authority, they make use of their factual knowledge, and they come to a deeper understanding of the way mathematical facts are related.

There are other reasons for teaching proof that proponents cite. The last main one is that proof is part of mathematical or cultural tradition and students should be familiar with it for that reason. That proof is part of mathematical tradition is clear; what that has to do with teaching it in the classroom is less obvious. Latin was also a part of a traditional curriculum until this century, but it is rare to find a high school or even a college student who has any knowledge of Latin. In today's world of high-tech computers and calculators, perhaps one could argue that students' time is better spent not on doing proofs but on learning to use computers as tools for doing mathematics. A reply to this might be that this sort of argument ignores the fact that proof is what mathematicians *do* and ignoring proof is ignoring an essential mathematical activity. Thomas Tucker [25] and others note, however, that the vast majority of pre-college, and indeed most college students, have no pressing need for mathematics as mathematicians do it; what they need is the mathematical knowledge that they can use at the grocery store, at tax time, or in the workplace. Proof is of value only to a few. The NCTM's reply to this is probably (and my reply is) that *proofs* may or may not be of value only to a few, but *proving* is valuable apart from any value proofs themselves have. The next chapter is an elaboration of this point.

Let me reiterate that proof certainly *is* a vehicle for communicating mathematical results and convincing us of the truth of theorems, and that these are important reasons for retaining it as part of the mathematics curriculum. The problem is that proof is not commonly taught in such a way as to promote such engaging classroom discourse, and Tucker communicates doubts as to the value of teaching proof unless it is used carefully, and not simply as exercises as is often the case now [25]. Tucker observes that proponents claim that proof will help students understand and believe the theorems they are proving. But he argues that, for the most part, it is not the case that students understand any better, and are no more apt to believe a proposition after proving it than they were by reviewing particular examples of it. Tucker does not give detailed evidence for his claim, but it is an observation echoed often in the literature: the problem with proof is not that it is of little value, but that students are poorly trained to appreciate its value because it is an exercise they perceive to be separate from the activities they take math to be, namely learning facts. This point will be revisited when I come to the discussion of the study by Healy and Hoyles.

The point is that although the reasons I have surveyed here are good ones, when these views of proof are the main views of the teacher, and consequently the students, the value of proof can be lost on students. In the next chapter I shall put forth a new argument in favor of proof which is distinct from any proposed here, though it is certainly related, and which alleviates the danger of proof's being seen as a superfluous exercise which un-motivates and frustrates students. Before doing so, however, I present some empirical evidence to back up the discussion of this chapter.

2.3 A study of students' conceptions of proof

2.3.1 Description

Healy and Hoyles [8] conduct a study in which they investigate students' understanding of proofs in mathematics. The study has limited power for the current project, since it is based on a curriculum in the United Kingdom, but I include it because the results help to illustrate my present point. The study is partially a response to statutory curriculum changes in England and Wales, in which proof is to be taught in such a way that students test and refine their own conjectures "to gain personal conviction of their truth alongside the experience of presenting generalizations and evidence of their validity" [8, p. 397]. The questions guiding the study which are also of interest to the present project are the following:

- What did students judge to be the nature of mathematical proof? What did they see as its purposes? Did they see proving as verifying cases or as convincing and explaining?
- Were students competent at constructing or evaluating a mathematical proof?

The study was centered around 14–15-year-olds (the U.S. equivalent of ninth graders) who were high achievers (the top 20–25% of the population), because high achievers would have had the most exposure to proof in their careers so far. The aim of the investigation was to learn about the characteristics of arguments the students recognized as proofs, the reasons for their judgments, and their own constructions of proofs.

The main instruments of the study were a student proof-questionnaire and a school questionnaire. Items on the questionnaire were of three types:

1. Students were asked to describe what a proof is and what it's for.
2. They were presented with a mathematical conjecture and examples of "proofs" of it. From these they were to select first the one that would be closest to their own approach, and second the one they thought the teacher would give the best mark to.
3. Students were asked to assess the arguments in terms of their validity and their explanatory power.

Students' responses to the first question were coded according to three categories: truth (verification), explanation (illumination and communication), and discovery (discovery and systematization).

For the second type of question the students were presented with two different mathematical conjectures. Each conjecture was presented with answers falling under each of the following descriptions:

1. empirical or concrete, with little or no explanation

2. generic case or common properties of specific cases

3. everyday-language narrative argument suggesting underlying reasons

4. valid formal-style deductive proof (i.e. one relying on symbols almost entirely)

5. invalid formal-style deductive proof

After the students chose which of the answers would be most like their own and which the teacher would like best, they were asked to evaluate each of the choices for its correctness, generality, and explanatory power. Students were to indicate agreement, disagreement, or indecision regarding statements such as “Bonnie's argument [that the sum of two even numbers is even] has a mistake in it”, “Only shows it's true in some cases”, or “Shows why it's true”. Generality judgments were further assessed by presenting the conjecture as proved and asking students whether further proof was necessary for subcases—e.g. supposing that the sum of two even numbers is always even, is the sum of square even numbers also an even number?

Finally, students were asked to construct their own proofs of one familiar and one unfamiliar conjecture. They were instructed to construct the argument that would earn the best possible mark from the teacher. These proofs were scored from 0 to 3: 0 for no proof, 1 for relevant information but no deductions, 2 for partial proof, and 3 for a complete proof. The form of the argument (empirical, narrative, or formal) was also recorded.

The school questionnaire is less important for the present purposes, but one aspect worth noting is that teachers were also presented with the two conjectures and answers, and asked to choose first their own approach and second the one they expected their students would believe would get the best mark.

In addition to the questionnaire, the researchers selected a sample of teachers and students to interview in order to elaborate on some of the findings.

2.3.2 Results: Proof is for explanation

Students turn out to be significantly better at choosing a correct argument than at constructing one themselves. In fact, there was a strong inverse relationship between the arguments students chose as their own approach on the multiple-choice section and the one they expected to get the best score. Among the teachers, however, there was a strong positive correlation between the approaches they would choose themselves and the ones they expected the students to believe were best. That is, according to Healy and Hoyles, "students judged that their teachers would reward any argument provided it contained some "algebra"[1] whereas teachers presumed that the logic of the argument would also be important" [8, p. 407]. Students preferred narrative and empirical forms of proof for themselves. From the interviews, there is some indication that the reason for this was not that they thought the empirical arguments were good proofs, but that they had no better arguments at their disposal. The students generally felt that empirical arguments had no explanatory value, so we can see that they were sensitive to the role of examples in proofs: examples help you gain access to a problem and convince yourself of a conjecture's truth, but they do not verify that truth. Explanatory value also seems to play a role in why students did not select algebraic proofs as their own approach: they felt algebraic proofs were difficult to understand and therefore were of no help when it came to explaining a proof to someone else.

This finding is interesting in light of students' views of proof. A view of explanation (to be discussed below) plays a large role in students' preferences for their own proving styles, and yet more than half of them indicated that they saw verification as the purpose of proof. Explanation was the next largest category, with 35%. (Students who gave answers which indicated more than one purpose for proof were counted in each category, so it may be that many students recognized both roles for proof.) Given their responses as reported above, however, we would expect students to believe that explanation played an even stronger role than they apparently do believe. But judging from their interviews with students, the researchers indicate that the belief in proof's value as explanatory and as a vehicle for understanding may have been more widespread than their coding shows.

2.3.3 Toward "symbol sense"

The questions guiding the study are of interest to us because they have to do with the belief aspects of metacognition. As I discussed in the last

[1]Healy and Hoyles speak of algebra, but what they mean is probably something more like symbolic logic, judging from the fact that they emphasize the clear display of structure and the generality of the arguments they call "algebraic."

chapter, Schoenfeld thinks students' beliefs about the nature of their tasks and about what they know have some influence on how well students perform in problem solving. It is easy to imagine approaching a proof differently depending on whether you see your task primarily as simply solving, explaining/communicating, verifying, or convincing (which may be a combination of the others). The argument I want to make concerning the role of proof in the curriculum relies heavily on communicating a certain conception of proof to the students. The Healy and Hoyles study reveals that students are more drawn to proofs they see as "understandable," which here means general and explanatory. What do the students mean by "explanatory"? Note that they also showed preference for arguments presented in words rather than entirely symbolic proofs, again because these are easier to follow and are therefore generally more convincing. The students apparently don't call algebraic arguments explanatory because algebra doesn't seem to them to be tied to the concepts (such as evenness) the way a narrative argument is. For students without a lot of experience, it is not obvious that "$x = 2y$, for some y" says the same thing as "x is even." Variables are non-entities to them; you don't run across x's and y's in real life anywhere, so how can they help explain things? The representation of variables as standing for unknowns seems to make students think that variables hold no content.

It is difficult to tell whether these implicit views of the students are simply the result of an oversight in teaching or whether perhaps the students are not at a cognitive level at which they can grasp the more abstract mode of thought that using algebra (in combination with logic) entails. Either way, it seems that the students at the level under investigation have not yet had sufficient time and experience to develop what Abraham Arcavi [1] calls "symbol sense"—a facility with symbols akin to the skills with numbers that are often described as "number sense." Symbol sense includes a recognition of the power and generality of algebraic (and logical) symbolism, when the use of symbols is and is not appropriate, how the choice of representation may influence the perception of a problem (e.g. consider choosing your variable to be s versus $\frac{p}{q}$), and so on. If students have not yet achieved this symbol sense, it is quite likely that they will prefer a narrative exposition to an algebraic one. There are, of course, ways to combine the ease of understanding gained by narration with the generality of symbols, and this is something which can be taught. In any event, the study's answers to its initial questions indicate that more needs to be done to show the students the relationship between the use of symbols and understanding—that is, proof needs to be taught in conjunction with some elementary logic, not only as a method of verification, but as a method of communication, explanation, and understanding mathematics.

The disconnect between students' perception of algebraic arguments and teachers' understanding of them is interesting. Students seem to think that algebra gets them the best marks, but they do not use that belief to construct their proofs. Clearly, they are not aware of the generality and power of algebra: they have not yet come to see it as a shorthand for natural-language explanations. Teachers indicate that they prefer algebraic proofs simply because they are clear and uncomplicated, and not particularly because of their symbolic content. This view of proof does not come through to the students. Students' double-view of proof could be brought into better focus if teachers taught some basic logic as a way to explain why algebraic arguments are good—the idea is to get the students to see that because mathematical arguments often have a generalizable logical structure, algebra *does* have explanatory power, particularly if combined with a narration.

In a conclusion similar to the one I made above, the researchers attribute the students' fractured view of proof at least in part to the fact that proof in the new curriculum is not explicitly taught in conjunction with algebra or geometry, but rather in a separate activity called "investigations." These investigations are exercises in which students collect and tabulate real data, which is then examined for a pattern, and this pattern is explained and proved. Thus students' most common encounter with proof is tied closely to the empirical approach, and this may account for the students' propensity to use empirical examples in their proofs. Furthermore, little connection is made between proof and mathematical "subjects" such as algebra and geometry; proof is a separate topic.

In addition to this, few students (only 1%) indicated discovery as one of the purposes of proof. I claim that this and the misconception enumerated in the previous paragraph can be changed by teaching students to "prove as they solve"—that is, the metacognitive questioning employed by Schoenfeld and by Mevarech, Kramarski and Lieberman will show students that not only does proof verify and explain, but it is also a method by which discoveries can be made. Problem solving by aiming to prove will promote the kind of metacognition that is so crucial to successful problem solving; it will provide the organizational framework which keeps students linking concepts and techniques they have learned and therefore shows them why their knowledge is useful. Students who are taught to solve problems with the aim of producing a proof that their solution is correct will tend to be better at keeping their goals and reasons for trying certain techniques and procedures clearly in mind. Then proving is not an extra thing over and above solving a problem, but is rather an integral part of the problem-solving process, and hence an integral part of *understanding* mathematics.

Chapter 3

Proof and problem solving

3.1 Proof as a method of promoting good problem solving

In this chapter I argue that proof and problem solving are in fact very closely linked. In the discussion of metacognition and its relation to performance in solving problems, we saw that Schoenfeld asks three questions of his novice problem solving students [21]:

1. What exactly are you doing?

2. Why are you doing it?

3. How does it help you?

These questions were designed to get students to think carefully about the architecture of their solution methods. Keeping the questions in mind helps prevent students from heading off on wild goose chases, curtailing unfruitful investigations before they have invested too much time and energy in pursuing them.

Now reflect on what a proof does. Earlier I discussed conventional classroom justifications and presentations of proving; these included communication and justification. Compare Schoenfeld's questions with the questions answered (usually implicitly) by a proof:

1. What did you do?

2. How or why does it show what you set out to show?

A good proof lays out cleanly what the prover did to accomplish her task and makes clear the reasoning by which each step is linked to its predecessor, by drawing on already-established principles and facts. That is, a proof demonstrates both *that* and *why* a theorem or proposition is true.[1] Proof does in retrospect what Schoenfeld's (metacognitive) questions are designed to do in prospect: namely, draw attention to a chain of relationships among mathematical facts. The proof may be carried out purely symbolically, but as we learned from Healy and Hoyles [8], proofs which at least incorporate natural language are much more effective from the students' point of view. In any case, a proof is an argument which makes use of the regimented language which gives mathematics its power to demonstrate and convince in an unambiguous fashion.[2] A good proof does so without mentioning any information which does no work in the argument. From now on, by "proof" I shall mean *good* proof unless otherwise specified.

"The" proof, by which I mean the narrative paragraph or the sequence of lines of symbols, is really the last step in a proof *process.* This conception of proving is too often overlooked in school mathematics, and I claim that this is because proof is taught as an activity separate from solving problems. In practice, arriving at a proof involves taking the kinds of steps one takes to solve other problems in which a proof is not demanded. Effective problem solving requires the solver to keep cognizant of the facts and relationships upon which he is drawing; at the same time, one of the crucial features of a proof is its demonstration of the relationships among mathematical facts and properties. A proof shows how one fact follows from another, and this is how it demonstrates truth and explains why something is true. But these relationships are not something we realize only at the end of a problem, as current teaching practice often leads students to believe. They are the very relationships which help us arrive at a solution in the first place. Therefore proving is an integral part of solving problems and should be taught as such.

My contention is that "the" proof is a summary of the successful part of the reasoning process. It takes the dead ends out, gets rid of the "fuzz,"

[1]It is a bit misleading to claim that any proof demonstrates *why* the theorem is true; philosophers of mathematics are still trying to analyze the notion of mathematical explanation. Particularly problematic is indirect proof, where it is not clear that such proofs show why something is true. All I mean here is that a proof shows why a theorem is true by presenting the mathematical reasoning which demonstrates its truth.

[2]Unambiguous at least at the level of proof we're talking about here; there is some discussion as to whether proofs are unambiguous over time.

picks out the facts on which the proposition to be proved rests, and shows how they are linked together. Compare a finished proof to the transcript of the protocol that produced it. The proof is much more readable; but aside from that, it often orders and presents facts very differently from the order in which they were thought of and used. "The" proof, however, is only possible after a course of *proving* has taken place. Proofs don't just spring fully formed from a mathematician's or student's head. Proving includes working toward a solution—problem-solving—and then presenting that solution in a clean manner, discarding dead ends. Viewed like this, it can be seen as a method of discovery as well as explanation and verification. That is, new relationships are discovered as old ones are exploited and built upon in the process of solving a problem.

As I said above, good proofs identify the successful steps in the problem solving process—that is, good proofs are based on the answers to questions someone exhibiting good metacognitive control would be able to answer if you stopped him at any point as he works on his solution to the problem. There is a difference between the problem solving process and the actual production of "the" proof which results from it, but a metacognitively guided problem solving process should lead easily to a proof, whereas a less controlled process will require further pruning and tightening, and may even produce a bad proof.

3.1.1 Examples

To better illustrate the claim that good proofs are metacognitively controlled ones, I now present a pair of proofs, good and bad respectively, and discuss the metacognitive steps these proofs might have come from.

A good proof

The first of these is taken from the "expert" protocol Schoenfeld discusses in *Mathematical Problem Solving* [20]. He is to solve the following problem:

> You are given a fixed triangle T with base B. Show that it is always possible to construct, with a straightedge and compass, a straight line that is parallel to B and that divides T into two parts of equal area.

For my current purposes, it isn't necessary to summarize the expert's problem solving session here. In his review of the protocol, Schoenfeld points out that the subject, GP, displays strongly goal-directed behavior in his solution and notes that "it is quite clear from the transcript that he did not know specifically what he was looking for as he worked through the problem." Yet a proof that the construction always works is easily obtainable from GP's solution. GP knows this and does not present a tidy proof. If he had, it might look like this (the sentences are numbered to facilitate analysis):

> (0) Let a and A denote the altitudes of the small and large triangles, respectively, and likewise b and B denote their respective bases. (1) Since a triangle's area is $\frac{1}{2}bh$, our aim is to find the location of the base of the small triangle such that $\frac{1}{2}ab = \frac{1}{2}AB - \frac{1}{2}ab$, i.e. $ab = \frac{1}{2}AB$. (2) Because triangles sharing an angle and having parallel bases are similar, and similar triangles have equal proportions, the ratio of the altitudes of the two triangles is equal to the ratio of their bases. (3) That is, $\frac{a}{A} = \frac{b}{B}$. (4) Using this proportionality along with the fact that we want $ab = \frac{1}{2}AB$, it is clear that we need to construct $A/\sqrt{2}$ and $B/\sqrt{2}$. (5) Now, since $1/\sqrt{2} = \sqrt{2}/2$, we can construct $A/\sqrt{2}$ by constructing an isosceles right triangle with legs of length A and bisecting its hypotenuse. (6) We then mark off this length on the altitude of the original triangle and construct the segment parallel to B through the point thereby determined. (7) This yields a triangle whose area is $\frac{1}{2}ab = \frac{1}{2} \cdot \frac{A}{\sqrt{2}} \cdot \frac{B}{\sqrt{2}}$, or $\frac{1}{4}AB$, as desired.

First, note that the proof begins by explicitly restating the goal of the problem; this shows that the solver has a clear idea of what his aim is. Sentences (2) and (3) note a crucial feature of the problem that will be necessary to exploit in order to find the solution. If stopped at this point, the solver might ask himself "What am I doing?" and answer, "Looking for important relationships I can exploit." These important relationships then set up sentence (4) as the answer to a the question "How does this relationship help me toward my goal?" Sentence (5) is again an observation of some mathematical facts, which may be known already or could be derived by asking oneself what in geometry might yield square roots and answering that a square root can be obtained by using triangles and the Pythagorean theorem. Then sentence (6) is an answer to the question "What do I do with this relationship?", and sentence (7) displays the results. Every sentence in the proof plays a role by highlighting relevant information or setting up a subgoal.

A bad proof

To provide a contrast, I have made up a bad proof based on the problem mentioned in Chapter 1 in which the task is as follows:

> Given lines l and m intersecting at point O and a point P on l, construct a circle tangent to both l and m passing through P.

A common first thought is that the diameter of the desired circle is the line segment from P to its symmetrical reflection Q on m. Even a careful enough rough sketch, however, will show that the circle with diameter PQ is not tangent to either line. Nevertheless, a solver might make the observation that PQ must be a chord of the circle and try to use this piece of information to produce the following proof:

> (0) Let Q be the symmetrical reflection of P onto m. (1) Note that the segment PQ cannot be the diameter of the desired circle, because the radii of a circle are perpendicular to tangents at the point of tangency, and angles QPO and PQO are not right angles, given that l and m intersect. (2) Segment PQ is, nevertheless, a chord of our circle. (3) Any chord forms an isosceles triangle with two radii of the circle. (4) Now, the altitude of such a triangle bisects the base, which in this case is the chord PQ. (5) Therefore, the center of the circle lies on the perpendicular bisector of PQ, which by symmetry is also the bisector of angle POQ. (6) To find out where, construct a perpendicular to l through P, and where it meets the bisector is the center of the circle, and we are done.

First, note that the goal, which turns out to be finding the center of the circle, is never explicitly stated; the initial thought about the PQ being the diameter preempts this, and so the global question "What am I doing?" is never explicitly answered. Now, since PQ is not the diameter, this line of reasoning seems to get the solver nowhere, but the realization that PQ must be a chord sidetracks her. It's likely that, were we to interrupt her at this point and ask what she was doing, she could tell us—but if we were to ask what the investigation was supposed to get her, she might have more difficulty. Nevertheless, in this case she is able to make use of the chord idea with the realization that it forms part of an isosceles triangle, and she can exploit symmetry to find the center of the circle. She does make the observation that the perpendicular bisector of PQ also bisects angle POQ, but she does not take the further step to realize that this observation could have provided a more efficient route to the proof—i.e. she doesn't notice that PQ is not doing any particular work for her. Neither, we should note, is the mini-proof that PQ can't be a diameter of the circle.

The lesson of the examples: Good metacognition, good proof

There is nothing wrong, of course, with generating ideas that turn out to be dead ends, nor even in playing around with ideas in order to see if they could lead somewhere. Experts certainly do that in their day to day work. But an experienced problem solver will check for other possible routes before choosing one she is not sure will lead anywhere, and she will certainly not leave the remains of those dead ends in the proof (at least in the fairly simple cases presented here). When pursuing an idea whose usefulness isn't clear, however, someone with good metacognitive skills should be able to say, "I don't know if this route is the right one, but I'm trying it because I think it will get me to *this*, which would contribute to the solution in *that* way…."

Thus, good metacognitive control does not necessarily prevent unproductive routes from being followed in the process of solving the problem; but as Schoenfeld notes, someone who is in control of that process will usually recognize (at least in retrospect), as GP does during his protocol, when he has done something silly or inefficient. The inefficiency is omitted from the final proof; a good finished proof is a presentation of all and only the relevant steps in the solution process, perhaps rearranged to make the most logical sense. Each of these steps is relevant because it is contributing to a solution. Metacognition is important in problem solving because it ensures that the solver is aware of how his activity is contributing to his ultimate goal. Therefore, teaching students to produce good proofs ought to improve their awareness of how their activities contributed to the solution they produced, and next time they're solving a problem, they should bring that awareness forward so that it contributes to an efficient problem-solving process from which a proof is more easily extracted.

3.1.2 Proof makes reasoning salient

It is certainly not true that proving is the *only* way to discover connections between old and new facts; sometimes simply playing around with transformations and variations on the same problem, without any thought of the principles which might undergird a problem's solution, will be a fruitful way to discover important mathematical relationships. My claim is that aiming at proofs is an efficient way of learning because of its link to metacognition. But *is* producing a proof always conducive to learning? In what sense is proving (for example) that the sum of two even numbers is always even a *problem*? That is, some propositions are so obvious that there is no "solving" going on in proving them—it's only a matter of unwinding definitions. On the other hand, there seem to be problems which are only problems, and don't require any proof. How is proof involved in practical problems such as deciding which pizza company to order from, given a certain budget?

The first question is fairly easy to answer. For so-called "obvious" propositions, the problem is simply to prove: to uncover and then present the facts which undergird even simple propositions. And by proving simple mathematical truths, we often learn to use relationships that might be useful later (e.g. that an even number can be represented as $2k$, where $k \in \mathbb{N}$).

The question regarding practical problems, however, is less transparent. The difficulty is that, apparently, all that is required is an answer; proof is really superfluous. But even an answer can require explanation and argument, which will involve drawing on mathematical facts. Not only that; the process of solving the problem also requires exploiting mathematical relationships. For example, take Kramarski, Mevarech and Lieberman's pizza task [9]: Students are given a certain budget for a class pizza party, and they gather information from a few pizza parlors on their sizes, prices and toppings. They are to determine the best use of the money (i.e., the goal is to buy the greatest amount of pizza they can with that budget) and then make an argument to the class treasurer that their plan is the best. In order to complete this task, they must make use of the relationship between the diameter or circumference of a circle and the circle's area, price per unit of area, and so on. They must do so in a reasoned way, so that they can present an argument—"the" proof—to the treasurer.

I want to bring out two aspects of this task. First, the students are required to provide their reasoning as part of the solution. This can be done with any practical problem: not only must you find the answer (a length, a time, a construction, etc.) but you must convince me that your answer is correct by explaining the facts on which you relied to produce the solution and why these facts support your conclusion. Thus, problem solving and proving can be linked even in problems which are, on the surface, only "practical."

Second, because they are asked to provide their reasoning, the students are more apt to make that reasoning explicit as they go, exploiting relationships they know they can defend. This is the lesson of Schoenfeld's classrooms. The research on metacognition indicates that explaining their reasoning at each step of the problem-solving process keeps students aware of why they are doing what they are doing. Perhaps they will even plan carefully before they begin following leads. In other words, when required to provide their reasoning students will need to be aware of their thought processes, or they will not be able reproduce them. That is, teaching students to write good proofs promotes metacognition and hence aids students in solving problems.

3.2 The relationship of proving and problem solving

My view of proof as a method of discovery forges a tight link between proving and solving problems, but it does not collapse them entirely. My view is that proving encompasses problem solving, and ought to be taught as such, but proving requires more than just arriving at a solution.

First, note that a proof and the solution to a problem can still be separate things. In the pizza task, the solution to the problem is simply the statement that the class should order their pizza from Company A. The *proof* of that statement shows why. But this need not be the case; with "obvious" propositions as mentioned above, producing a proof simply *is* the solution to the proposed problem.

A lot hinges on the way "problem solving" and "proving" are used. Much of the time, what teachers mean by "problem solving" is anything that is not simply drill practice—in other words, story problems. These "problems" set up a little scenario and then ask a question about it, e.g., "Mary and Susie have fourteen stickers total. Eight of them are Mary's. How many are Susie's?" The students are expected to extract an equation from the problem statement and then solve it for the appropriate quantity. Such problems are supposed to relate drills to real life. But because they are so closely related to drills, they quickly become routine and are not in fact *problems* at all, for they neither arise from nor arouse students' curiosities. As Leone Burton [2] puts it, students end up with answers to questions they don't possess. Because arriving at an answer is the perceived goal of the task, students learn to proceed by rules, and do not stop to think about the reasons behind the rules. According to Magdalene Lampert, "there is a tendency to use rules as reasons for action, without recognizing that using a rule is different from explaining why the rule works or why it is legitimate to use it in a particular case" [10, p. 56]. For these story problems, it is legitimate to use the technique at hand, whether it is elementary arithmetic or a more advanced algebraic or geometric technique, usually because that is what the students have been studying. Thus problem solving as it is often taught does not demand of students that they think creatively and forge links between units.

Proving, likewise, is traditionally a separate activity with its own unit or context. Think about it: geometry is difficult because you have to prove things in that subject. You don't have to do it anywhere else until you finish college-level calculus, and then only if you are taking upper-level math courses. Then, suddenly, proving things is almost all you do. Only when a student is on the way to becoming a professional mathematician does the truth come out: proof *is*, in large part, what mathematics is about.

So I, along with many others, am saying first of all that the notions of proving and problem solving as they are usually used in classroom mathematics need to be re-conceptualized so that they are not separated, use-starved activities. But this is only a matter of revising classroom practice.

In a more theoretical realm, proving and problem solving do seem to be at least conceptually separable activities. In problem solving, we might say that there is a question to be answered. This could be "What is the volume of the solid which. . . ?", demanding some concrete quantity as an answer, or it could be "Is it true that every. . . has the property that. . . ?", demanding a negative or affirmative answer. A solution can be given without actually providing a proof. In proving, however, we invoke the apparatus which allows us to answer these questions. A proof cannot be given unless a problem is solved, and in this way proof is something beyond just problem solving.

To what extent, however, *are* there two different activities that mathematicians do, one called "proof," and one called "solving problems"? It seems easy at first to grant that writing a finished proof—"the" proof, to put it as I did above—is not the same thing as problem solving; any writer will tell you that there is still a great deal of work to be done even after all the research and planning—the problem solving—for an essay or book is finished. The actual writing involves not only filling in details, but also a great deal of revision. Likewise, once a solver has arrived at a solution, there is work to do in presenting the proof: explicitly stating the reasoning and cutting out the unnecessary and unhelpful ideas generated during the solving process. Nevertheless, answering the question the problem poses and showing why that answer is correct are an interwoven process, and so in practice proving and problem solving probably cannot be disentangled, though we may still speak of them as separate things.

3.3 Proof, problem solving, and "understanding"

In the last section I argued that proof and problem solving are closely linked, because the questions answered by a proof are nearly the same as those used to promote metacognition, which has been shown to be a factor in students' success at solving problems. Now I turn to a discussion of why this link is important in making the case for the use of proof in school mathematics.

In my introduction, I provided an overview of the vision the National Council of Teachers of Mathematics puts forward in their *Standards* documents. Their aim is to create a mathematically literate population, who learns mathematics with understanding. "Understanding," to the NCTM, is a broad notion encompassing many different facets of learning mathematics. One especially important component of the council's notion of understanding, and the one on which I will focus here, is problem-solving ability. In particular, the claim is that an ability to solve problems is integral to a student's understanding of mathematics. To see why this is so, I canvass the NCTM's notion of understanding and show how my discussion up to now is related to it.

Let me first provide a bit of background. After the unsuccessful "New Math" and "Back to Basics" trends in mathematics education, a new vision of mathematics was needed. In 1989 the National Council of Teachers of Mathematics came out with that vision in its *Curriculum and Evaluation Standards* [13]. The centerpiece of this vision of mathematics is a distinction between "knowing" and "doing" mathematics: "knowing" math, according to the NCTM, is just collecting a set of mathematical tools and facts in a propositional sort of way, whereas "doing" mathematics involves using those skills to "think mathematically." Thinking mathematically involves making connections among mathematical facts as well as between those facts and everyday life, communicating in mathematical language, and solving problems. The NCTM's ambitious goal was at that time, and still is, mathematical literacy for all students, and in addition to the methodological goals already mentioned, the 1989 *Standards* contained content specifications as well. The 2000 version of the Standards, *Principles and Standards for School Mathematics* [14], retains and updates this vision of mathematics, and is the version on which my discussion will be based.

3.3.1 Problems vs. exercises

The first point is that mathematical understanding is best gained by the use of problems which are *problems* and not just exercises. A problem, according to Schoenfeld [20] and John Hayes [7], is a situation which raises a question for the student and for which the path to the answer is not known in advance. It is something non-routine. Thus activities such as elementary "story problems" in which two children have apples or baseballs and students are asked to find the total or the difference are generally not problems, but exercises. Note that activities which start as problems can quickly become exercises as students gain familiarity with a routine for solving the problems. After a number of apple or baseball problems, students have learned the algorithm for solving the "problem," and it becomes an exercise.

Exercises typically do not promote the kind of understanding the NCTM desires. Once a scheme is picked up, a student no longer needs to engage metacognitively with the exercises, but simply executes them with little thought. This can cause some problems, as for example in the famous case Schoenfeld describes [21] (and which I mentioned in Chapter 1) in which students had routinized their division story problems to the extent that they answered the question of how many buses were needed to transport a group of soldiers as "31 remainder 12," without thinking about the fact that remainders make no sense in this context. You can't (generally!) drive the remainder of a bus, and if they had done a similar problem out on the playground, Schoenfeld speculates, no one would have made such a silly mistake.

So in the classroom teachers need to place emphasis on problems which challenge the students to use what they already know in novel ways. "Problem solving means engaging in a task for which the solution method is not known in advance," according to the NCTM. "In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so" [14, p. 52]. In this claim the NCTM is backed up by research such as Schoenfeld's ([20], [21]) and Kramarski, Mevarech and Lieberman's [9].

3.3.2 Playing the game

Nobody, it is surely safe to say, likes math drills. They seem pointless—and they probably are, if students are never given an opportunity to draw on the facts they're memorizing. Drills of math facts are as pointless as practicing scales if you never get to learn a piece of music, or practicing jump shots if you never get to play in a basketball game, as Thomas Romberg puts it [17]. Problem solving is a way to alleviate this tedium. Depending on the way a teacher wants to approach it, problems can provide a rationale for skills practice before drilling, as they did in a lesson described by Magdalene Lampert [10]. Lampert was introducing her fifth graders to exponents. She began the unit by having the students explore patterns in square numbers, and they made and proved the conjecture that the last digits of the square numbers follow a pattern of 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 over and over. From here she challenged them with questions concerning fourth powers, and the discussion there led them to generalizations about how exponents work and how exponentiation is related to multiplication and addition. She got the students to develop strategies to assert answers to questions about last digits without consulting a calculator. "The activity of developing such strategies," she claims, "engages students in clarifying the distinction between exponentiation and multiplication and leads to evidence that supports the mathematically legitimate shortcut of finding products by adding exponents" [10, p. 46]. Having developed the general motivation behind the algorithms, specific drill practice makes much more sense.

Problems can also provide the arena in which students can use their drilled knowledge to play around, discovering patterns which lead them to ask questions, make conjectures, and prove those conjectures. Deborah Schifter [18] describes a second-grade teacher's experience with students noticing what they called "turn-arounds", e.g. that $6 + 4 = 4 + 6$, what we would call the commutative property of addition. Their basic addition facts showed them that the turn-arounds worked for the small numbers the students worked with all the time, so the teacher decided that a lesson on commutativity was appropriate. Not all of the students were convinced that turn-arounds always worked, so the teacher had students experiment with larger numbers. This empirical work seemed to convince some, but others were still unsatisfied. These students developed other methods of checking the conjecture, such as arranging groups of counters. One student finally realized that any way you divide up 105 blocks into two groups to add them, there are always 105 blocks. From there, another student generalized away from 105 to any number, since "[s]he will always get the same answer because she is always starting with the same number of cubes" [18, p. 78]. These students took the specific knowledge they had and generalized it.

Such activities probably do not make drilling any more interesting, but it is hoped that students will see that without their basic skills, they would not be able to do the more interesting problem-solving activities. Drilling makes sense when there is a context in which it is useful to have a large store of facts from which to draw.

The NCTM designed their Standards in large part to solve the problem of students' disjointed knowledge of mathematics. They blame this disjointedness largely on an overemphasis on basic skills drilling, which, they claim, results in students who have many facts at hand but know nothing about how to use them or why it is important to know them. Such deficiencies in knowledge constitute a lack of understanding. To the NCTM, it is in the *use* of mathematical knowledge that understanding lies. Using drilled facts to arrive at and solve problems takes the activity of collecting and memorizing a bunch of superficially related facts and turns it into learning the specifics of some general properties. By "problems" here I don't necessarily mean the sort of problem I referred to earlier as "practical" and "story" problems. A problem can be a question which arises naturally as in the exponential and commutative investigations above. Investigating such problems turns the lists of exponential or logarithmic properties we all have to memorize in precalculus courses into a well-motivated and interconnected body of knowledge. Thus, problem solving fulfills one of the NCTM's goals for understanding.

3.3.3 Using knowledge

According to the NCTM, "Effective problem solvers constantly monitor and adjust what they are doing. They make sure they understand the problem. ... Research indicates that students' problem-solving failures are often due not to a lack of mathematical knowledge but to the ineffective use of what they do know" [14, p. 54]. Schoenfeld's work, as I discussed earlier, provides vivid evidence of this assertion.

As Schoenfeld shows (for example, with the students who could prove that the center of the circle tangent to two intersecting lines lies on the angle bisector but could not use this proof in a related construction— see Chapter 1), it is often the case that students who do have the mathematical skills and knowledge required to solve certain problems are often unable to use them effectively. Schoenfeld teaches heuristics to use to understand and explore the problems at hand, such as trying simple cases and looking for patterns. Such strategies are designed to get students to make conjectures on how to solve the problem, which they will then have to carry out, and even conjectures which extend the problem's main idea. For instance, Schoenfeld describes an example in which the given problem deals with Pythagorean triples [19]. The exploring the students did to understand that problem led them to discover and prove for themselves much more than what was required to solve the problem at hand. They found patterns they thought would be useful in their task, but in order to use them they needed to confirm that they did hold in all cases. This kind of exploration and proving on carefully chosen problems teaches students to loosen up their thinking and draw on many different resources—not just the ones immediately studied.

During the problem-solving process, Schoenfeld thinks explanation is important. In his protocols, he does not, on the surface, take "explanation" to mean the explanation of a result in terms of mathematical principles and facts; that is, he is not looking for an answer to a question such as "Why must the center of the circle lie on the bisector of the angle?" Rather, he is probing students' ability to explain their activities and use those explanations to direct further activity. We might ask, "Why are you trying to construct the angle bisector to find the center of the circle?" or "What would constructing the bisector get you?" But this sort of process explanation does involve mathematical principles; students who are explaining why they are investigating certain relationships will have to give reasons such as, "Well, the center of the circle is going to be halfway between the two lines, we think, since the radius is constant. An angle bisector is always the same distance from the two rays that make up the angle. So if we have the angle bisector, we just have to find the place where the perpendicular from P intersects it, and that should be the center of our circle." They may not always be correct, but if they are thinking on this level, it is likely they will soon discover what's wrong with their scheme and either abandon or revise it. What they are doing by answering the process question is linking the process of solving the problem to the principles which underwrite a solution's correctness. Showing after the fact that the solution (in this case a construction) is correct is very closely linked to justifying the steps involved in doing the construction in the first place.

This on-line justification will require students to draw on their background knowledge and use it to make connections. By doing so they build new knowledge, namely the things they prove, directly and immediately upon old knowledge. In this way, proving becomes not only a method of justification, and even more than a method of explanation; it is also a method of discovery, as I have said before. The proof written at the end is the clean version of all the thinking that has taken place. It skips over the false starts and dead ends and presents an organization of a thought process that didn't necessarily—in fact, rarely ever, if the problem is a genuine problem and not an exercise—proceed in the order presented. But it is not this clean version which really requires the metacognition instrumental to solving the problem. It is the "proving-as-you-solve" which does this. When the problem has been solved by aiming at a proof, the final proof will be fairly simple.

Note Schoenfeld's remark that knowing a lot of mathematics doesn't do the students much good when their beliefs keep them from using it. Neither Schoenfeld nor anyone else would (or could!) go as far as to say that the development of such control is more important than mathematical content, but there is a sense in which such a statement would fit the NCTM's vision of mathematical understanding. The kind of "understanding" ultimately sought in a mathematics classroom is very much bound up with *using* what the students know, and not just knowing it. Good control (demonstrable by skill in and the resulting success at problem solving) is evidence that a student can make use of all of the mathematical information he has acquired—can exploit the connections among concepts and procedures. And the NCTM takes this as a sign of understanding.

3.3.4 A picture of mathematics

The NCTM's vision for school mathematics includes a desire that the mathematics students learn resemble more closely the way professional mathematicians think of and do mathematics. This aspect of the vision is more prominent in the 1989 Standards than it is in the 2000 revision, but it is not ignored in the later version. The view is echoed in the literature (see, for example, Jerry Uhl and William Davis' "Is the Mathematics We Do the Mathematics We Teach?" [26], as well as Romberg [17], and Schoenfeld [23], [22]). The general idea is that mathematicians find math interesting and students typically do not. The remedy is supposed to be that teaching math in ways that reflect the processes mathematicians actually use will be more effective in engaging students and getting them to understand mathematics for what it is (i.e. a connected body of knowledge, rather than a collection of loosely related facts).

We know that mathematicians in large part spend their time proving theorems, and I have been arguing that teaching students mathematics by teaching them to prove will improve their problem solving abilities and engage them in their learning better along the way. What does this have to do with the NCTM's Standards and their notion of understanding? As Yehuda Rav argues in his article "Why Do We Prove Theorems?" [16], proofs rather than theorems are the bearers of mathematical knowledge. His thesis is that "the essence of mathematics resides in inventing methods, tools, strategies and concepts for *solving problems*" (emphasis his), and that "[p]roofs... are the heart of mathematics, the royal road to creating analytic tools and catalyzing growth" [p. 6]. To support his claim he cites examples of un- or recently-resolved problems such as the Goldbach conjecture, the continuum hypothesis, and Fermat's last theorem. Each of these problems has generated or been attacked using an enormous amount of mathematical apparatus, and this apparatus has been useful in many other areas of mathematics.

If Rav is right, the lesson we can draw from his thesis is that students who are taught mathematics through the method of problem solving and proving will have a greater amount of truly mathematical knowledge than will students who memorize and apply facts and theorems do. "Theorems," Rav says, "are the headlines, proofs are the inside story" [p. 22]. The commutativity of addition example discussed above shows students inventing their own concepts which they see as relevant to their task, and this is doing mathematics if anything is.

3.3.5 Summary

Over the course of this thesis I have discussed the importance of problem solving for understanding, because in solving problems one must forge and make use of connections among known mathematical facts. I have stressed the importance of metacognition for problem solving, and hence for understanding. I have also built a case for the importance of proof in promoting metacognition. Putting all of these elements together, the overall argument is essentially this:

- If you can prove,
- then you can explain,
- which means you have active metacognitive processes,
- so you can solve problems,
- and therefore you understand mathematics.

Chapter 4

Conclusion

4.1 Teaching proof

How should we teach proof, then, if the above description of proof is the appropriate one for school mathematics? The key is that proving is not an activity that is separate from the rest of mathematics, as another topic in a math book—and it is most certainly not related only to geometry. Students of all ages can be asked to give reasons for the things they say. Second graders can argue and prove (probably quite informally) that $6+3=3+6$ and extend that argument to the general $a+b=b+a$. Writing a paragraph which is "the" proof need not be introduced until later, perhaps eighth or ninth grade. If this activity of giving reasons and letting the search for reasons guide mathematical problem solving has been used throughout their education, then learning to write the proofs themselves will be an easily-motivated activity and students will see the relevance, importance, and—perhaps most importantly—*usefulness* of proof in mathematics.

And what is that usefulness? As mentioned in the last chapter, Yehuda Rav argues that proofs are the very vehicles of mathematical knowledge: theorems tell us very little, and even if we had an oracle who would tell us the truth or falsity of any mathematical proposition we could conjecture, mathematics would still continue much as it does today. Teaching proof as an integral mathematical activity would show students the interconnected nature of mathematics. It requires them to uncover the underlying support for every assertion they make, which in turn should give them the view that what matters in mathematics is not the individual facts, but the structure of how those facts fit together and make up a coherent body of knowledge. People like Schoenfeld and Kramarski, Mevarech and Lieberman are already working on teaching mathematics this way.

4.2 For further investigation

In this thesis I have put forward the hypothesis that proof is valuable in the school curriculum because of its instrumental role in the cognitive processes which constitute "learning with understanding"—that is, proof will help increase problem-solving skills. This role is not one that supersedes, but rather supplements, the traditional arguments for teaching proof. The evidence I present here draws on those traditional arguments, such as proof's verification and communication of mathematical results, as well as evidence from cognitive psychology concerning the role of metacognition in learning. The picture of proof that emerges emphasizes a role in mathematical discovery which mathematicians have noted but which is overlooked in educational literature. The strength of this argument is that it provides students with motivation for the mathematical activities of proof and problem solving, which Thomas Tucker considers to be lacking in other arguments for the use of proof.

This body of evidence is only the beginning, however. All I have done here is argue for a connection between proof and problem solving by clarifying what educational researchers mean by understanding, metacognition, and proof. It is taken for granted that problem solving is an important way to learn to understand mathematics, and I have pointed out that students will learn to solve problems more effectively when they are taught to prove as they do so. The metacognitive difference between knowing *that* and knowing *why* touches on some epistemological issues as well; knowledge *why* seems "deeper" or more significant than knowledge *that*. It will take more work to say with precision what the epistemological difference is, because (among other things) there is the obvious complication that not all proofs seem to explain *why* a result is true; reductio proofs and perhaps also "methodologically impure" proofs seem to tell us no more than *that* something is true. Despite these obvious difficulties, the present thesis indicates that there is something to be said here regarding the old problem of how and why some proofs seem to be more explanatory than others: namely, that the explanatory proofs are those which display the cognitive and metacognitive steps of the problem-solving process. Besides this philosophical work, there is also empirical work to be done. A study based on my arguments here could be conducted to confirm whether proof is, in practice, a viable way of improving problem-solving skill and hence mathematical understanding.

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