# From Reducibility to Extensionality
## The two editions of *Principia Mathematica*

Jessi Berkelhammer

September 26, 2003

# Contents

# Overview

Bertrand Russell's original goal in writing *Principia Mathematica* was to show that mathematics could be reduced to logic. He aimed to present a general theory of logic which captured all of mathematics. In this paper, we will explain Russell's system and consider how it progressed through the two editions of *Principia Mathematica*. We will also look at Russell's later responses to Frank Ramsey's criticism.

To complete this logicist goal of showing that all mathematics follows from logical premisses and all mathematical concepts are definable in logical terms alone, Russell had to both characterize a logical system and interpret mathematics in these terms. In the first edition of *Principia Mathematica*, Russell aimed to present a general logical system. In order for it to be general, his logical system needed to account for all functions, including intensional ones. Two propositional functions are said to be co-extensive if their values have the same truth value for any argument. In particular, two propositions are co-extensive if they have the same truth value. For example, Russell believed that "The sky is blue" and "Clouds are white" are different propositions, but they are co-extensive. Extensional propositional functions are those that have co-extensive values given co-extensive arguments. An intensional function is one which is not extensional: such a function may have different values given co-extensive arguments. "Bertrand believes that $P$" is an example of an intensional function. Russell interpreted mathematics, which is extensional, within his intensional logic by showing how one can interpret extensional classes in terms of intensional propositional functions. In a system which allows intensional functions, propositional functions can take on different values given co-extensive arguments; a consequence of this is the extensions of such functions depend on more than the extensions of their arguments.

In order to avoid paradoxes arising from "vicious circles," Russell devised a ramified theory of types. His ramified theory of types took account of the quantified variables in a function's definition. In such a theory of types, one could not quantify over all propositional functions which some object satisfies, or makes true. This quantification is necessary for mathematics, and he introduced the axiom of reducibility in order to allow such quantification indirectly.

Russell and others thought the assumptions of this system, most notably the axiom of reducibility, compromised its logical validity. For the second edition of *Principia Mathematica*, Russell minimized his reliance upon the axiom of reducibility. However, in order to do so he had to reduce the scope of the system by limiting the role of intensional phenomena. In this edition, he only allowed that propositions be arguments to truth functions, or those whose truth value depends only on the truth value of its arguments. More generally, it is only a function's extension which plays a role in determining the extension of any function of that function. Russell did not fully endorse an extensional view, for he retained his ramified theory of types, which distinguishes functions based on their order, even when they are extensionally equivalent. Although the logical nature of this system was less doubtful, it was insufficient as a basis for mathematics. Russell did not provide an alternative powerful enough to allow one to generalize over functions satisfied by some object, or made true when given the object as argument, within a ramified theory of types.

In both editions, Russell did not clearly distinguish his system's syntax from its semantics. Instead of specifying its syntax, he presented the semantic domain of individuals, propositions, and propositional functions, and then referred to such objects as if they were syntactical objects. Ramsey, a critic of Russell's system, more clearly distinguished the syntax and semantics of his system. The referents of the symbols in his language are individuals, propositions, and propositional functions. Ramsey's variables range over propositions themselves, not linguistic representations of them, and he saw no need to introduce a ramified type theory. Ramsey endorsed an extensional view: objects, such as propositional functions, are defined based on their values, and intensional distinctions between them which are not reflected in different values are of no consequence to his logical theory. This means that co-extensive propositional functions with different representations are nonetheless the same function.

Russell was not satisfied with the second edition, and later he endorsed an extensional view in which logic need not account for intensional phenomena. Although he disagreed with Ramsey on certain points—most notably identity—he accepted Ramsey's criticism that the stratifications of Russell's theory of types were unnecessarily severe and that simple type theory suffices to prevent the paradoxes. By replacing ramified type theory with simple type theory, neither the axiom of reducibility nor an alternative axiom is necessary to allow quantification over functions satisfied by some object. Since this view does not account for intensional functions, it is less general than Russell's initial goal; however, it does give him the desired result of reducing mathematics to a logical system.

# Chapter 1

# The system of *Principia Mathematica*'s first edition

## 1.1 Basic principles of the logical system

Alfred North Whitehead and Bertrand Russell's *Principia Mathematica* is a monumental contribution to logic and the foundations of mathematics. The three volumes of the first edition were published between 1910-1913, and those of the second edition between 1925-1927. Although the work was a collaboration, the philosophical foundations which we will consider here were primarily the work of Russell.[1]

### Logicism

*Principia Mathematica* had the logicist goal of reducing mathematics to logic. In Russell's words: "The principle aim of *Principia Mathematica* was to show that all pure mathematics follows from purely logical premisses and uses only concepts definable in logical terms."[2] By defining mathematical concepts in

[1]In [44], Whitehead states: "The great labour of supervising the second edition of the *Principia Mathematica* has been solely undertaken by Mr. Bertrand Russell. All the new matter in that edition is due to him, unless it shall be otherwise expressly stated. It is also convenient to take this opportunity of stating that the portions in the first edition—also reprinted in the second edition—which correspond to this new matter were due to Mr. Russell, my own share in those parts being confined to discussion and final concurrence. The only minor exception is in respect to *10."(Section *10 gives the "The Theory of Propositions containing one Apparent Variable.")

[2][38] pg. 57

terms of logical ones, we can justify them by logical principles alone, and deduce all mathematical theorems from logical axioms using logical inference rules. Russell also believed that one may not stand outside the logical system. This perspective does not allow one to make meta-logical statements, for all logical reasoning falls within the system itself and is subject to its rules. In order to show this reduction of mathematics to logic, *Principia Mathematica* must both explain a logical system and show how to interpret mathematics within such a system.

Although Russell did not give a precise specification of characteristics that he took a logical system to have, he did discuss general traits which he associates with a logical system. Such traits include generality and necessity. Logical propositions should be general, and not involve any specifics. They should be expressible using just variables and logical constants, such as logical connectives and quantifiers, which express the form of a proposition. Additionally, logical propositions should be tautological. He stated that the axiom of infinity is an example of a proposition which can be stated in just logical terms but is not a logical proposition for it is not necessarily true. In *Principia Mathematica*, he stated that the "proof of a logical system is its adequacy and its coherence."[3] A system is coherent if it does not lead to contradictions. Russell offered a circular explanation of what it means for a system to be adequate: it "must embrace among its deductions all those propositions which we believe to be true and capable of deduction from logical premises alone, though possibly they may require some slight limitation in the form of an increased stringency of enunciation."[4]

### Formal system

Russell builds up his system from undefined concepts and basic propositions that govern them. Such concepts and propositions are *primitive*. Using primitive propositions and concepts, we can prove all of the theorems of the system. Russell writes "Pp" to identify a primitive proposition when it is introduced.

The notational conventions of *Principia Mathematica* are different from other logical systems. Dots are used both as conjunction symbols and as brackets separating propositions. A greater number of dots indicates a broader scope of the separation. Parentheses and brackets are also used

[3][45] pg. 12
[4][45] pgs. 12-13

to separate the arguments to a propositional function and to separate the portion of a proposition to be negated. For example, Russell writes

$$\vdash \colon .p \supset q . p \supset r . \supset \colon p . \supset . q . r$$

where, in modern notation, we would write

$$\vdash ((p \to q) \land (p \to r)) \to (p \to (q \land r)).$$

He writes

$$\vdash \colon .(x).\phi x : (x).\psi x :\supset .(x).\phi x.\psi x$$

where we would write

$$\vdash (((\forall x)\phi x) \land ((\forall x)\psi x)) \to ((\forall x)\phi x \land \psi x).$$

The language of *Principia Mathematica* uses *unrestricted* variables, symbols which stand for any object the name of which can be substituted in place of the symbol to make a significant statement. We shall return shortly to what makes a statement significant. Russell gives three principles governing the use of variables[5]: a variable ambiguously denotes, a variable's identity does not shift within a single context, and the possible values, or determinations, of two variables may be identical. Two variables may also have a different range of possible values, in which case it is meaningless to substitute a name for a possible value of one variable for the other.

### Violation of the use/mention distinction

Throughout *Principia Mathematica*, Russell frequently violates the distinction between the use and mention of signs. He does not talk about formulas as linguistic symbols denoting objects, as one presenting a modern formal system would, but instead conflates a concept with its representation. While outlining his system, he often gives an intuitive motivation for a concept, and then gives axioms to reflect what is intuitively expressed by such quasi-linguistic entities. He then talks about the syntactical properties of these quasi-linguistic entities as if they were objects in a formal syntax. For example, he might characterize a proposition, not an expression of the proposition, by whether it contains some variable. He does not give a rigorous and formal

[5] [45] pg. 4

syntactical specification of how formulas are put together, and assumes that it is clear based on their intended interpretation. The signs are not themselves objects in the formal theory, but their referents are. However, he is not clear in distinguishing the signs from their referents. An example of this violation is that Russell frequently states that some vague concept is equivalent to an axiom, even though the concept is given semantically and the axiom syntactically, and does not discuss the relation between the semantics and the syntax. This is further complicated because Russell's system is an intensional one, where variables range over generalized expressions, not the extensions of the expressions. For example, a variable which ranges over propositions ranges over more than the referents of propositions, which are either 'true' or 'false.' However, Russell never specifies what it means for the variable to range over propositions, and whether these are formulas or objects.

Kurt Gödel criticizes Russell for failing to provide a rigorous formal syntax:

> It is to be regretted that this first comprehensive and thoroughgoing presentation of a mathematical logic and the derivation of Mathematics from it is so greatly lacking in formal precision in the foundations [..], that it presents in this respect a considerable step backwards as compared with Frege. What is missing, above all, is a precise statement of the syntax of the formalism. Syntactical considerations are omitted even in cases where they are necessary for the cogency of the proofs, in particular in connection with the "incomplete symbols." These are introduced not by explicit definitions, but by rules describing how sentences containing them are to be translated into sentences not containing them. In order to be sure, however, that (or for what expressions) this translation is possible and uniquely determined and that (or to what extent) the rules of inference apply also to the new kind of expressions, it is necessary to have a survey of all possible expressions, and this can be furnished only by syntactical considerations.[6]

[6] [13] pg. 114

### Primitive concepts and propositions

One primitive concept is that of *elementary propositions*[7]. These involve no variables, and hence no quantifiers. This concept is an example where Russell confuses use and mention. The intuitive idea is that an elementary proposition is one which is expressed by a formula containing no variables; however, Russell only mentions propositions themselves, and does not talk about formulas expressing some proposition. Elementary propositions may be combined using the propositional connectives to form other elementary propositions. In the first edition of *Principia Mathematica*, the connectives *disjunction* and *negation* are primitive. To say that proposition $p$ is true, we *assert* it; this is written "$\vdash .p$."[8]

A proposition must unambiguously denote to have a truth value. Statements such as $\phi x$ (where $x$ is a variable) have no truth value, for $x$ is ambiguous. An example is a statement such as "$x$ is red"—we will have a proposition once we substitute an individual for $x$. This makes the statement an ambiguous value of a propositional function. A unary propositional function, or a statement which will become a proposition once a name for a definite object is substituted for its variable, is expressed as $\phi\hat{x}$. In the above example, "$x$ is red" is an ambiguous value of the propositional function"$\hat{x}$ is red". In Russell's notation, $\phi\hat{x}$ and $\phi\hat{y}$ refer to the same propositional function. Elementary propositions are formed from propositional functions by assigning values to all of the variables within a propositional function. A propositional function may also be an argument to another propositional function. $\psi(\phi\hat{x})$ denotes the result of applying $\psi\hat{x}$ to the argument $\phi\hat{x}$.

W.V. Quine notes Russell's confusion between a sign and its referent, and comments on Russell's "failure to focus upon the distinction between 'propositional functions' as attributes, or relations-in-intension, and 'propositional functions' as expressions, viz., predicates or open sentences."[9] Propositional functions are an essential concept in *Principia Mathematica*; as is common throughout the work, Russell provides an intuitive understanding of what he means by the concept of propositional function, but does not give a formal specification. Although he never specifies the syntax, he gives enough information to allow us to figure it out, and talks about propositional functions as if they were syntactic entities.

[7] [45] pg. 91
[8] [45] pg. 92
[9] [35] pg. 383

Some symbols are defined in terms of other symbols whose meaning is understood within the system. Definitions serve to simplify the system, for they allow certain combinations of symbols to be expressed in shorter forms. For example, implication is defined in terms of negation and disjunction, which are primitive. Implication is not itself primitive, but once it is defined the symbol may be used in the system. When a symbol or notation is first introduced, Russell writes "Df." The left of the equals sign is the *definiendum*, which is what is being defined; the *definiens* is to the right of the equals sign and is the meaning of the *definiendum*.

All propositional functions have a range of values which results from substituting for variables. A value satisfies a propositional function if taking this value as argument makes the propositional statement true. If propositional function $\phi\hat{x}$ has values such that all satisfy $\phi\hat{x}$, then we have that $(x).\phi x$ is true. This tells us the resulting proposition is true for any significant argument. If the range contains some true propositions, then we have $(\exists x).\phi x$ is true, and if the range is only false propositions we have $(x). \sim \phi x$ is true. In these cases, $x$ is an *apparent variable*, or, in modern terminology, a *bound variable*. $(\exists x).\phi x$ is not taken as a primitive, but is defined as equal to $\sim (x). \sim \phi x$.

One primitive proposition corresponding to the inference rule *modus ponens* states: "Anything implied by a true elementary proposition is true."[10] The related syntactical primitive proposition states: "When $\phi x$ can be asserted, where $x$ is a real variable, and $\phi x \supset \psi x$ can be asserted, where $x$ is a real variable, then $\psi x$ can be asserted, where $x$ is a real variable."[11] In formal notation, this means that if we have $\vdash .\phi x$ and $\vdash .\phi x \supset \psi x$ we can derive $\vdash .\psi x$. This can be generalized for functions of more than one variable.

The primitive proposition which serves as an inference rule for elementary propositions may be extended for propositions containing apparent variables. This primitive proposition states that "What is implied by a true premiss is true."[12] This is the more general rule corresponding to *modus ponens*, and lets us infer $\vdash .\psi x$ from $\vdash .\phi x$ and $\vdash .\phi x \supset \psi x$, even in cases where the values of $\phi\hat{x}$ and $\psi\hat{x}$ contain apparent variables. This also holds in the more general case of propositional functions taking more than one argument.

Another concept is *formal equivalence*: functions are formally equivalent

[10] [45] pg. 94
[11] [45] pg. 95
[12] [45] pg. 132

when their values imply each other. The definition of equivalence of propositions is:

$$p \equiv q. = .p \supset q.q \supset p \ \text{Df.}^{13}$$

For propositional functions, $\phi x \equiv_x \psi x$ is defined to mean that $(x).\phi x \equiv \psi x$.[14] These functions share an *extension*, for whatever satisfies one satisfies the other.

### Incomplete symbols

Certain symbols within *Principia Mathematica* are *incomplete*. This means that they only have meaning in a particular context, not independently of a defining context. The three most important groups of incomplete symbols are descriptions, classes, and relations; we will later discuss classes, which are relevant to the current discussion. Incomplete symbols differ from symbols which also have meaning in isolation, and thus need not have the same characteristics. Most importantly, incomplete symbols need not adhere to formal rules of identity; whether they do adhere to such rules is dependent on the particular interpretation. Once we have determined their meaning in a context, we can consider their equivalence as we would consider whether symbols are identical in isolation.

## 1.2 The theory of types

In his 1908 paper "Mathematical Logic as Based on the Theory of Types," Russell's main motivation is to present a type theory which is not troubled by contradictions. The particular contradictions in question are the self-referential ones, such as the liar paradox, the paradox of the least indefinable ordinal, and the paradox of the class of classes that are not elements of themselves.

A type theory is a syntactical specification of a language where the intended domain of discourse is divided into kinds, or "types." Variables and constants in the language have a type, and a statement is well-formed if each of it constituents is of the proper type, as given by the type theory. Variables are restricted to range over exactly one totality—this is the variable's type. This means if some variable $x$ ranges over individuals, then in

[13][45] pg. 115
[14][45] pg. 139

this context $x$ can only stand for an individual. Violations of type lead to meaningless statements—they are neither true nor false. There are different type theories. Simple type theory is one such theory, which I will explain below. Russell's, as given in "Mathematical Logic as Based on the Theory of Types" and *Principia Mathematica*, is another such theory. Russell's type theory is more discriminant than simple type theory, for it does not allow certain statements which are legitimate within simple type theory.

### Simple theory of types

In the 1974 paper *Russellian Simple Type Theory*, Alonzo Church gives a formulation of simple type theory which is similar in ontology to Russell's logic. Namely, "the propositional and functional variables have intensional values but the values of the individual variables remain extensional."[15] We will later consider what it means for a variable to have an intensional value. Church's presentation is similar in formulation to Church's presentation of Russell's type theory, which we will look at below. His presentation accounts for the fact that a propositional function may take more than one argument. In this presentation, individuals are of type $i$. Whenever $\beta_1, \beta_2, ..., \beta_m$ are types, $(\beta_1, \beta_2, ..., \beta_m)$ is a type, intended to denote the type of the propositional function that takes objects of the types $\beta_1, \beta_2, ..., \beta_m$ as arguments. $f(a_1, a_2, ..., a_m)$ is well-formed whenever $f$ is a variable or constant of type $(\beta_1, \beta_2, ..., \beta_m)$, and $a_j$ is a variable or constant of type $\beta_j$ for all $j \leq m$.

Simple type theory prevents any propositional function from directly taking itself as argument. For example, $\phi(\phi\hat{x})$ will never be a legitimate statement. $\phi\hat{x}$ has some type $(\beta)$; this means its argument must be of type $\beta$. Since $\phi\hat{x}$ is of type $(\beta)$, it cannot be an argument to $\phi\hat{x}$. However, we will later consider a case allowed in simple type theory where a function $\phi\hat{x}$ takes as argument another propositional function which was defined using $\phi\hat{x}$.

### Russell's theory of types

In *Principia Mathematica* Russell gives a theory of types different from the simple type theory mentioned above. Critics, including Hermann Weyl, Ramsey, and Quine, have claimed that Russell's theory is an unnecessarily complicated version of simple type theory. However, Russell developed his theory

[15] [5] pg. 25

independently of a simple theory of types and one should approach it as a self-contained theory. Hylton, for example, states:

> I have avoided the expression 'ramified type theory' because it seems to me tendentious. It strongly suggests that the theory is best understood as the result of of imposing complications (or ramifications) upon a simpler underlying theory. Once this is accepted, it becomes natural to ask whether we cannot avoid the complications by using only the simpler theory.[16]

### Vicious circle principle

The self-referential paradoxes which motivated Russell to create his type theory result from a vicious circle. A vicious circle can occur when an element of a collection is only definable in terms of the entire collection. Russell[17] gives the example of statements about "all propositions." Such statements refer to the totality of propositions, so presuppose their existence. However, they are themselves propositions, so should belong to this totality. Vicious circles result from cases such as this, where a collection contains elements which presuppose the entire collection. In order to avoid these, a system must not allow such statements to be legitimate. While simple type theory does not allow a propositional function to take itself as argument, it does not prevent the formation of totals whose members presuppose the total. Russell aims to avoid the formation of such totals. According to Russell, simple type theory, which allows one to speak of all propositional functions taking some object as argument, does not discriminate enough to prevent vicious circles.

> Propositions... must be a set having no total. The same is true... of propositional functions, even when these are restricted to such as can significantly have as argument a given object $a$. In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting.[18]

Using quantifiers, we can form statements in simple type theory where a propositional function's argument is formed from this function. This violates

[16] [17] pg. 311
[17] [45] pg. 37
[18] [45] pg. 37

versions of the vicious circle principle. If $y$ ranges over individuals and $\phi$ is a constant, type $((i))$ propositional function,

$$\psi(y) \equiv (\exists\chi)(\exists\theta)(\theta(\chi) \wedge \phi(\chi) \wedge \chi(y))$$

is a legitimate propositional function of type $(i)$, for its only argument, $y$, ranges over individuals. $\chi$ ranges over propositional functions of type $(i)$, for it takes an individual as argument; $\theta$ ranges over propositional functions of type $((i))$ and $\phi$ is a particular propositional function of type $((i))$, for each takes a type $(i)$ propositional function as argument. We can thus state $\phi(\psi\hat{x})$, because $\psi\hat{x}$ is of the proper type to be a legitimate argument for $\phi$. However, $\psi\hat{x}$ is constructed from $\phi$. This is an example where the argument to a function is constructed using the function. Propositional functions such as $\psi\hat{x}$, which quantify over propositional functions of the same (or higher) type (in this case $\theta$), allow for an argument to a function to depend upon the function.

By adhering to the vicious circle principle—"whatever involves *all* of a collection must not be one of the collection" or "whatever contains an apparent variable must not be a possible value of that variable"—Russell's theory of types denies the legitimacy of the paradoxical propositions. Although some of these statements may appear to involve classes or relations, Russell shows how the paradoxes can be understood in terms of propositional functions, and makes sure that they are not allowed in his system.

### 1.2.1 A formal presentation of Russell's type theory

#### Church's presentation

Church's formulation of Russell theory of types follows. This differs slightly from Russell's presentation in ways which will be discussed. This presentation accounts for the fact that propositional functions may take many arguments, and treats propositions as propositional functions with no arguments. In this presentation, the most fundamental stratification is what Church calls *r-types*. To say that two propositional functions are in the same place in the hierarchy is to say that they share an *r-type*. We consider the *r-types* of a statement's variables and constants to determine whether it is well-formed. This formulation also requires us to identify the *level* and the *order* of a variable, which will be defined shortly.

Individuals are of *r-type* $i$. Whenever $\beta_1, \beta_2, ..., \beta_m$ are *r-types*, $m \geq 0$, and $n \geq 1$, $(\beta_1, \beta_2, ..., \beta_m)/n$ is an *r-type* of level $n$. The variable with such an *r-type* is an $m$-ary propositional function.

Individuals are of *order* 0. A variable ranging over propositional functions of *r-type* $(\beta_1, \beta_2, ..., \beta_m)/n$ is of order $N + n$, where $N = 0$ if $m = 0$ and otherwise $N$ is the greatest order of $\beta_1, \beta_2, ..., \beta_m$. A functional variable's *level* is thus the difference in *order* between the variable in question and the order of its highest-ordered argument. Since $n \geq 1$, the functional variable's order must be at least one larger than the order of its largest argument. However, $n$ may be larger than 1, in which case there is a larger gap in *order* between the propositional function and its arguments.

We can compare the *r-types* of different variables; $(\alpha_1, \alpha_2, ..., \alpha_m)/k$ is directly lower in *r-type* than $(\beta_1, \beta_2, \ldots, \beta_m)/n$ if $\alpha_j = \beta_j$ for $j \leq m$ and $k < n$. Thus *r-types* with the same arguments can be compared by comparing their *level* or *order* (since the *r-types* of the arguments are the same, these *r-types* will differ from each other the same amount in both *order* and *level*).

The main way in which Church's presentation diverges from Russell's is that a variable ranges over values of its *r-type* and directly lower ones. Because of this feature, Church's formulation is cumulative, while Russell's is not.

Church provides an abbreviation schema for this recursive formulation. A string of $i$'s (the *r-type* of individuals) can be replaced by the number of $i$'s within the set of parentheses. For example, a level $n$ proposition will have no arguments. Its *r-type* is thus $(\ )/n$ , or $0/n$. A binary propositional function of level 3 which takes individuals as arguments is *r-type* $(i, i)/3$, or $2/3$. A level 1 binary propositional function, whose arguments are a level 2 proposition and a unary, level 2 propositional function taking an individual as argument, is *r-type* $((\ )/2, (i)/2)/1$, or $(0/2, 1/2)/1$. This is a third order propositional function. This means that the range of such a variable is binary propositional functions of level 1 with arguments of *r-types* $0/2$ and $1/2$.

Variables of *r-type* $0/n$ are well-formed formulas for all $n$. $f(a_1, a_2, \ldots, a_m)$ is well-formed exactly when $f$ is a variable or constant of *r-type* $(\beta_1, \beta_2, ..., \beta_m)/n$, $m > 0$, and $a_j$ is a variable or constant with *r-type* equal to or directly lower than the *r-type* $\beta_j$ for all $j \leq m$. Well-formed formulas may be combined using propositional connectives to form other well-formed formulas. Well-formed formulas may also be formed by instantiating apparent variables with variables or constants of appropriate *r-type*, and by quantifying over real variables or constants.

In Russell's presentation, one determines a variable's *r-type* by analyzing a well-formed formula containing the variable; Church gives comprehension axioms. The first axiom, which is the specific case for propositions, is:[19]

$$(\exists p).p \equiv P.$$

Here, $p$ is a propositional variable, meaning that its *r-type* is $0/n$, where $n \geq 1$. In this scheme, the apparent variables of $P$ must be of order less than $n$, and the other variables and constants in $P$ are of order less than or equal to $n$. The more general comprehension axiom is:

$$(\exists f).f(x_1, x_2, \ldots, x_m) \equiv_{x_1,x_2,\ldots,x_m} P.$$

Here, $f$ is a functional variable with *r-type* $(\beta_1, \beta_2, \ldots, \beta_m)/n$, $x_1, x_2, \ldots, x_m$ different variables of *r-types* $\beta_1, \beta_2, \ldots, \beta_m$, $P$'s apparent variables are of order less than the order of $f$, and $P$'s free variables and constants are of order less than or equal to the order of $f$. These axioms illustrate the importance of order within Russell's type theory—the order of all constituent variables and constants is essential to determining the *r-type* of another variable.

For further clarification of these distinctions, here are some examples of unary propositional functions of different *r-types*. If $\phi$ is a variable over functions which take as argument individuals, $\phi$ is of *r-type* $1/n$, where $n$ is this functional variable's level. The order will be the same as the level. Any value (greater than or equal to 1) is a possible value for $n$. If $n$ is 1, then individuals are the only totality presupposed by $\phi$. The theory of types was motivated to avoid vicious circles, and distinguish propositional functions taking the same arguments but presupposing different totalities. A propositional function presupposes a totality of higher order by quantifying over such a totality. If $n$ is 3, then $\phi$ is a propositional function of order and level 3. This means that it presupposes the totality of order 2 variables; for example, its definition may have been obtained by quantifying over some second order variable. This function takes individuals as arguments, but it is distinct from a variable of *r-type* $1/1$ because it has a different order and level. The *r-type* $1/1$ is directly lower than $1/3$.

A function of *r-type* $(1/1)/2$ is of level 2 and order 3. It takes as argument order 1 (and level 1) propositional functions taking individuals as arguments. Since its level is 2, we know that its order is two larger than that of its

[19][7] pg. 750

argument. In this case, this means that it presupposes the totality of second order propositional functions even though its argument is first order. An example of such a propositional function is: $(\forall\phi)(\phi(\psi\hat{x}))$, where $\psi$ is of *r-type* $1/1$, and $\phi$ of some second order *r-type.* A function of *r-type* $(1/2)/1$ is also of order 3, but its level is 1. As argument, it takes a propositional function of the second order (and level) which accepts individuals as argument. This propositional function is of the lowest possible order for such an argument. We have so far mentioned three *r-types* of order 3. There are exactly four unary, third order *r-types*, because there are 4 ways to sum up to 3 using this formulation. The one not yet mentioned is *r-type* $((1/1)/1)/1$. A variable of this *r-type* has level 1, and ranges over first-level propositional functions whose arguments are second order, first level propositional functions taking as argument first order, first level propositional functions which take individuals as arguments.

### Avoidance of a vicious circle

Let us return to the example from simple type theory where the argument to a propositional function depended upon the function. In Russell's type theory, such a function is illegitimate. The example we had was $\psi(y) \equiv (\exists\chi)(\exists\theta)(\theta(\chi) \wedge \phi(\chi) \wedge \chi(y))$ where $y$ ranges over individuals and $\chi\hat{x}$ is a propositional function of *r-type* $1/n$. We do not need to specify $n$'s value; we know that the *r-type* will fit this form because $\chi\hat{x}$ takes one individual as argument. Assuming that $\phi(\chi)$ is well-formed, $\phi\hat{x}$ must be of *r-type* $(1/n^+)/m$, where $n^+$ is some number equal to or greater than the $n$ of $\chi\hat{x}$'s level, and $m$ is some number greater than or equal to 1. $\phi\hat{x}$'s order is $n^+ + m$. $\phi\hat{x}$ is a constant, and we can say that the apparent variable $\theta\hat{x}$, which also takes $\chi$ as argument, is of the same *r-type* as $\phi\hat{x}$. The *r-type* of $\psi\hat{x}$ is $1/b$, where $b$ is some number greater than or equal to $n^+ + m + 1$. $\psi\hat{x}$ takes individuals as arguments, but its level must be large enough to account for $\theta\hat{x}$'s order. We shall see that $\psi\hat{x}$ is a not a legitimate argument to $\phi\hat{x}$. $\phi\hat{x}$ is *r-type* $(1/n^+)/m$. For $\phi(\psi\hat{x})$ to be well-formed, $\psi\hat{x}$'s *r-type* must be equal to or directly lower than *r-type* $1/n^+$. This is not the case, for the lowest possible level for $\psi\hat{x}$ is $n^+ + m + 1$, making the *r-type* neither equal to nor directly lower than $1/n^+$. We previously saw that $\phi(\psi\hat{x})$ is well formed in simple type theory. Unlike simple type theory, Russell's type theory does not allow such statements where a propositional function's argument depends upon the function itself.

### Predicativity

Predicativity is an important concept in *Principia Mathematica*. Using Church's formulation, predicative propositional functions are those of the first level. This means the function's order is one more than the highest order of any of its arguments, which is the lowest possible order for the function given these arguments. The highest order of the arguments is the highest order totality presupposed by a predicative function. By definition, non-predicative propositional functions presuppose some totality which is higher in order than any of their arguments. Of the third order propositional functions mentioned above, *r-types* $((1/1)/1)/1$ and $(1/2)/1$ are both predicative *r-types*, while the others are not. Both of these third order *r-types* take second order arguments. In the former case, this second order argument is itself a predicative propositional function—that of *r-type* $(1/1)/1$. However, in the latter case the second order argument—$1/2$—is not itself predicative. As predicativity is defined in the Introduction to *Principia Mathematica*, this distinction is unimportant. Predicative propositional functions must be of the first level, but their arguments may have higher levels. Russell gives a definition of predicativity in terms of assumed totalities:

> [A] predicative function of a variable argument is one which involves no totality except that of the possible values of the argument, and those that are presupposed by any of the possible arguments.[20]

We write that a propositional function $\phi\hat{x}$ is predicative by writing $\phi!\hat{x}$. $\phi!x$ represents a value of predicative propositional function $\phi!\hat{x}$.

This is the definition of predicativity generally attributed to Russell. However, in section *12 of *Principia Mathematica*, Russell gives a more restrictive characterization of predicativity. It is not a formal characterization, and is a case where he violates the use/mention distinction, for he characterizes propositional functions based on their properties as symbols. He calls functions with no apparent variables *matrices*, and states that a "function is said to be *predicative* when it is a matrix."[21] This sense of predicative function is stricter, for it requires that a predicative function be a function that cannot be written with any apparent variables. This bars the comprehension

[20] [45] pg. 54

[21] [45] pg. 164

axiom $(\exists\psi)(\forall x)(\psi!(x) \equiv \phi(x))$ where $\phi$ has any quantified variables. An elementary proposition also contains no apparent variables, and an elementary function only has values which are elementary propositions.[22] If we take this definition of predicativity, then predicative function, elementary function, and matrix are different names for the same thing—a function without any apparent variables.

### 1.2.2 The axiom of reducibility

Russell's theory of types tells us that if $x$ is a variable, then it has some *r-type* $\beta$, and $x$ is a legitimate argument to all propositional functions of *r-type* $(\beta)/n$, where $n \geq 1$. In this theory, we are not allowed to make statements about "all properties of $x$," because these properties are defined by propositional functions of various *r-types*. We can generalize over all statements of a particular *r-type*, but any variable is a legitimate argument to propositional functions of infinitely many *r-types*. This means that we cannot talk about *all* propositional functions taking some variable as input, but only about all propositional functions of a particular *r-type*.

If the system of *Principia Mathematica* is the basis for mathematics, then it is essential that we can speak of all statements which are true of some argument. For example, we shall see that rules of identity require us to say that two distinct objects cannot share all of their properties, or satisfy all of the same propositional functions. Induction and analysis also require us to generalize over propositional functions which an object satisfies. Simple type theory allows us to generalize in this way, for a particular object is only a valid argument to one type of function (if the object is of type $\beta$, then it is only a legitimate argument to functions of type $(\beta)$). However, the distinctions made by Russell's theory of types prevent us from generalizing over all functions taking some type of arguments. Russell solves this problem by introducing the axiom of reducibility.

The axiom of reducibility states that any function $\phi\hat{x}$ is formally equivalent to a *predicative* function. In Church's notation, this means that any propositional function is co-extensive with, or satisfied by the same arguments as, some level 1 propositional function. The axiom does not claim that we can always define such a propositional function, but that it exists. As we saw, there are different notions of predicativity. This axiom is a stronger

[22][45] pg. 127

assumption if we take the more restrictive sense of predicativity.

The axiom for unary propositional functions is:[23]

$$\vdash: (\exists f) : \phi x. \equiv_x .f!x \quad \text{Pp}$$

There are analogous axioms for propositional functions of more than one input. For a binary propositional function $\phi(\hat{x}, \hat{y})$ it is:

$$\vdash: (\exists f) : \phi(x, y). \equiv_{x,y} .f!(x, y) \quad \text{Pp}$$

Bernard Linksy gives a general form of the axiom using Church's notation[24]:

$$(\exists f^{(\beta_1,\ldots,\beta_n)/1})(\forall x_1)\ldots(\forall x_n)[\phi(x_1,\ldots,x_n) \equiv f(x_1,\ldots,x_n)]$$

where the *r-types* of $x_1, \ldots, x_n$ are $\beta_1, \ldots, \beta_n$ respectively. The axiom thus says that $\phi$, a propositional function of *r-type* $(\alpha_1, \alpha_2, ..., \alpha_m)/k$, is co-extensional with $f$, one of *r-type* $(\beta_1, \beta_2, ..., \beta_m)/1$, where $\alpha_j = \beta_j$ for $j \leq m$. The order of $f$ is 1 plus the order of the highest-order argument; the order of $\phi$ is unspecified, but it is at least as large as the order of $f$. This is an instance of Russell's notion of 'typical ambiguity'. Typical ambiguity means that the *r-type* of variables is not specified and thus a variable may be of any *r-type*; the statement is well-formed because we assume that any variable is of one particular *r-type*. In this case we are not quantifying over functions of different *r-types*, but are asserting a proposition involving $\phi$, a function of unspecified *r-type*.

## 1.3 Identity

Identity is a defined concept in *Principia Mathematica*, not a primitive one. Definition **13.01** states that:

$$x = y. =: (\phi) : \phi!x \supset \phi!y \quad \text{Df}$$

This says that $x$ and $y$ are identical when every predicative function satisfied by one is satisfied by the other. Russell's theory of types prevents us from making statements about *every* function satisfied by some variable, because a variable may satisfy functions of different orders so we cannot quantify over

[23] [45] pg. 167

[24] [26] pg. 90

them. However, the axiom of reducibility suffices to tell us that, if $x = y$, a function satisfied by $x$ is also satisfied by $y$. Using the axiom of reducibility, we can prove proposition **13.101**:

$$\vdash : x = y . \supset . \psi x \supset \psi y$$

In this case, $\psi$ is not necessarily a predicative function. We can make this statement within the system of *Principia Mathematica* because it does not require us to quantify over all propositional functions of arbitrary order. This is another instance of 'typical ambiguity', for the *r-type* of $\psi$ is unspecified. The axiom of reducibility guarantees that there is a predicative propositional function co-extensive with a propositional function of arbitrary *r-type*. This means that if there is some higher-order propositional function satisfied by $x$ but not by $y$, then there is some predicative function satisfied by $x$ but not by $y$. Thus if $x$ and $y$ satisfy all of the same predicative functions, there is not a higher-order propositional function which one satisfies but the other does not.

### Properties of identity

*Principia Mathematica* has propositions stating that identity is reflexive, symmetrical, and transitive. There are also propositions, for one and two variables, asserting that the statement that $x$ has some property is equivalent to the statement that something identical with $x$ has that property. For conventional use, there are definitions related to identity, such as $x \neq y$.

## 1.4 Classes

We can think of a class as a collection of objects with a certain property. One of Russell's main motives for assuming the axiom of reducibility in the first edition of *Principia Mathematica* is that it allows him to make no assumptions about class existence. Instead of talking about a class of objects with some property, we can interpret this idea in terms of the propositional function defining this property. Russell states that the axiom of reducibility replaces the assumption that classes exist, and gives the same results without other problems which arise in a system which assumes that classes exist. These problems include "the one and the many":

> If there is such an object as a class, it must be in some sense *one* object. Yet it is only of classes that *many* can be predicated. Hence, if we admit classes as objects, we must suppose that the same object can be both one and many, which seems impossible.[25]

Russell saw this problem as an example of the paradoxical situations which can arise by assuming that classes are actual objects. He thought it was a paradox that a collection was simultaneously *one* class and *many* objects forming the class.

### Role of classes

Although Russell does not assume class existence, classes still play a role in the system of *Principia Mathematica*. This role is a pragmatic one—they allow us to speak in a certain way, without asserting anything about the existence of such objects. Russell aims to "[reduce] statements that are verbally concerned with classes and relations to statements that are concerned with propositional functions."[26] Once we have a way to represent classes, we can explore their properties, and apply mathematical principles to them (such as combining them and looking at complements). Propositional functions are the basic constituents of the semantics of Russell's system, but we can speak of whatever satisfies a propositional function. A propositional function is satisfied by whatever arguments make it true. We can think of a class as everything which satisfies a particular propositional function; this means every propositional function determines a class (although it may be the empty class). Classes are key to interpreting mathematics, which is extensional, within Russell's intensional logic. We write $\hat{x}(\phi x)$ to represent the class which satisfies the propositional function $\phi\hat{x}$. This is an *incomplete symbol*, and does not have a meaning in isolation.

### Extensionality

An extensional function $\psi\hat{x}$ of an argument $\phi$, where $\phi$ is an individual or a propositional function, is one whose truth depends only on the extension of $\phi$. This means that $\phi$'s truth value is a factor in determining $\psi(\phi)$'s, but no other properties of $\phi$ are relevant. In the language of *Principia Mathematica*,

[25][45] pg. 72

[26][45] pg. 38

we express that a function $f$ of functions is extensional as follows:[27]

$$\phi x . \equiv_x . \psi x :\supset_{\phi,\psi}: f(\phi\hat{z}) . \equiv . f(\psi\hat{z})$$

This tells us that for $\phi\hat{x}$ and $\psi\hat{x}$ which are equivalent for legitimate arguments, $f$ holds of the function $\phi\hat{x}$ exactly when it holds of $\psi\hat{x}$. This function $f$ will have the same values so long as the class determined by $\phi\hat{x}$ is constant. In other words, only the values that satisfy $\phi\hat{x}$, and not the definition of $\phi\hat{x}$, are important in determining the truth of $f(\phi\hat{x})$.

Functions which are not extensional are intensional; an example is the function $\psi(\hat{\phi})$: "Bertrand believes that $\phi$". $\phi$ ranges over propositions, so has *r-type* $0/n$. $\psi\hat{x}$ ranges over propositional functions taking a proposition as argument, so has *r-type* $(0/n^+)/m$, where $n^+$ is a natural number greater than or equal to $n$. If $F$ is the constant standing for 'France is less populated than India', and $G$ represents 'Canada is less populated than the United States', they are each of *r-type* $0/1$ so we can substitute $F$ or $G$ for $\phi$ in $\psi(\hat{\phi})$. Bertrand may believe that France is less populated than India, but not believe that Canada is less populated than the United States. In both cases, our argument $\phi$ has a positive truth value, but $\psi(F)$ is true while $\psi(G)$ is not, so they have different extensions. With intensional functions, it may be the case that arguments with the same extension give different values (the extension will be a truth value in the case of arguments which are propositions). This is never the case with extensional functions—arguments with the same extension will give the same value. The function "If Bertrand lives in Britain, then $\phi$" is an extensional function. Given any propositions with the same truth value as argument, the value of the function will be the same.

Definition **20.01** defines what it means to apply a function $f$ to an incomplete symbol representing a class:

$$f\{\hat{z}(\psi z)\} . =: (\exists\phi) : \phi!x . \equiv_x . \psi x : f\{\phi!\hat{z}\} \quad \text{Df}$$

Saying that the function $f$ holds of the class of arguments which satisfy $\psi$ is defined to mean that some predicative function $\phi$ holds of the same arguments as $\psi$, and the function $f$ (which is not necessarily extensional) holds of $\phi$. Using the formal rules of propositional and quantifier logic as given in *Principia Mathematica* and assuming that $\psi x . \equiv_x . \chi x$, we can prove

[27][45] pg. 73

that

$$f\{\hat{z}(\psi z)\}. \equiv .f\{\hat{z}(\chi z)\}$$

This tells us that if $f\{\hat{z}(\psi z)\}$ is viewed as a function of $\psi\hat{z}$, then this function of functions is extensional. Russell explains:

> every proposition about a class expresses an extensional property of the determining function of the class, and therefore does not depend for its truth or falsehood upon the particular function selected for determining the class, but only upon the extension of the determining function.[28]

Russell's concept of classes requires that the value of $f$, when given $\{\hat{z}(\psi z)\}$ as argument, depends only on the extension of $\psi\hat{z}$. Although Definition **20.01** gives us that a function such as $f$ is extensional with the class defined by the function $\psi\hat{z}$ as argument, we have not shown that all functions of functions are extensional.

Russell uses the definition of identity ($x = y. =: (\phi) : \phi!x. \supset .\phi!y$) to prove proposition **20.15**:[29]

$$\psi x. \equiv_x .\chi x :\equiv .\hat{z}(\psi z) = \hat{z}(\chi z)$$

This is what distinguishes classes—two propositional functions satisfied by the same objects determine the same class. The proof of the reverse direction of this equality—$\hat{z}(\psi z) = \hat{z}(\chi z) :\supset . \vdash: .\psi x. \equiv_x .\chi x$—uses the axiom of reducibility to guarantee the existence of a co-extensive predicative function.[30]

### Class membership

Russell uses Peano's notation; $x\epsilon\hat{z}(\psi z)$ expresses that $x$ is a member of the class determined by propositional function $\psi\hat{z}$. Definition **20.02** says[31]:

$$x\epsilon(\phi!\hat{z}). = .\phi!x \quad \mathrm{Df}$$

Interpreting $x\epsilon(\phi!\hat{z})$ analogously to the treatment of the function application in definition **20.01**, we obtain:

[28] [45] pg. 191
[29] [45] pg. 189
[30] [45] pg. 191
[31] [45] pg. 188

$$x\epsilon\hat{z}(\psi z). \equiv: (\exists\phi) : \psi y. \equiv_y .\phi!y : \phi!x$$

The axiom of reducibility tells us that for all functions $\psi y$, there is a co-extensive predicative propositional function; this is the left conjunct of the *definiens* of the above definition. Since this is an axiom it always holds, so $x\epsilon\hat{z}(\psi z)$ must be formally equivalent to $\phi!x$, which is formally equivalent to $\psi x$. This substitition gives us **20.3**[32]:

$$x\epsilon\hat{z}(\psi z). \equiv .\psi x$$

This proposition is the statement which we would like to hold of classes—to say $x$ is a member of the class determined by some function is to say that this function holds of $x$.

### Properties of classes

Russell states that characteristics about classes are of three sorts: the basic characteristics, those involving classes and descriptions, and the propositions which show that the characteristics of classes of individuals apply to classes of classes as well.

The basic characteristics have to do with requirements for class identity and consequences of this. Although classes are incomplete symbols and do not have meaning in isolation, we can still talk about their equivalence within a context. The main proposition about classes tells us that formally equivalent functions determine identical classes and identical classes are defined by formally equivalent functions. This is proposition **20.15**, $\psi x. \equiv_x .\chi x :\equiv .\hat{z}(\psi z) = \hat{z}(\chi z)$, which we saw above.

Using the axiom of reducibility, we can use this proposition to prove that:

$$(\exists\phi).\hat{z}(\psi z) = \hat{z}(\phi!z)$$

This means that any class is defined by a predicative function, and we can quantify over all the classes to which a term belongs, for it is a legitimate totality. We can think of classes as "quasi-things."[33] We represent them with a name and think of them as having an *r-type*, although they are not actual objects so the name and *r-type* are only conveniences. Since the axiom of

[32] [45] pg. 189
[33] [45] pg. 81

reducibility guarantees that a class can be defined by a predicative function, a class's *r-type* is entirely determined by the *r-types* of its members, those objects which satisfy the predicative function defining the class.

Other basic propositions about classes tell us that identical classes share all members (and all classes with this set of members are identical) and all properties, and that something is a member of a class exactly when it satisfies the function which defines the class. Class identity is reflexive, symmetrical and transitive.

The second sort of propositions have to do with descriptive expressions; such as $(\iota z)(\phi x)$, which stands for "the $x$ satisfying $\phi\hat{x}$". Russell explores this in depth in his Theory of Definite Descriptions. These expressions may be treated as typical members of a class in certain circumstances. Also, we can use a descriptive phrase about a class to express something about the class satisfying some function. A class variable is an abbreviation for an expression about a defining function, so the properties are analogous.

The third set of propositions state that there is nothing unique about classes of individuals, and classes of classes share their properties. These proofs are analogous to previous ones, and tell us that the relation between a class and a class of classes is analogous to the relation between an individual and a class of individuals.

# Chapter 2

# The axiom of reducibility and the second edition of *Principia Mathematica*

## 2.1 Justification in the first edition for the axiom of reducibility

In the first edition of *Principia Mathematica*, Russell admits that one should be hesitant to accept the assumption of the axiom of Reducibility. Nonetheless, he offers a pragmatic justification of the axiom:

> That the axiom of reducibility is self-evident is a proposition which can hardly be maintained. But in fact self-evidence is never more than a part of the reason for accepting an axiom, and is never indispensable. The reason for accepting an axiom, as for accepting any other proposition, is largely inductive, namely that many propositions which are nearly indubitable can be deduced from it, and that no equally plausible way is known by which these propositions could be true if the axiom were false, and nothing which is probably false can be deduced from it... In the case of the axiom of reducibility, the inductive evidence in its favour is very strong, since the reasonings which it permits and the results to which it leads are all such as appear valid. But although it seems very improbable that the axiom should turn out to be false, it is

> by no means improbable that it should be found to be deducible from some other more fundamental and more evident axiom.[1]

He then provides an example in which the assumption seems a safe one. Consider the higher-order assertion: “Napoleon had all the qualities that make a great general.” We can speak of the totality of predicative properties held by great generals. Since there are a finite number of great generals, we can form a predicative property unique to great generals by disjoining a predicative property unique to each great general. This predicative property is co-extensive with “holding all the qualities of a great general,” for it will hold in the same instances. The axiom of reducibility is just the assumption that there is always such a predicative property equivalent to a higher-order property. Russell uses the word *order* to mean what Church calls *level* in Church’s formulation of Russell’s type theory—the amount of slack between a propositional function and its arguments. Predicative propositional functions have no slack; we will call these first order to correspond to Russell’s usage. In Church’s usage, such propositional functions are first level (and also first order in cases where the arguments are individuals).

Also in favor of the axiom is the fact that, according to Russell, it is a weaker assumption than that of class existence, which would require augmenting the system with variables and axioms to specify the use of classes as independently existing objects. Assuming class existence would allow us to reduce a propositional function’s order—taking a propositional function $\phi\hat{x}$ of any order, the assumption of class existence gives us a first order statement which asserts that $x$ is an element of the class of objects satisfying $\phi\hat{x}$. This means that we could reduce all propositional functions which a certain variable satisfies to a minimal order, such that each propositional function has order one larger than the variable itself. Such propositional functions are predicative, and we could then generalize over them for they share an order (and also, in Church’s formulation, an *r-type*). Russell introduced the axiom of reducibility to allow this generalization over all functions which an object satisfies, so this axiom would not be necessary were there other ways to make such generalizations.

Both the axiom of reducibility and the assumption of class existence allow us to assume that a propositional function is co-extensive with a predicative one, but the axiom of reducibility is a weaker assumption, for it does not

[1] [45] pgs. 59-60

require including additional objects in the system. Were Russell to claim that predicative functions were extensional, then he could no longer distinguish between classes and propositional functions, and his assumption would be no weaker than that of class existence. However, the axiom of reducibility only guarantees that there is some predicative propositional function co-extensive with any other propositional function, not that such a predicative function is an extensional one. The intensional characteristics of Russell's propositional functions are not negated by the axiom of reducibility.

Russell states that the "axiom of classes" is an alternate name for the axiom of reducibility. However, the axiom avoids complications that arise from assuming classes exist, such as the problem of "the one and the many," which was discussed above. Russell believes that this problem arises from assuming that classes are actual objects, and does not arise if classes are merely convenient ways to talk about what satisfies a propositional function. The axiom of reducibility minimizes the ontological assumptions of Russell's logical system, for it allows him to remain agnostic on the issue of class existence. Similarly, the axiom of reducibility for binary functions may be called the 'axiom of relations,' for the assumption is like the assumption that a statement of any order about two variables gives us a relation between the variables.

### 2.1.1 *Introduction to Mathematical Philosophy*

Between the two editions of *Principia Mathematica* Russell published *Introduction to Mathematical Philosophy*, a book on logic for a more general audience. He expressed doubts about the axiom of reducibility in this book as well, but did not entirely reject it. He states that the axiom is a "generalised form of Leibniz's identity of indiscernibles."[2] Two objects are Leibniz identical if they share all properties. This means that two distinct objects cannot have all their properties in common, for then they would be the same object. Leibniz identity is a special case of the axiom of reducibility, for it states that if some property distinguishes $x$ and $y$, then there is a predicative propositional function which holds of one and not the other (for example, the property of "being $x$"). Since the axiom of reducibility implies this, the axiom is a stronger assumption.

Russell does not intend the comparison to Leibniz identity as justification

[2] [39] pg. 192

of the axiom, for he believed that Leibniz's identity of indiscernibles was not a logical truth, but an empirical one. Since the axiom of reducibility implies Leibniz's principle, this implies that the axiom could contingently be false. If the axiom is not necessarily true, then he did not think it should be assumed within a logical system. However, at the time of *Introduction to Mathematical Philosophy*, Russell still included the axiom in his system.

> I do not not see any reason to believe that the axiom of reducibility is logically necessary, which is what would be meant by saying that it is true in all possible worlds. The admission of this axiom into a system of logic is therefore a defect, even if the axiom is empirically true. It is for this reason that the theory of classes cannot be regarded as being as complete as the theory of descriptions. There is need of further work on the theory of types, in the hope of arriving at a doctrine of classes which does not require such a dubious assumption. But it is reasonable to regard the theory outlined in the present chapter as right in its main lines, *i.e.* in its reduction of propositions nominally about classes to propositions about their defining functions. The avoidance of classes as entities by this method must, it would seem, be sound in principle, however the details may still require adjustment.[3]

## 2.2 Reaction to the first edition

The completion of *Principia Mathematica* was a tremendous achievement, and was recognized as such by members of the international logical community. In accordance with Russell's own take on the project, the axiom of reducibility was the most controversial aspect of the work. Ramsey speaks of the "serious objections which have caused its rejection by the majority of German authorities, who have deserted altogether its line of approach."[4] These objections centered around the axiom of reducibility.

### 2.2.1 Hilbert school

In a 1914 lecture, David Hilbert's student Heinrich Behmann took *Principia Mathematica* to be the first unification of two traditions: constructing logic

[3] [39] pg. 193
[4] [37] pg. 164

by mathematical means and analyzing logic's role in mathematics' construction.[5] However, Behmann was critical of aspects of Russell's work, particularly the theory of types. He believed the stratification was too severe, and prohibited propositions which are not paradoxical, such as the proposition "All propositions are either true or false."[6]

Behmann was not dismissive of the axiom of reducibility:

> "The axiom thus states that in this one order there already are enough functions to define all possible classes; it can therefore be viewed as a kind of completeness axiom for predicative functions.[7]

This comparison was to Hilbert's completeness axiom, which is a maximality axiom stating that the system of the real numbers is the largest system satisfying the properties of arithmetic.

Behmann was also sympathetic towards Russell's "no classes" theory; like Russell, he preferred minimal assumptions about ontological existence:

> The classes—and by the same token, incidentally, the numbers—are thus... nothing else than figures of speech extremely useful for ease and clarity in presenting arithmetic, but can nonetheless become quite problematic as soon as one takes them seriously and, in violation of their nature, takes them for the names of objects.[8]

In his understanding of the foundations of mathematics, Behmann aimed to minimize ontological assumptions about abstract entities such as classes and emphasize the role of concrete objects like individuals and empirical facts. He believed this approach could ground the approach of *Principia Mathematica*. Like Russell and Hilbert, he held individuals to be basics which could not be further analyzed, and aimed for a "de-ontologization of mathematics."[9] He preferred minimal assumptions about the existence of abstract objects, such as natural numbers.

Initially, Hilbert was also not dismissive of the axiom of reducibility. In 1917-1918, his goal was still the logicist one of reducing mathematics to logic,

[5] [29] pg. 306
[6] [29] pg. 306
[7] [29] pg. 307
[8] [29] pg. 313
[9] [29] pg. 319

and he believed Russell had been successful in axiomatizing logic, a key step in the logicist program.[10] In notes from these years, he agreed with Russell's introduction of the axiom as a way to retain a type theory which avoids vicious circle paradoxes in a system powerful enough to serve as a basis for mathematics. He did recognize that the axiom does not fit a constructivist picture, for it assumes that predicative propositional functions exist independently, even in cases where it may not be possible to give a definition of the predicative function.[11] In 1918, Hilbert expressed agreement with Behmann's comparison of the axiom to Hilbert's own arithmetic completeness axiom.[12]

With time, Hilbert became more critical of the axiom of reducibility. By the end of 1921, he rejected the axiom, and also the logicist goal of reducing mathematics to logic. He rejected the axiom for reasons he earlier noted: it assumes that the totality of predicative propositional functions is prior, and there must be predicative propositional functions which satisfy the demands of the axiom—namely, predicative propositional functions co-extensive with all other functions. The necessity of such constraints, namely the existential character of the axiom, conflicts with the logicist goal, and compelled Hilbert to abandon the logicist goal, and instead return to the goal of establishing consistent formal theories.[13]

Hilbert continued to criticize the axiom of reducibility. In a 1929 article, Hilbert claimed his consistency proof allows the description of arithmetic propositions without requiring, as Russell's did, "the help of the very problematic axiom of reducibility."[14] In 1931, as justification for his proof theory, Hilbert criticized other programs for the assumptions which they require. His proof theory does not require axioms such as Russell's axioms of infinity and reducibility, which he called "real, contentual presuppositions, not compensated for by proofs of consistency, and of which the latter [the axiom of reducibility] is not even plausible."[15]

In a 1930 paper, *Die Philosophie der Mathematik und die Hilbersche Beweistheorie*, Paul Bernays was also critical of the axiom of reducibility. He denied the possibility of it being a logical law—this would mean that the assumption would hold independently of the range of propositional functions,

[10] [42] pgs. 11-12
[11] [42] pg. 19
[12] [29] pg. 317
[13] [42] pg. 19
[14] [30] pg. 230
[15] [30] pg. 273

and that impredicative definitions could not enlarge the domain of predicative ones. He claimed that one could construct counter-examples, where an impredicative definition is not co-extensive with any predicative one. If the axiom of reducibility is not a logical law, then Bernays claimed it must be an assumption outside of the bounds of logic; as such it limits the construction of predicates to adhere to this assumption. He proposed to use the concept of *logical function* instead of predicate, and stated that we can speak of the totality of logical functions so the axiom of reducibility is unnecessary.

### 2.2.2 Weyl

In 1918, Weyl wrote *Das Kontinuum*. Weyl was not a logicist, and his goal was not to reduce mathematics to logic. In *Das Kontinuum*, Weyl developed a version of a predicative foundational stance, on which mathematical concepts could be defined in terms of more basic ones, but he held that the natural numbers were primitive. Recursive definitions and inductive proofs were also basic principles, and natural numbers could be treated like a completed totality. In *Das Kontinuum*, Weyl criticized aspects of Russell's view. He objected to Russell's theory of types, stating a " 'hierarchical' version of analysis is artificial and useless."[16] Instead he proposed a "narrower iteration procedure"; this is consistent with his view that the natural numbers are primary and may be assumed. Weyl also stated that Russell missed the "crucial point"—Weyl believed that the "the principles of definition must be used to give a precise account of the sphere of the properties and relations to which the sets and mappings correspond."[17] Russell does not use such "principles of definition"—he never gives a formal syntactic presentation of the ranges of the symbols in his language. Weyl was particularly critical of the axiom of reducibility and Russell's defining the natural numbers as equivalence classes; according to Weyl, this indicates the "veritable abyss" separating him from Russell.[18] Given Weyl's different primitives, the complications of Russell's type theory are unnecessary. Russell only needs to assume the axiom of reducibility because Russell's type theory prohibits certain generalizations—the stratifications of his type theory prevent quantifying over all propositional functions satisfied by some individual or propositional function. Were it not for the stratifications of his type theory, there would be no need to assume

[16][43] pg. 32
[17][43] pg. 47
[18][43] pg. 47

the axiom of reducibility. Thus, any position, such as that of Weyl, which does not espouse a ramified theory of types has no need of the axiom of reducibility.

In a paper from 1925-1927, Weyl considered the problems of Russell's ramification of type theory. He discussed the problem of the least upper bound—the upper bound of a set of first order numbers would itself be of higher order. He notes that the dilemma would be avoided if there were a co-extensional predicative property, which is the issue which motivated Russell's to introduce the axiom of reducibility. Weyl questioned the proposition stating the existence of such a co-extensive predicative propositional function:

> Yet, a proof for this has never been attempted, and there is not the slightest indication that one could put up construction principles for the properties of first level that would be far-reaching enough to ensure the correctness of the proposition; indeed this is from the outset so monstrously improbable that no one could reasonably be expected to look for such principles. Russell has chosen a rather abstruse way out, namely to postulate this completely unintuitive proposition as an axiom (axiom of reducibility).[19]

### 2.2.3 Wittgenstein

In his *Tractatus Logico-Philosophicus*, Ludwig Wittgenstein took issue with aspects of *Principia Mathematica*. These criticisms center around Russell's confusion of the use and mention of signs, and Russell's attempts to avoid meta-logical claims: "Russell's error is shown by the fact that in drawing up his symbolic rules he has to speak about the things his signs mean."[20] Wittgenstein thought this was an impossible task: "[it] is clear that the laws of logic cannot themselves obey further logical laws."[21] In particular, there is only one law of contradiction. This differs from the views of Russell, who believed that the law of contradiction is itself a statement within the system without special status and thus type theory requires that there be a law of contradiction for every type. Additionally, Wittgenstein criticized the axiom

[19] [30] pg. 132
[20] [46] pg. 42
[21] [46] pg. 95

of reducibility in a similar manner to Russell's questioning in *Introduction to Mathematical Philosophy*—Wittgenstein claimed that the axiom is not a logical truth. While the axiom may happen to be true, it is not a logical proposition and could just as well be false.[22]

### 2.2.4 Other criticism

Another criticism of the axiom of reducibility is that it undoes the ramification of Russell's theory of types, and reinstates the self-referential paradoxes which motivated Russell's type theory. Russell's type theory avoids the paradoxes because, within the theory, such seemingly contradictory definitions are not well-formed statements. He claimed that the order hierarchy was necessary to bar the paradoxes, and that a hierarchy which does not take account of quantified variables would allow vicious circles. However, the axiom of reducibility guarantees that there is a predicative propositional function co-extensive with any higher-order one. This may seem to negate any role played by the order hierarchy, for it states that any propositional function is co-extensive with a propositional function of the minimal order allowed by its arguments. One might argue that there is no reason for a type theory to characterize certain propositional functions as of a higher order if the theory guarantees that there always exists a lower-order propositional function satisfied by the same arguments (although we may not know what such a predicative function is). If the order hierarchy is necessary to bar vicious circles and, with the axiom of reducibility, the order hierarchy is shown to not add anything to the system, then it would seem that Russell's system still allows the paradoxes. This argument is fallacious because the axiom of reducibility does not remove the importance of the order hierarchy—the axiom does not state that there is an equivalent predicative propositional function to every propositional function, but only a co-extensive one. Although there are no extensionally unique propositional functions which cannot be expressed as a predicative function, there are propositional functions which are not intensionally equivalent to any predicative function, and the order hierarchy is necessary to express such functions.

In his 1976 paper, Church retracted his earlier criticism that the axiom of reducibility reinstates the paradoxes and undoes the ramification of type theory. He notes that the higher order propositional functions do differ in-

[22] [46] pgs. 95-96

tensionally from the co-extensive predicative ones guaranteed by the axiom of reducibility. Russell's type theory, with the axiom of reducibility, does differ from simple type theory; although objects of higher order are formally equivalent to predicative ones, they are not identical and may differ regarding intensional properties: "it is only in intension that we are to think of additional values of the functional variables as arising at each new level."[23] In a 1984 article, Church states that this criticism of the axiom of reducibility was "based on a confusion between material equivalence... and identity of propositions."[24] Thus, even with the axiom of reducibility, type theory still intensionally distinguishes between propositional functions of different orders and prevents intensional vicious circles, but no longer distinguish extensional ones: "the rejection of impredicative definition is annulled in extensional but not intensional matters."[25]

## 2.3 Changes made for the second edition

The first volume of the second edition of *Principia Mathematica* was published in 1925. Due to these criticisms and personal doubts, Russell made alterations for it. He did not change the body of the text, but added an introduction and three appendices. However, some of these changes have broad consequences for the logical system of *Principia Mathematica*. We will consider the motivations and effects of Russell's suggestions.

First, we will consider the relatively minor changes. Due to Sheffer's discovery that a single propositional connective suffices to define the others, Russell took this to be his only primitive connective.[26] Russell also denied any distinction between real and apparent variables. Thus $\vdash .fx$, where $x$ is an arbitrary variable, means $\vdash .(x)fx$. He realized that asserting something about 'any' value is equivalent to asserting it about 'all' values (of a single *r-type*). Any real variables in the first edition are to be read as apparent variables where the scope is the entire asserted proposition.

[23] [7] pg. 758

[24] [9] pg. 521

[25] [7] pg. 758

[26] Goldfarb, in [14], takes Russell's emphasis of the importance of the Sheffer stroke as evidence of Russell's disdain for meta-theory. Goldfarb claims that Russell thought contributions to logic must be internal to the system, so an internal discovery such as the Sheffer stroke was an important advancement.

The crucial change in the second edition involved the axiom of reducibility. In his philosophical autobiography, Russell states:

> My chief purpose in this new edition was to minimise the uses of the 'axiom of reducibility.'[27]

His doubts about the axiom impelled him to find an alternative for the axiom; in this edition, he instead assumed a principle of extensionality which allowed him to reach similar results as in the first edition. It seems he was motivated more by the lack of reasons to keep the axiom than by specific reasons to discard it. The principle of extensionality does not give all the results which the axiom of reducibility did, but Russell still found it to be a preferable alternative. In the introduction to the second edition, he says of the axiom of reducibility:

> This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content. On this subject, however, it cannot be said that a satisfactory solution is as yet obtainable.[28]

### 2.3.1 The principle of extensionality

In 1922, Russell wrote the Introduction to Wittgenstein's *Tractatus Logico-Philosophicus*. In this introduction, Russell explained the principle of extensionality. First, let us consider a definition of truth functions:

> A truth-function of a proposition $p$ is a proposition containing $p$ and such that its truth or falsehood depends only upon the truth of falsehood of $p$.[29]

Wittgenstein's view, the principle of extensionality, is that all functions of propositions are truth-functions and all functions of functions depend only on the extension of their arguments to determine their extension. It may seem that the functions of propositions which Russell labeled as intensional are exceptions to this. But Wittgenstein denies this:

[27][38] pg. 89
[28][45] pg. xiv
[29][46] pg. 13

> ...it is clear that "A believes that $p$", "A thinks $p$", "A says $p$", are of the form " '$p$' says $p$": and here we have no co-ordination of a fact and an object, but a co-ordination of facts by means of a co-ordination of their objects.[30]

Russell explains Wittgenstein's argument in terms of propositional attitudes. In such propositions as "A believes $p$," the proposition $p$ occurs "as a fact on its own account,"[31] which is distinct from the the fact of the proposition's truth value. The role played by proposition $p$ in "A believes $p$" is distinct from its role in "$q \supset p$"; in the former $p$ occurs as a fact, while in the latter it occurs as a proposition whose meaning results from the meaning of its parts. In modern terms, we would say that $p$ is mentioned in the first case, and used in the second. Wittgenstein aims to give a general account of propositions. If one accepts that all functions of propositions are truth-functions, then propositions can be defined in terms of atomic propositions. An atomic proposition is a proposition which asserts an atomic fact, or "a fact which has no parts that are facts"[32]—such as "Bertrand is British."

### The assumption that a function only occurs through its values

In the introduction to the second edition of *Principia Mathematica*, Russell credits Wittgenstein with offering a principle of extensionality as a replacement for the axiom of reducibility. Russell expresses some doubts about this, but overall finds it to be satisfactory. "There are difficulties in the way of this view, but perhaps they are not insurmountable...We are not prepared to assert that this theory is certainly right..."[33]

By assuming that a function of propositions is always a truth function, and that a function only occurs in a proposition through its values, we are saying that functions of functions are extensional. To say that a function only occurs in a proposition through its values means that we can substitute a co-extensive value of a function into a proposition without altering the extension of the proposition. If some formula $\phi$ contains the propositional function $f$, and $f$ is co-extensive with propositional function $g$ (which means that $(\forall x)(f(x) \equiv g(x))$), we can substitute $g$ for $f$ in $\phi$ and this will not

[30] [46] pg. 85
[31] [46] pg. 20
[32] [46] pg. 12
[33] [45] pg. xiv

affect the value of $\phi$. To evaluate $\phi$, we must evaluate the extension of its constituents; since $f$ and $g$ are co-extensive, their values are equivalent and hence $\phi$ will have the same values in either case. If functions of functions are extensional, "A believes $p$" is not a function of $p$, for it may have a different value if we substitute a co-extensive proposition in place of $p$.

Russell states the assumption which replaces the axiom of reducibility as: "a function can only appear in a matrix through its values."[34] This assumption is another case where Russell blurs the use and mention of signs. The definition of a matrix, which follows, is key to his presentation of the assumption. However, Russell's definitions characterize atomic propositions, elementary propositions, general propositions and matrices, all of which he treats as semantic entities, by their syntactic properties.

Atomic propositions cannot be further broken down into propositions, and do not contain quantifiers.[35] Elementary propositions[36] include both atomic and molecular propositions, which are composed of atomic propositions and logical connectives. A matrix is a function whose values are elementary propositions.[37] Thus once we supply the appropriate number of arguments to a matrix, we have an elementary proposition. Matrices cannot contain apparent variables; however, we may generalize over a matrix by iteratively quantifying over one or more variables. The result is a general proposition, which may contain apparent variables.

Despite Russell's failure to distinguish the use and mention of a sign, the assumption that a function only occurs through its values involves the distinction between the use and mention of a propositional function. A propositional function may be mentioned (and not used) in some context, but only its values, not the propositional function itself, can actually occur (or be used) in a logical statement. For example, the propositional function $\phi x$ is mentioned in the statement "Bertrand believes that $\phi x$", but only its values occur in the logical statement $\phi x \supset \psi x$. This assumption tells us that $\phi ! x$, a propositional function, cannot itself occur as an argument in a logical matrix; it can only occur as a generalization of its individual values. Thus there cannot be a logical matrix of a form like $f!(\phi!\hat{z})$ because the "only matrices in which $\phi!\hat{z}$ is the only argument are those containing $\phi!a$, $\phi!b$, $\phi!c$, $\ldots$, where $a$, $b$, $c$, $\ldots$ are constants; but these are not logical matrices, being

[34][45] pg. xxix
[35][45] pg. xv
[36][45] pg. xvii
[37][45] pg. xxii

derived from the logical matrix $\phi!x$."[38]

### A new axiom

The assumption that a function only appears in a matrix through its values tells us that a general proposition resulted from generalizing over some matrix (the proposition was possibly the result of more than one generalization over different variables). In Russell's representation, we can use the new variable $\phi_1 x$ to express a propositional function which was derived by generalizing over a matrix containing only individual variables; its values are what he calls "first-order propositions." An example of such a function is $(y).\phi!(\hat{x}, y)$.[39] This is derived from the matrix $\phi!(x, y)$ by making $y$ an apparent variable. We also may have cases where we generalize over a variable of higher order. $\phi_2 x$ represents a function with argument $x$ which contains $\phi_1$ as apparent variable; it is a "second-order function."

Russell again blurs the distinction between the use of a sign and its mention in his attempt to formalize this assumption. He gives the general intuitive idea of his assumption, and then gives a formal axiom corresponding to this intuition. Russell claims that, given our assumption, a function $\phi_1$ does not express anything that some matrix does not express. Instead of writing $\phi_1$, we could express the propositional function as an infinite conjunction or disjunction of matrices. This is similar to the axiom of reducibility, for it tells us that there is always a corresponding matrix. Russell expresses this as:[40]

$$(\phi_1).f!(\phi_1\hat{z}, x). \equiv .(\phi).f!(\phi!\hat{z}, x)$$

This is not a primitive proposition, or axiom, but an equivalence that Russell claims can be proved for any particular function from the assumption that only a function's values are used to determine a matrix. The axiom, or primitive proposition, which he introduces is for the general case:[41]

$$\vdash: (\phi).f!(\phi!\hat{z}, x). \supset .f!(\phi_1\hat{z}, x) \quad \text{Pp.}$$

In the first edition, there was no need for this axiom, since it can be derived from the axiom of reducibility. Having given up the axiom of reducibility in the second edition, Russell now has to add this an independent axiom.

[38] [45] pg. xxxi
[39] [45] pg. xxxiii
[40] [45] pgs. xxxvi-xxxvii
[41] [45] pg. xxxvii

Russell then discusses propositional functions whose values may be second order propositions, which we may write as $\phi_2 x$. Such a function either equals $(\phi).f!(\phi!\hat{z}, x)$ or $(\exists\phi).f!(\phi!\hat{z}, x)$. He considers whether we can derive $(\phi_2).f!(\phi_2\hat{z}, x)$ from $(\phi).f!(\phi!\hat{z}, x)$. In cases where $(\phi).f!(\phi!\hat{z}, x)$ is true because it is the value of a tautological stroke function, then $(\phi_2).f!(\phi_2\hat{z}, x)$ is also true. However, this is not always this case, so the inference does not always hold.

### Classes

The extensional assumption of the second edition—a function can only occur through its values—alters the concept of classes in *Principia Mathematica*. Two co-extensive propositional functions will provide the same values when they are passed as an argument to another function $f$; since we have assumed that any function of functions $f$ is extensional, its value will be the same when it takes as argument these co-extensive functions. We can formally express the assumption that an arbitrary function $f$ of functions is extensional as:[42]

$$\phi x \equiv_x \psi x. \supset .f(\phi\hat{z}) \equiv f(\psi\hat{z})$$

Assuming $\phi\hat{z}$ and $\psi\hat{z}$ have the same value when used as arguments to extensional functions, there is nothing to distinguish $\phi\hat{z}$ from $\psi\hat{z}$. Thus, given our definition of identity, we can prove that they are equal:[43]

$$\phi x \equiv_x \psi x. \supset .\phi\hat{x} = \psi\hat{x}$$

The assumption of extensionality thus removes any difference between functions and classes—$\hat{x}(\phi x)$ now expresses the same concept as $\phi\hat{x}$. "Thus classes, as distinct from functions, lose even that shadowy being which they retain in *20."[44] In the first edition, classes were incomplete symbols distinct from propositional functions. Since it is not meaningful to talk about the identity of incomplete symbols for they have no meaning in isolation, a class could not be equal to a propositional function. Their status as incomplete symbols prevented such statements: "$\hat{z}(\phi z) = \psi!\hat{z}$ is not a value of $x = y$."[45] In the second edition, a class does not retain any characteristics

[42] [45] pg. xxxix

[43] [45] pg. xxxix

[44] [45] pg. xxxix. Section *20 is where Russell gives his theory of Classes in the first edition of *Principia Mathematica*.

[45] [45] pg. 84

distinguishing it from the propositional function which determines it; classes are thus no longer incomplete symbols and can be used interchangeably with propositional functions in any context.

In the first edition of *Principia Mathematica*, classes were not treated as existing objects, but as a convenient notation. Although a class did not have an *r-type*, we could think of it as having an *r-type*, which was just the *r-type* of its members (which was always the same, for the members were what satisfied some predicative propositional function whose existence was guaranteed by the axiom of reducibility). In the second edition, this is more complicated for we have classes containing members of the same *r-type* but themselves having different *r-types*. For example, we could have a class of individuals (*r-type* 1) satisfying some predicative function (*r-type* 1/1) and another class of individuals (also *r-type* 1) satisfying a second-order function (*r-type* 1/2). These two classes have members of the same *r-type* but are defined in terms of propositional functions of different *r-types*.

In most cases, this complication is inconsequential. Some proofs from the first edition must be rewritten, but the same theorems may be proven. However, Russell is unable to prove that $2^n > n$ without the axiom of reducibility in cases where $n$ is infinite.

### Truth-functions

In Appendix C of the second edition, Russell defends the assumption that a function can only occur in a matrix through its values. He does not claim that the assumption is always true, but that mathematics is confined to cases where it holds. Thus, mathematics only contains extensional functions of functions, or those whose value depends only on the values of the functions it takes as arguments. In Appendix C, he considers whether it is the case that the assumption is generally true, and that all functions of functions are extensional. We will first consider the specific case of whether all functions of propositions are truth-functions.

Russell states that there is a distinction between "propositions considered factually" and "propositions as vehicles of truth and falsehood."[46] This is the same distinction Russell made in the Introduction to Wittgenstein's *Tractatus Logico-Philosophicus*. A proposition such as " 'Canada' occurs in the proposition 'Canada is larger than France' " is one in which we consider

[46][45] pg. 406

the word 'Canada' factually. We are not talking about a specific occurrence of the word 'Canada', but a member of a class of similar occurrences. These similar occurrences include mentions of words which appear like the word 'Canada.' To assert something *about* a proposition—for example that it is false or long or occurs in some context—is distinct from asserting the proposition. When one asserts something *about* a proposition, the proposition should be considered factually, or as a member of the class of occurrences of this proposition. When one asserts a proposition, the truth values alone should be considered.

Logic is only concerned with propositions as vehicles of a truth-value. In these cases, we are concerned with a particular occurrence of the proposition. We assert the proposition itself—we do not assert any properties of an instance or representation of the proposition (for example, its length or color or even its truth). The proposition itself does not occur when asserted, but its components do.

Russell states that another key feature of the role of propositions in truth functions is that they occur *transparently*.[47] By this Russell means that we do not assert anything about the actual proposition, but use it to say something about something else. When a symbol (or proposition) occurs in a truth-function, we are not speaking *about* the asserted symbol (or proposition), but using the symbol (or proposition) to speak about another thing, its components.

Russell emphasizes the distinction between factual and assertive propositions to show that propositions only occur in truth functions. His argument is that it only seems that propositions occur in other functions because we confuse assertions of a proposition and assertions about a proposition. Once we limit ourselves to the assertion of propositions, we only consider transparent occurrences of propositions, and thus only occurrences in truth-functions. We can then generalize to Russell's assumption that a propositional function, "a symbolic convenience in speaking about certain propositions,"[48] only occurs through its values. If propositions, the extensions of propositional functions, are only arguments to truth-functions, then it is only the extension of a propositional function which is relevant in determining the extension of another function taking the propositional function as argument.

[47] [45] pg. 407

[48] [45] pg. 408

### 2.3.2 The success of the second edition

The question remains whether it is satisfactory to replace the axiom of reducibility with a principle of extensionality. Extensionality is an assumption which Russell and others find to be less offensive; does it lead to the necessary results?

Russell does not think so:

> It appears that everything in Vol. I remains true (though often new proofs are required); the theory of inductive cardinals and ordinals survives; but it seems that the theory of infinite Dedekindian and well-ordered series largely collapses, so that irrationals, and real numbers generally, can no longer be adequately dealt with. Also, Cantor's proof that $2^n > n$ breaks down unless $n$ is finite. Perhaps some further axiom, less objectionable than the axiom of reducibility, might give these results, but we have not yet succeeded in finding such an axiom.[49]

[49] [45] pg. xiv

# Chapter 3

# Further critiques of *Principia Mathematica*

## 3.1 Ramsey's views

### 3.1.1 Background considerations

Frank Ramsey's critique of *Principia Mathematica* influenced Russell in making changes for the second edition. Russell explicitly credits only Wittgenstein for proposing the principle of extensionality, but Bernard Linsky, a contemporary Russell scholar, states that Ramsey "served as an intermediary"[1] between Russell and Wittgenstein, and that it was the influence of Ramsey, who helped Russell as a proof-reader,[2] which encouraged Russell to take an extensional approach in the second edition. In a later review and in his philosophical autobiography, Russell accepts many of the proposals Ramsey offers to alter the system of *Principia Mathematica*.

Ramsey's principle work relating to *Principia Mathematica* was *The Foundations of Mathematics*. Although it was published in 1925, after Ramsey had read the changes Russell made for the second edition of *Principia Mathematica*, much of the criticism focuses on the axiom of reducibility. Ramsey criticizes the details of *Principia Mathematica*, but is in general sympathetic to the logicist goal. However, he notes a flaw with Russell's general goal: namely, Russell's view that all true general propositions are logical ones.

[1] [26] pg. 8
[2] [16] pg. 443

Ramsey believed that the logical status of a proposition is based on its form, for certain propositions may happen to always be true without being necessarily true. He believed that logic should consist of tautologous general statements—those true by virtue of their form, not their content. Such statements are a subset of true general statements.

Like Wittgenstein, Ramsey believed that logic is extensional. At the basis of his system are *atomic propositions*, which cannot be further broken down. He gives the example that, if we consider 'Socrates' and 'wise' as names, 'Socrates is wise' is an atomic proposition.[3] A group of $n$ atomic propositions has $2^n$ truth possibilities, for each one may be true or false. We form propositions which are not atomic by asserting some combinations of atomic propositions are true. For example, to assert the non-atomic proposition $P \vee Q$ is to assert the combinations which assert either the atomic proposition $P$ or the atomic proposition $Q$. A truth-function of propositional arguments (possibly infinitely many) is a proposition which is true or false depending upon the truth values of the propositions which it takes as argument. For example, $P \vee Q$ is the same truth-function of $P$ and $Q$ as $R \vee S$ is of $R$ and $S$. General propositions, or those formed by quantifying over a variable, are truth functions in which the arguments are not explicitly enumerated. For example 'All birds have feathers' should be interpreted as the logical product of all values of 'If $x$ is a bird, $x$ has feathers.'[4]

The two ways of building up propositions—by combining atomic propositions with logical connectives and by generalizing, lead to truth functions. This supports the extensional view that all propositions are truth functions of atomic propositions. This does not mean that a proposition like "A believes $p$" is a truth-function of $p$, but that it is a truth-function of some atomic propositions. Since there are $2^n$ truth possibilities of $n$ atomic propositions, there are $2^{2^n}$ subclasses of truth possibilities, each corresponding to a truth function. Following Wittgenstein, he allows propositions to be truth functions of infinitely many propositions. The truth-functions of these arguments which come out true given all possible truth values for the arguments are tautologous propositions, and those which always come out false are contradictory ones. Ramsey's view is that mathematics consists of only tautologies.

If two symbols standing for propositions agree with the same sets of

[3][37] pgs. 168-169
[4][37] pg. 171

truth-possibilities of atomic propositions, then they are instances of the same proposition, even if they are expressed in different ways. For example, $P$ and $P \wedge P$ are ways of writing the same proposition. This view distinguishes the symbol from the proposition which it represents, and we can thus say that $P$ and $P \wedge P$ stand for the same proposition, even though they are symbolically different. Russell does not clearly differentiate a symbol from its referent; although he would say that $P$ and $P \wedge P$ are formally equivalent, he would not say that they are instances of the same proposition. An intensional system like Russell's distinguishes what is represented by different symbols. In contrast, Ramsey's system is extensional. He believes that a propositional function's extension is a class, and that mathematics involves classes, not propositional functions.[5] The propositions of mathematics assert propositions regarding extensions, so mathematics is a calculus of extensions. Ramsey's goal is to reduce the calculus of extensions to a calculus of truth-functions. The demonstration of this reduction would show that the extensional propositions of mathematics are just tautological statements; by showing that mathematics just consists of tautologies, we have filled the logicist goal of reducing mathematics to logic.

### 3.1.2 Ramsey's critique of *Principia Mathematica*

Ramsey believed that *Principia Mathematica* offers a generally successful method of showing that the propositions of mathematics are tautologies. However, in *The Foundations of Mathematics* he lists three main faults of *Principia Mathematica*. Ramsey claimed that his approach solves these problems, and in doing so successfully reduces a calculus of extensions to one of truth-functions, and shows that mathematics consists of only tautologies.

The first of these faults of *Principia Mathematica* stems from Russell's belief that every class is defined by a propositional function. Ramsey thought it was possible that some infinite class has no defining property, but such a class should still be included when we quantify over classes.

Ramsey claimed that the second fault of *Principia Mathematica* is Russell's Theory of Types. Ramsey discussed the paradoxes which motivated Russell to provide his theory of types. Ramsey claimed that the paradoxes can be divided into two sorts: those involving logical and mathematical terms, such as the paradox of "the class of all classes which are not members

[5] [37] pg. 177

of themselves," and those involving linguistic, epistemic or symbolic notions, such as the liar's paradox, or the paradox of "the least integer not nameable in fewer than nineteen syllables."[6] Ramsey believed that simple type theory suffices to avoid the first group of paradoxes, for it prevents a propositional function from taking itself as argument. He claimed that the paradoxes of the second sort result from epistemic or linguistic confusions, and the ambiguity of epistemic and linguistic terms. Once such terms are given a formal meaning, the contradictions are solved without requiring a ramified type theory. In order to avoid the paradoxes of the second sort, Russell presented a ramified type theory, or one which discriminated more finely than simple type theory based on quantified variables. If a logical system alone need not account for these paradoxes (because we give special consideration due to the peculiarities arising from the non-logical epistemic components), then type theory need not discriminate more than simple type theory. This simplification removes the need for the axiom of reducibility, which was introduced to allow generalizations barred by the ramifications of Russell's type theory.

Ramsey claimed that the ramifications of Russell's type theory discriminates propositional functions based on their presentation as well as properties of the functions. The presentation is not an essential property of a function, and should not affect its characterization. A variable's order is, in Peano's terminology, a "pseudo-function."[7] Ramsey distinguishes the use and mention of a sign, and thinks that the symbolic presentation of a propositional function is just an instance of the function itself, and a propositional function can be symbolized in different ways. He draws an analogy with fractions. The value $\frac{2}{3}$ can have infinitely many different numbers in its numerator without changing its value so long as the denominator's value changes accordingly; it would be wrong to characterize this number by the value in the numerator, which is merely a result of this particular reduced presentation. A propositional function of *r-type* $1/3$ takes individuals as arguments. Such a function is characterized as *r-type* $1/3$ because it quantifies over second order propositional functions (and possibly first order ones as well). According to Ramsey, this is contingent upon the particular presentation; particular instances of a function, not the function itself, have an order. Other instances of it are of different orders, just as other instances of the value of $\frac{4}{6}$ have numerators other than 4. We can think of trivial cases where characterizing a proposi-

[6][37] pg. 183
[7][37] pg. 210

tional function as of a higher order does not affect its value. For example, the function $R(x) \equiv (\forall\phi)((\phi(x) \vee \neg\phi(x)) \wedge \chi(x))$ (where $x$ ranges over individuals, $\chi$ is *r-type* 1/1, and $\phi$ *r-type* 1/2) is *r-type* 1/3, but is co-extensive with $\chi\hat{x}$. Although this quantifies over the totality of second order propositional functions, its value is not dependent on this totality. Although the functions $\chi\hat{x}$ and $R\hat{x}$ are co-extensive, we cannot say that they are intensionally equivalent.

Ramsey's third and final criticism of *Principia Mathematica* is its use of identity. Ramsey claimed that identity is misinterpreted in *Principia Mathematica*—Russell's definition does not capture what is meant by identity. Ramsey's claim is that it is logically possible for distinct objects to be indistinguishable, and a logical system must not exclude the possibility of such objects.

### 3.1.3 Ramsey's proposal

In attempting to solve the problems of *Principia Mathematica*, Ramsey is more careful than Russell to avoid violating the use/mention distinction. He notes that:

> [The] expressions 'function of functions' and 'function of individuals' are not strictly analogous; for, whereas functions are symbols, individuals are objects, so that to get an expression analogous to 'function of functions' we should have to say 'functions of names individuals'. On the other hand, there does not seem any simple way of altering 'function of functions' so as to make it analogous to 'function of individuals', and it is just this which causes the trouble. For the range of values of a function of individuals is definitely fixed by the range of individuals, an objective totality which there is no getting away from. But the range of arguments to a function of functions is a range of symbols, all symbols which become propositions by inserting in them the name of an individual. And this range of symbols, actual or possible, is not objectively fixed, but depends on our methods of constructing them and requires more precise definition.[8]

Russell makes no distinction between symbols and objects, and thus treats functions of individuals and functions of functions in the same manner.

[8] [37] pgs. 199-200

Ramsey realized that while the range of individuals is fixed, the symbols which are the range of arguments to a function of functions is not fixed, and must be defined. Russell defines this range by how functions are constructed; the domain of a function $f$ cannot include any function whose definition presupposes a totality that includes $f$. In other words, every function $f$ is constructed at some level, or stage of development; all variables occurring in the definition of $f$ must range over objects constructed at a lower level, or at a prior stage of development. In particular, the arguments to $f$ must be of a lower level. To avoid vicious circles in such a theory, Russell introduced his ramified theory of types, whose restrictions require him to introduce the axiom of reducibility. Ramsey aims to treat functions of functions similarly to functions of individuals, while noting that they are not analogous notions. Instead of determining which symbols are possible arguments based on how they are constructed, he wants to determine the symbols in the range based on the meanings of the symbols, just as the signs which are possible arguments to a function of individuals are determined based on their meaning as the names of individual objects. The meanings of function symbols result from the propositions which are their values when supplied with arguments. This method also allows us to include propositions which are meaningful but which we do not know how to construct.

A key defined term in Ramsey's theory is a *predicative function*, which has a different meaning than that given by Russell. Ramsey's definition states: "A *predicative function* of individuals is one which is any truth-function of arguments which, whether finite or infinite in number, are all either atomic functions of individuals or propositions."[9] He notes that "Before 'propositions' we could insert 'atomic' without narrowing the sense of the definition. For any proposition is a truth-function of atomic propositions, and a truth-function of a truth-function is again a truth function."[10] Ramsey claims that this includes more functions of individuals than any range in Russell's system, for it includes both what Russell, in the first edition of *Principia Mathematica* calls *predicative functions* (these are functions which Ramsey calls *elementary functions*—functions whose values are truth-functions of finitely many atomic propositions) and also functions with an infinite number of arguments, which Russell does not allow. The range of allowable arguments to a function is not determined by the way the function is defined or con-

[9] [37] pg. 202
[10] [37] pg. 202, fn. 1

structed, but by the meaning of the definition. He says that he will "call such functions 'predicative' because they correspond, as nearly as a precise notion can to a vague one, to the idea that $\phi a$ predicates the same thing of $a$ as $\phi b$ does of $b$."[11]

Ramsey's goal is to deal with functions of functions in a similar manner. A predicative function of functions is a truth-function of the values of its arguments and constant propositions. Given this definition, it is only the values of a function which are relevant in determining the values of a predicative function which may contain it. Generalization over a variable can never create a non-predicative function, because generalizing a truth-function just groups together its instances. If we start with a predicative function, we cannot generalize to form a non-predicative one. Russell's justification of the axioms of *Principia Mathematica* still stands even if we restrict our attention to the functions of individuals which are predicative in Ramsey's sense. We can generalize over such functions which an individual satisfies, so there is no need for an axiom like Russell's axiom of reducibility.

Ramsey has defined the range of functions not by what we, as humans, can express, but "by how the facts their values assert are related to their arguments."[12] Generalization neither forms non-predicative functions nor leads to vicious circles. Although a propositional function may seem to presuppose some totality of propositional functions of which it is a possible member, this is a result of how we have described the propositional function, not a vicious circle. Any value of the propositional function is some proposition, whose meaning does not presuppose a totality of functions. For example, $(\phi).\phi a$ asserts all propositions of the form $\phi a$, and is itself a member of that totality. However, we have only expressed it as a universal generalization because of the empirical impossibility of writing a conjunction of infinite length. There is no vicious circle, but use of a convention for expressing the logical product of a set of propositions. Similarly, we may talk about the tallest man in a room, which refers to him by reference to a totality containing the man.

### 3.1.4 Functions in extension

So far we have only considered functions of individuals which are predicative. As mentioned above, Ramsey disagreed with Russell's take on identity

[11] [37] pg. 212-213
[12] [37] pg. 205

in *Principia Mathematica*. Ramsey believed that two objects could satisfy all of the same atomic functions, and hence all predicative ones, but still be different objects. To account for this, he introduces non-predicative functions. In order to introduce such functions, Ramsey makes them entirely extensional, "[dropping] altogether the notion that $\phi a$ says about $a$ what $\phi b$ says about $b$".[13] A function in extension of individuals is an arbitrary correlation between an individual and a proposition. For example:

> $\phi$(Socrates) may be Queen Anne is dead,
>
> $\phi$(Plato) may be Einstein is a great man;
>
> $\phi\hat{x}$ being simply an arbitrary association of propositions $\phi x$ to individuals $x$.[14]

We can note that a function is a function in extension by writing it as $\phi_e$. Predicative functions are a subset of functions of extension. It is important to distinguish this subset for higher-order cases, as we shall see.

Ramsey claimed that Wittgenstein's suggestion to remove the $=$ sign from the logical system and use the convention that different symbols stand for different objects does not allow us to define classes not definable by a predicative function. Ramsey claimed that the introduction of non-predicative functions in extension allows him to express that two symbols stand for the same object, or satisfy all of the same functions, which is necessary to define all classes. It solves the problem of Russell's approach, for quantifying over all functions in extension of individuals includes *all* correlations between an individual and a proposition, not just those functions which correspond to describing some property. We can define $x = y$ to mean:

$$(\phi_e).\phi_e x \equiv \phi_e y$$

This is a tautology when $x = y$, and a contradiction otherwise; this definition of identity guarantees that $x$ and $y$ satisfy *all* of the same functions, not just predicative ones.

Introducing functions in extension also solves the first fault Ramsey identified within *Principia Mathematica*—the issue of undefinable infinite classes. When considering functions of individuals, the range of functions to consider is the functions of extension. Any class will be defined by some function in

[13] [37] pg. 215
[14] [37] pg. 215

extension (although we may not be able to describe it), and the totality of classes is the totality of functions in extension. This means that we include all classes, even those whose members do not have some property in common.

### Functions of functions

The case is different with functions of functions, and Ramsey shows there is no need to include functions of extension in their range, for predicative functions suffice. We do not need to consider identity in such cases—identity of classes defined by functions of functions can be reduced to the issue of functional equivalence. While it is necessary to distinguish different individuals which share all properties in order to define what it means for individuals to be identical, there is no need to define what it means for functions to be identical. Instead, we can consider identity of classes, or equivalence between the functions defining the classes. The logical equivalence of functions can be defined without the introduction of arbitrary extensional functions.

Another reason why predicative functions of functions suffice as the range of functions of functions is that we need not consider classes of functions, for instead we can consider classes of classes. Although we require extensional functions of individuals to define all classes of individuals (which includes those without a property in common), we can define any class of classes with a predicative function, although it may not be finitely expressible. Here is such a predicative function $f$:[15]

$$f(\phi_e \hat{x}) = \Sigma_\psi(\phi_e x \equiv_x \psi_e x).$$

Here $\psi \hat{x}$ in the summation ranges over exactly those functions in extension which make $f$ true. The extensional functions of individuals $\psi_e \hat{x}$ define any class, even those without a common property. We can define a class of such classes (which may have no common property) by taking the logical sum of functions equivalent to functions $\psi_e \hat{x}$; each function $\psi_e \hat{x}$ defines a class which is an element of this class of classes.

Limiting ourselves to predicative functions of functions includes all of the functions of *Principia Mathematica*. These functions are extensional as defined in *Principia Mathematica*—if the range of $f(\hat{\phi}\hat{x})$ is that of predicative function of functions, it is a truth-function of the values its arguments, so we have that:

$$\phi_e x \equiv_x \psi_e x :\supset: f(\phi_e \hat{x}) \equiv f(\psi_e \hat{x}).$$

[15][37] pg. 218

Ramsey states that by treating functions of individuals and functions of functions differently—the former's range being functions in extension and the latter's being predicative functions—he obtains a complete theory of classes and the general system of *Principia Mathematica.* This is in keeping with his general goal of reinterpreting Russell's system:

> And in thus preserving the form while modifying the interpretation, I am following the great school of mathematical logicians who, in virtue of a series of startling definitions, have saved mathematics from the skeptics, and provided a rigid demonstration of its propositions. [16]

## 3.2 Russell's response to Ramsey

In a 1931 review in *Mind* and in *My Philosophical Development*, Russell commented on Ramsey's analysis of *Principia Mathematica.* In the journal review, he outlines *The Foundations of Mathematics.* He agrees with Ramsey's point that logic should include only tautologous generalizations. He also admits the first two faults Ramsey finds with his work: the assumption that all classes can be defined by propositional functions and that distinguishing the two sorts of contradictions allows type theory to be greatly simplified. In the second edition of *Principia Mathematica*, Russell was not satisfied with his proposal, for retaining his type theory without the axiom of reducibility prevents analysis dependent upon Dedekind sections. Ramsey's alternative solves this problem by proposing a simple type theory. Russell defended Ramsey's introduction of functions in extension, realizing that although we cannot give examples of such functions, they still occur when we generalize over all functions.

However, Russell disagreed with Ramsey's criticism of the treatment of identity within *Principia Mathematica.* Their disagreement comes down to a disagreement regarding the identity of indiscernibles—Ramsey believed it was possible for two things to have all properties in common, while Russell believed that by definition two things must differ in regards to some property.

[16] [37] pg. 219

**_My Philosophical Development_**

In his philosophical autobiography, Russell discussed Ramsey's criticism in more depth. He stated that Ramsey's main point—"that mathematics must be rendered purely extensional and that the troubles of the *Principia* arose from an illegitimate intrusion of an intensional point of view"[17]—should be further studied. He did not got so far as to agree with the argument, but noted that it is worthy of consideration. Russell also claimed to be undecided regarding Ramsey's introduction of functions of extensions.

> I feel that a correlation of entities to propositions which is wholly arbitrary is unsatisfactory. Take, for example, the inference from '$fx$ is true for all values of $x$' to '$fa$'. With Ramsey's explanation of the concept '$fx$' we cannot tell what '$fa$' may be. On the contrary, before we can know what '$fx$' means, we have to know '$fa$' and '$fb$' and '$fc$' and so on, throughout the whole universe. General propositions thus lose their *raison d'être* since what they assert can only be set forth by enumeration of all the separate cases. Whatever may be thought of this objection, Ramsey's suggestion is certainly ingenious, and if not a complete solution of the difficulties, is probably on the right lines. Ramsey himself had doubts.[18]

In both cases above, Russell held back from fully endorsing Ramsey's views, but takes them to be worthy criticism. Regarding the theory of types, Russell fully endorsed Ramsey's views. Russell stated that the distinctions between order are unnecessary, and categorizing propositional functions based on the simple type hierarchy suffices to avoid the paradoxes. Since the axiom of reducibility was only introduced to allow generalizations prevented by Russell's ramified theory of types, neither it nor an alternative axiom is necessary.

## 3.3 Other responses to the second edition

### Church

We saw that in his 1976 paper "Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski," Church defends the axiom of

[17][38] pg. 91
[18][38] pg. 93

reducibility, noticing that it claims the existence of a co-extensive predicative function within an intensional logic, and hence does not undo the distinctions of type theory. In a 1928 review of the second edition of *Principia Mathematica*, he states disagreement with the proposed changes of the second edition:

> If for no other reason than that it leads to important results to which the other does not, the axiom of reducibility seems to be distinctly preferable to the postulate which Whitehead and Russell now propose as a substitute for it.[19]

He believed that if Russell wants to retain his type theory, he should also retain the axiom of reducibility, which allows analysis in a ramified type theory. In his 1984 paper, he stated:

> If, following the early Russell, we hold that the object of an assertion or a belief is a proposition and then impose on propositions the strong conditions of identity which this requires, while at the same time undertaking to formulate a logic that will suffice for classical mathematics, we therefore find no alternative except ramified type theory with axioms of reducibility, and with axioms and axiom schemata 1-15 appropriately modified.[20]

The axioms and axiom schemata Church mentions are his axiomatization of Russell's theory. The strict identity conditions he mentions are those of the first edition of *Principia Mathematica*, where two propositions with the same truth value are not identical, for an intensional function may have different values when given the propositions as arguments.

### Bernard Linsky

Bernard Linsky argues that the principle of extensionality limits the ramification of type theory. The first edition of *Principia Mathematica* includes such extensive ramification of propositional functions because propositional functions are characterized by their intension. However, if a function only depends upon the extensions of its arguments, the need for such ramification is limited. The distinction between identity and equivalence only arises in

[19] [8] pg. 240
[20] [9] pg. 521

an intensional logic; in an extensional one, two objects which are formally equivalent, or co-extensive, are identical. Bernard Linsky makes the point that the theory of types is affected by the principle of extensionality:

> With the principle of extensionality anything true of one propositional function will be true of every coextensive one, so the only thing on which a propositional function can depend is its extension, and so the type theory will not be so extensively ramified. A function may be identified by the functions used to define it, but that really means only on the extensions of the functions used to define it.[21]

Linsky emphasizes the intensionality of Russell's system. He believes that the full ramified theory is necessary to "capture intensional phenomena."[22] He believes that in the first edition of *Principia Mathematica* all propositional functions, including predicative ones, are intensional, and the predicative ones are a subset which act as a theory of classes. These predicative propositional functions are sufficient to define all the properties of objects in the world, but the higher-order propositional functions still have a purpose: "Higher type propositional functions do not really introduce new properties of things. They may characterize new ways of thinking of or classifying them, but they do not introduce any new real universals."[23] Linsky believes that the changes introduced for the second edition—denying the intensional aspects of the system by accepting the principle of extensionality—was too extreme and undid what made *Principia Mathematica* such an important work: namely that it provided a system which both "captured intensional phenomena" and provided a logical system to which mathematics could be reduced.

### Other reviews

In 1926, B.A. Bernstein reviewed the second edition of *Principia Mathematica.* His main criticism was one which was made about both editions of the work: "the distinction between propositional logic as a mathematical system and as a language must be made."[24] In a 1931 review, Rudolf Carnap

[21] [26] pg. 97
[22] [26] pg. 101
[23] [26] pg. 106
[24] [2] pg. 713

notes that much of the generality of *Principia Mathematica* is lost by the assumption of extensionality in the second edition.[25] Russell's original goal was to give a system more general than just the extensional domain of mathematics but, with the principle of extensionality, he limits his domain to an extensional one.

In a 1936 paper Quine argued that the axiom of reducibility and the principle of extensionality together undo the ramification of Russell's type theory. However, we should note Irving Copi's point that Russell never simultaneously defended the axiom of reducibility and the principle of extensionality.[26] Quine and Gödel agreed with Ramsey, and later Russell, that simple type theory is sufficient to avoid the paradoxes. The necessity of the axiom of reducibility is no longer an issue without a ramified theory of types.

[25] [16] pg. 514

[26] [11] pg. 103

# Concluding Remarks

We can summarize the historical developments discussed in this thesis as follows. *Principia Mathematica* was an ambitious attempt to culminate the logicist goal. The work was successful to some degree—it outlined a system which reduced mathematics to basic axioms of a logical character. However, Russell did not provide a formal syntactic specification of the system, and this flaw led to some looseness in his arguments. Also, the first edition relied upon the axiom of reducibility, which both Russell and critics thought compromised the system as a logical one.

In hopes of retaining the overall system, Russell discarded this axiom for the second edition, and instead assumed a principle of extensionality. This alternative was not fully satisfactory to either Russell or to critics. Not only did the system no longer allow methods of analysis, but it compromised much of the generality of the first edition by limiting propositional functions to extensional ones, and no longer capturing intensional phenomena as the first edition did.

Another alternative, proposed by Ramsey and eventually accepted by Russell, was to simplify the theory of types, and hence remove any need for the axiom of reducibility. This simplified system no longer characterizes propositional functions by their definition, and allows analysis and other methods of mathematics without requiring a means to reduce the order of propositional functions (for functions no longer have an order). This view does not preclude that the full ramified theory may be necessary to capture all intensional phenomena; however, it assumes that such intensional phenomena are not inherent in propositional functions, but only in their representation, and that these intensional phenomena need not be considered as a part of mathematics.

# Bibliography

[1] Bernays, P. (1930). *Die Philosophie der Mathematik und die Hilbersche Beweistheorie.* 2003 Revision of P. Mancosu and I. Mueller translation by S. Awodey, B. Buldt, D. Schlimm, and W. Sieg.

[2] Bernstein, B. (1926). Review of *Principia Mathematica Second Edition. Bulletin American Mathematical Society 32.* 711-713.

[3] Carnap, R. (1947). *Meaning and Necessity.* The University of Chicago Press, Chicago.

[4] Chihara, C. (1973). *Ontology and the Vicious-Circle Principle.* Cornell University Press, Ithaca, NY.

[5] Church, A. (1974). Russellian Simple Type Theory. *Proceedings and Addresses of the American Philosophical Association 47.* 747-760.

[6] Church, A. (1956). *Introduction to Mathematical Logic,* Vol. I. Princeton University Press, Princeton, NJ.

[7] Church, A. (1976). Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski. *The Journal of Symbolic Logic Vol 41.* 747-760.

[8] Church, A. (1928). Review of *Principia Mathematica Second Edition. Bulletin American Mathematical Society 34* 237-240.

[9] Church, A. (1984). Russell's Theory of Identity of Propositions. *Philosophia Naturalis 21.* 513-522.

[10] Cocchiarella, N. (1989). Russell's Theory of Logical Types and the Atomistic Hierarchy of Sentences, in *Rereading Russell: Essays on*

*Bertrand Russell's Metaphysics and Epistemology.* Savage. C.W. and Anderson, C.A., ed. Minnesota Studies in the Philosophy of Science, Volume XII, University of Minnesota Press, Minneapolis, 41-62.

[11] Copi, I. (1971). *The Theory of Logical Types.* Routledge and K. Paul, London.

[12] Feferman, S. 2000. The Significance of Herman Weyl's *Das Kontinuum*, in *Proof Theory.* F. Hendricks, et al., eds. Kluwer Academic Publishers, Dordrecht, 179-194.

[13] Gödel, K. (1944). Russell's Mathematical Logic, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 113-134.

[14] Goldfarb, W. (1979). Logic in the Twenties: The Nature of the Quantifier. *The Journal of Symbolic Logic Vol. 44.* 351-368.

[15] Goldfarb, W. (1989). Russell's Reasons for Ramification, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 232-249.

[16] Grattan-Guinness, I. (2000). *The Search for Mathematical Roots, 1870-1940.* Princeton University Press, Princeton.

[17] Hylton, P. (1990). *Russell, Idealism, and the Emergence of Analytic Philosophy.* Clarendon Press, Oxford.

[18] Hylton, P. (1990). Logic in Russell's Logicism, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 196-231.

[19] Irvine, A.D. (1989). Epistemic Logicism and Russell's Regressive Method, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 172-195.

[20] Kleene, S. (1964). *Introduction to Metamathematics.* North Holland Publishing Company, Amsterdam.

[21] Landini, G. (1998). *Russell's Hidden Substitutional Theory.* Oxford University Press, NY.

[22] Lewis, C.I. (1928). Review of *Principia Mathematica Second Edition. American Mathematical Monthly 35.* 200-205.

[23] Lewis, C.I. and Langford, C.H. (1959). *Symbolic Logic.* Dover Publications, New York.

[24] Linsky, B. (1988). Propositional Functions and Universals in *Principia Mathematica. Australisian Journal of Philosophy Vol. 66.* 447-460.

[25] Linksy, B. (1990). Was the Axiom of Reducibility a Principle of Logic?, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 250-264.

[26] Linsky, B. (1999). *Russell's Metaphysical Logic.* CSLI Publications, Stanford, CA.

[27] Linksy, L. (1987). Russell's "No-Classes" Theory of Classes, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 285-304.

[28] Linksy, L. (1983). *Oblique Contexts.* The University of Chicago Press, Chicago.

[29] Mancosu, P. (1999). Between Russell and Hilbert: Behmann on the Foundations of Mathematics *The Bulletin of Symbolic Logic Vol. 5. Number 3* 303-330.

[30] Mancosu, P. (1998). *From Brouwer to Hilbert.* Oxford University Press, New York.

[31] Muehlmann, R. (1969). Russell and Wittgenstein on Identity. *Philosophical Quarterly 19.* 221-230.

[32] Myhill, J. (1979). A Refutuation of an Unjustified Attack on the Axiom of Reducibility, in *Bertrand Russell Memorial Volume.* Roberts, G., ed. Humanities Press, NY, 1979, 81-90.

[33] Quine, W.V. (1936). On the Axiom of Reducibility. *Mind 45.* 498-500

[34] Quine, W.V. (1966). Russell's Ontological Development. *The Journal of Philosophy 63.* 657-667.

[35] Quine, W.V. (1963). Russell's Theory of Types, in *Essays on Bertrand Russell.* Klemke, E.D., ed. University of Illinois Press, IL, 1970, 372-387.

[36] Quine, W.V. (1938). On the Theory of Types, in *Bertrand Russell: Critical Assessments Volume II: Logic and Mathematics.* Irvine, A.D., ed. Routledge, NY, 1999, 78-95.

[37] Ramsey, F. (1990). *Philosophical Papers.* Mellor, DH, ed. Cambridge University Press, Cambridge.

[38] Russell, B. (1959). *My Philosophical Development.* Routledge, NY.

[39] Russell, B. (1919). *Introduction to Mathematical Philosophy.* George Allen and Unwin Ltd, London.

[40] Russell, B. (1908). "Mathematical Logic as Based on The Theory of Types" in *Logic and Knowledge: Essays 1901-1950*, Robert Marsh, ed. The Macmillan Company, NY, 1956.

[41] Russell, B. (1931). Review of *The Foundations of Mathematics and other Logical Essays. Mind 40.* 476-482.

[42] Sieg, W. (1999). Hilbert's Programs: 1917-1922. *The Bulletin of Symbolic Logic Vol. 5. Number 1.* 1-44.

[43] Weyl, H. (1918). *The Continuum.* Dover Publications, Mineola, NY, 1994.

[44] Whitehead, A.N. (1926). Letter to the Editor of *Mind. Mind 35.* 130.

[45] Whitehead, A., Russell, B. (1962). *Principia Mathematica to *56.* The Syndics of the Cambridge University Press, NY.

[46] Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus.* Dover Publications, Mineola, NY, 1999.