Gnomes in the Fog: The Reception of Brouwer's Intuitionism in the 1920s
by Dennis E. Hesseling
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Reviewed by Jeremy Avigad

The early twentieth century was a lively time for the foundations of mathematics. This ensuing debates were, in large part, a reaction to the set-theoretic and nonconstructive methods that had begun making their way into mathematical practice around the turn of the twentieth century. The controversy was exacerbated by the discovery that overly naïve formulations of the fundamental principles governing the use of sets could result in contradictions. Many of the leading mathematicians of the day, including Hilbert, Henri Poincaré, Émile Borel, and Henri Lebesgue, weighed in with strong views on the role that the new methods should play in mathematics. Dennis E. Hesseling's book, *Gnomes in the Fog*, documents reactions to the "crisis of foundations" that was inaugurated by the attempts of L.E.J. Brouwer's to re-found modern mathematics on "intuitionistic" principles.

Brouwer's interests in foundational issues extend at least as far back as his 1907 doctoral dissertation at the University of Amsterdam. In that work, he tried to ground mathematics, especially the mathematics of the continuum, on the basis of a priori intuition of time together with processes of mental construction. According to Brouwer, such constructions precede language, which provides only a flawed means of communication *ex post facto*. Since logic is typically presented as a collection of linguistic principles, Brouwer took logic, as well, to be secondary to mathematical understanding. In doing so, he countered tendencies, seen in Russell and Hilbert, to view mathematics as based essentially on logic and language. Brouwer's dissertation also includes discussion of the law of the excluded middle and the nature of mathematical existence, two topics that, as Hesseling makes clear, were to become central to the foundational debates later on.

It was, however, in the newly emerging field of topology that Brouwer first made a name for himself. Between the years of 1909 and 1913, he not only clarified basic terminology and helped put the subject on a rigorous foundation, but also introduced many of the field's central methods. His results on fixed-points of continuous transformations, degrees of mappings, and invariance of dimension were groundbreaking and seminal. But issues having to do with the grounding of point-set topology in set-theoretic terms were closely related to foundational issues in his dissertation. He returned to these themes in 1918, with a paper, "Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten" ["Foundation of set theory independent of the logical law of the excluded middle"]. Here he put forth a radical alternative to Cantorian set theory, in conformance with his intuitionistic views of mathematics, based on the notion of a "choice sequence." In 1923, he developed an intuitionistic theory of real-valued functions, which showed how notions of continuity, measurability, and differentiability could be developed in intuitionistic terms. A remarkably clear and concise introduction to Brouwer's philosophical and foundational contributions can be found in Mark van Atten's *On Brouwer* [2].

The transition to twentieth century mathematics involves two important tendencies:

1. the shift of focus from symbolic expressions to abstract objects; and
2. the use of structural, set-theoretic, and infinitary methods to describe such objects, operations on them, and their properties.

Brouwer was by no means rejecting the first tendency, or advocating a return to the algorithmic sensibilities of the nineteenth century. We have already noted his emphasis on mental constructions and skepticism towards the symbolic or linguistic devices used to describe them. Neither did he reject the second tendency; his 1927 proof of the "bar theorem" used transfinite induction, and he treated proofs as infinitary objects on which mathematical operations can be performed. But, on Brouwer's view, the proper methods of *doing* mathematics were to reflect an appropriate "constructive" or "intuitive" understanding. Between 1925 and 1934, despite Brouwer's antipathy towards formalism, various aspects of intuitionistic reasoning received formal treatments in the hands of Kolmogorov, Glivenko, and Brouwer's own student, Heyting. With the appearance of formal definitions of computability in the 1930's, it was not long before mathematical logic had uncovered connections between intuitionistic mathematics and computability, providing ways in which various intuitionistic methods can be understood in computational terms. Among modern practitioners of constructive mathematics, such computational aspects of the practice tend to be emphasized more often than Brouwer's philosophical doctrines. (See, however, [2, Chapter 5] for an aspect of Brouwer's intuitionism that cannot be understood in computational terms.)

In any event, whatever the philosophical presuppositions, Brouwer's methodological proscriptions would have constituted a radical shift in mathematical practice. Coming from a mathematician of Brouwer's stature, this challenge to set-theoretic foundations could not be ignored. But, as Hesseling observes, Brouwer's challenge didn't become a movement until Hermann Weyl's publication of "Über die neue Grundlagenkrise der Mathematik" ["On the New Foundational Crisis in Mathematics"] in 1921. (This and related foundational essays are translated and gathered in [1].) Weyl, a student of Hilbert's, has made lasting contributions to diverse branches of mathematics, including function theory, analytic and algebraic number theory, representation theory, and mathematical physics. From an early stage in his career, he was influenced by Edmund Husserl's phenomenology, and traces of this influence can be discerned in his landmark 1918 treatment of general relativity in *Raum, Zeit, Materie* [*Space, Time, and Matter*]. In that same year, he published a work, *Das Kontinuum* [*The Continuum*], in which he developed a foundational approach to replace the contemporary set-theoretic foundations of analysis. Referring to the latter, he wrote that "every cell... of this mighty organism is permeated by the poison of contradiction and ... a thorough revision is necessary to remedy the situation." In 1921, however, he proclaimed "I now renounce my attempt and join Brouwer's," and joined forces with the new "revolution."

A good deal has been written about Brouwer, including a recent biography [3] by van Dalen. In *Gnomes in the Fog*, Hesseling has made the interesting choice of documenting *reactions* to Brouwer's manifestos rather than focusing on Brouwer himself. He has been extraordinarily thorough. The introduction tells us that he has analyzed more than 1,000 primary sources, including published papers in mathematical and non-mathematical journals, newspaper articles, correspondence, and unpublished manuscripts, drawing on archives from all over Europe. An appendix lists more than 250 published works, almost all of which appeared between 1921 and 1933. Many of these works are discussed explicitly in the text.

Hesseling draws some interesting conclusions. At a symposium on the foundations of mathematics in Königsberg in 1930, Rudolf Carnap, Heyting, and Johann von Neumann presented papers on logicism, intuitionism, and formalism, respectively. Ever since then, there has been a tendency to characterize the "crisis of foundations" as a struggle among these three. Hesseling notes, however, that in the range of foundational writings he considered, logicism, as characterized by Carnap, plays at best a minor role. Furthermore, Hesseling argues that formalism, as characterized by von Neumann, was not a clearly articulated position from the start, but, rather, was gradually shaped in response to intuitionistic challenges.

Hesseling's three opening chapters serve to provide background, to describe Brouwer's work, and to present the opening salvos of the debate. His last, somewhat speculative chapter, explores connections between intuitionism and analogous political and cultural currents. Two chapters in between are entitled "Reactions: existence and constructivity" and "Reactions: logic and the excluded middle." These aim primarily to summarize the reactions to intuitionism found in the corpus of documents that Hesseling has analyzed. The chapter titles are somewhat misleading: it would be a mistake to draw a sharp distinction between the intuitionistic views of existence statements (formally, $\exists x\ \varphi$) and the law of the excluded middle (formally, $\varphi \vee \neg\varphi$), since the two are intertwined. The two chapters, rather, aim to separate metaphysical questions from logical ones. From a metaphysical point of view, one is concerned with the task of giving a general account of mathematical knowledge and mathematical objects, while from a logical perspective, one is concerned with identifying the proper principles of mathematical reasoning. These are certainly related; one would expect a metaphysical stand to have bearing on the proper principles of reasoning, and, conversely, one would expect logical principles to be justified by reference to a metaphysical view. It is, nonetheless, often useful to keep these two foci distinct.

Despite Brouwer's misgivings about formalism, mathematical logic proved to be a valuable tool in clarifying the differences between intuitionistic and classical practice. Although he is occasionally imprecise with the details, Hesseling does an able job of documenting these developments. For example, formalizations of intuitionistic logic show that the principle $\neg\neg\varphi \rightarrow \varphi$ of "double-negation elimination," which allows one to prove an assertion by showing that its negation implies a contradiction, is equivalent, as a schema, to the law of the excluded middle, $\varphi \vee \neg\varphi$. Instances of the latter can also be used, together with intuitionistic logic, to prove $\neg\forall x\ \varphi \rightarrow \exists x\ \neg\varphi$. This provides a similarly indirect means of proving an existence statement. In fact, classical first-order logic can be characterized as the result of adding the law of the excluded middle to intuitionistic logic. Various "double-negation translations," from Kolmogorov and Glivenko to Gödel and Gerhard Gentzen, provide ways in which classical forms of reasoning can be interpreted in intuitionistic terms. The differences between Brouwer's proposed methods and those sanctioned by set theory, however, went beyond the axioms and rules governing the logical connectives; set theory sanctions nonconstructive definitions of infinitary sets and sequences that are intuitionistically forbidden. Here, too, logical analysis helped clarify Brouwer's intuitionistic methods, and their contrast to set-theoretic ones.

Hesseling's account of the progress that was made towards clarifying the metaphysical differences between the two conceptions is less satisfying. It is notoriously difficult to find a clear, rational basis from which to address questions having to do with mathematical meaning and existence, since the very project presupposes a general conception of meaning and existence with respect to which one situates the mathematical versions. Since there are no uniformly accepted candidates for the former, one is left addressing weighty issues without a firm ground to stand on. But one can at least try to be clear about one's presuppositions and the philosophical framework on which one's analysis is based, and such clarity and precision is absent from Hesseling's narrative. To be sure, the fault often lies with the original authors; but even the most carefully articulated position sounds flimsy when summarized in a few sentences, and the benefit of hindsight should have afforded a clearer articulation of the philosophical issues at stake.

Hesseling's overview does manage to convey a sense of the issues. Whereas the intuitionistic notion of existence is variously described in terms of definability, describability, algorithmic computability, intuitive construction, or phenomenological experience, the notion of classical existence is typically understood as somehow "independent" of all these. So, the general philosophical problem for classical logic was taken to be finding some sort of justification for the associated knowledge claims. In the debates of the 1920's, one sees traces of the idea, later to become a central part of Quine's views, that theoretical statements are to be justified holistically rather than at the sentential level. In particular, classical existence claims are to be justified in terms of global properties, like consistency, of the theory in which they are a part, rather than in terms of a more "local" analysis of their meaning.

Hesseling's presentation also falls short in its analysis of the mathematical context in which the debates took place. Brouwer's work in topology merits only a brief discussion, with the conclusion that "Brouwer's dissertation and his topological work mostly follow naturally from the same basic principles." It would have been nice to have a better sense of how Brouwer's treatment of the continuum, for example, compared to that of rival foundational approaches, and the substantive mathematical issues the different developments were meant to address. Similarly, it would have been nice to have more insight into the relationship between Weyl's core mathematical work and his foundational and philosophical views. It is important to keep in mind that Brouwer's critique cut to the core of what it means to do mathematics, and was designed to bear upon the day-to-day practice of every working mathematician. Separating the critique from an understanding of how it affects that practice gives the debate the character of a rhetorical exercise, belying its pragmatic relevance to a subject that plays such a central role in human thought and experience.

In sum, Hesseling has aimed for breadth rather than depth in his analysis. The result is an interesting sociological study of the cultural processing of new ideas in an important scientific discipline. The work also provides a thorough survey of historical data that should be used to support a better mathematical and philosophical understanding of the foundational issues. It would be unfair to be overly critical of Hesseling for not making further progress towards developing such an understanding, when the mathematical and philosophical communities, themselves, have not done much better. One gets the sense that, over time, mathematicians simply got used to the new set-theoretic methods, and, finding them convenient, grew tired of debate as to whether they are appropriate to mathematics. While this was going on, philosophers of mathematics, understandably reluctant to make declarations as to the proper practice of mathematics, honed the ability to frame issues in such a way that philosophical analyses can't *possibly* have any bearing on what mathematicians actually do. These tendencies towards apathy, on the one hand, and irrelevance, on the other, have been effective in severing substantial communication between the two communities. The sharp disciplinary separation has, in turn, reinforced these tendencies.

Readers of Hesseling's book are likely to be left with conflicting emotions. After seeing the methodological and foundational confusions of the early twentieth century recounted at length, it is hard not to feel a sense of relief that these days are now behind us; coupled, perhaps, with a touch of pity for the mathematicians and philosophers who had to struggle through them. But reflecting on the historical development might also shake us out of our complacency, and following the tortuous path that has led to our current understanding might lead us to wonder whether alternative paths may better suit our purposes. To be sure, the modern set-theoretic methods for reasoning about the mathematical objects have become central to mathematical thought. But we can imagine that one day mathematical sensitivities may evolve so that questions having to do with symbolic representations and algorithms once again form the core of the subject. Or, perhaps, a better synthesis will make it possible to enjoy the heuristic value of modern conceptual methods while preserving the subject's algorithmic content. Mathematicians of the future may well look back at the philosophical and methodological confusions on the early twenty-first century and take pity on *us*, just as we pity our forebears.

Lacking a time machine, we can only speculate as to what the future will bring. But we also play a role in shaping that future, and so it is important that we engage in the subject with an appropriately reflective attitude. The merit of historical works like Hesseling's is that, by bringing our preconceptions to the fore, they help us understand them better.

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References

[1] Paolo Mancosu, editor. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's.* Oxford University Press, Oxford, 1998.

[2] Mark van Atten. *On Brouwer.* Wadsworth Publishing, Belmont, California, 2003.

[3] Dirk van Dalen. *Mystic, Geometer, and Intuitionist: The Life of L. E. J. Brouwer.* Two volumes. Oxford University Press, New York, 1999 and 2005.