Plato's ghost: The modernist transformation of mathematics
by Jeremy Gray
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reviewed by Jeremy Avigad

It only takes a few minutes on Amazon.com or MathSciNet to make the case that Jeremy Gray is among the most prolific historians of mathematics working today. Winner of the 2009 AMS Albert Leon Whiteman Prize for notable exposition and exceptional scholarship in the history of mathematics, his books and articles, and the many collections of essays that he has edited, cover just about every aspect of mathematics in the nineteenth- and early-twentieth centuries. It is therefore no small assertion to say that the book under review, *Plato's ghost*, is his most far-reaching and ambitious work to date.

Gray's goal is to clarify the sense in which modern mathematics is "modern," and explore the historical process by which the subject attained that character. This goal is set out in the opening words of the introduction:

> In this book I argue that the period from 1890 to 1930 saw mathematics go through a modernist transformation. Here, modernism is defined as an autonomous body of ideas, having little or no outward reference, placing considerable emphasis on formal aspects of the work and maintaining a complicated—indeed, anxious—rather than a naïve relationship with the day-to-day world...

This is about as close to a definition of "modernism" as Gray provides, but the rest of the introduction does manage to fill out the picture considerably. In art history, the term "modernism" is used to characterize a cultural movement in the late nineteenth and early twentieth century, with a strong tendency towards abstraction, which self-consciously aimed to distance itself from enlightenment views and values. In a review in the *New Yorker* (February 23, 2009), Louis Menand conveys a sense of the movement as follows:

> How ... did people like Picasso and Joyce change the game? They did it by shifting interest from the what to the how of art, from the things represented in a painting or a novel to the business of representation itself. Modern art didn't abandon the world, but it made art-making part of the subject matter of art...

> Modernism was formally difficult and intellectually challenging.
> Its thrills were not cheap.

This description comes fairly close to characterizing what Gray takes to be "modern" about modern mathematics as well. Many of his central themes have to do with formal aspects of mathematics, and he pays particular attention to developments in logic and our understanding of language, the rise of the axiomatic method, and mathematical and philosophical attempts to come to terms with the nature of mathematical reasoning itself. At the same time, he is keenly interested in the way modern mathematics gradually broke free of its empirical moorings, as, for example, one ceased to view geometry as the study of space but rather as an exploration of the many structures that could possibly serve as useful representations of space; and in the increasing focus on these representations, rather than what was being represented. By the middle of the twentieth century, it had become common to view mathematics as the study of abstract structures that stand independent of the empirical world but yet, paradoxically, play an essential role in our scientific theorizing. Gray ably traces the evolution of this viewpoint from the incipient glimmerings in the nineteenth century to its mature form.

There are many ways to write the history of mathematics. Even if one felt that history proper should just be an assemblage of bare, unadorned facts (a view which Gray does *not* subscribe to), one would still have to make choices as to *which* facts are relevant or important. One can comb through archival material and letters to determine who first proved what theorem when, where an idea or method first originated, or who learned what from whom and how. One can focus on the lives of mathematicians, filling out their personalities and ambitions, and chronicling their struggles, hardships, rivalries, triumphs, and failures. One can, instead, write the history of mathematics as a self-standing history of ideas, describing the research agendas and central problems, and the ways that mathematical theories and methods developed in response to "internal" mathematical pressures. Or one can situate the mathematical ideas in a broader philosophical context, focusing on the ambient (either implicit or explicitly stated) views as to the nature and goals of the subject. With an even broader scope, one can focus on mathematics as an institution, and situate all the above in the context of the social, national, political, and economic factors that bear upon the mathematical profession and its research agendas. One can expand the circle of ideas even further, and view mathematics as a part of a broader cultural history, including developments in literature, the arts, and even theology.

The striking thing is that in this book, Gray does it all, and the focus on the development of mathematics from 1890 to 1930 is perhaps the only sense in which the narrative is constrained. The Library of Congress classification characterizes the subject matter as follows:

1. Mathematics–History–19th century.
2. Mathematics–Philosophy.
3. Aesthetics, Modern–19th century.

Even that doesn't do justice to full scope of the book, whose topics include the history of projective geometry, from the late eighteenth century "descriptive geometry" of Gaspard Monge to the algebraic perspective of Klein's *Erlangen Program*; the eighteenth century philosophical views of Immanuel Kant, post-Kantian interpreters from Herbart and Fries to Cassirer, and reactions to Kant from Frege to Helmholtz to Poincaré and Russell; the history of non-Euclidean geometry, through the work of Riemann, and its bearing of the developments on the foundations of physics, through the writings of Poincaré, Duhem, Hertz, Minkowski, and Einstein; the rival methodological approaches to algebraic number theory by Dedekind and Kronecker; the history of set theory from Cantor to the set-theoretic paradoxes, and then on to Zermelo's axiomatization; the development of the axiomatic tradition from the British algebraists through the American axiomatic school and Hilbert; developments in logic from Boole, Peirce, Frege, and Schröder to Russell and Hilbert; developments in linguistics; developments in psychology, and views on the relationship between psychology and logic, especially those of Helmholtz and Wundt; developments in the foundations of analysis, including the infinitesimals of Du Bois-Reymond and Stolz and the "five letters" between Baire, Borel, Lebesgue, and Hadamard; the early twentieth century "crisis of foundations," and the manueverings of Brouwer, Weyl, and Hilbert, as well as many of the lesser players; analogies between Cauchy's contributions to analysis and the history of music; comparisons to Catholic modernism, a late nineteenth century movement that tried to reconcile the view of the Church with post-enlightenment science; and much, much more.

Gray's focus on the period from 1890 to 1930 may at first seem odd, since historians of mathematics typically take the "birth" of modern mathematics to have occurred in the late nineteenth century. The resolution to this apparent anomaly reminds me of a joke that made its rounds in the New York Jewish community in the mid-1980's, amidst debates surrounding recent supreme court rulings on abortion. Question: according to Jewish tradition, when does a fetus become a viable human being? Answer: when it graduates from law school. Whatever the joke tells us about overbearing Jewish parents, it can also serve to remind us that the passage from birth to maturity is a tortuous process. Whereas mid-twentieth century mathematics would have been barely recognizable to most late nineteenth century mathematicians, today's mathematics would seem perfectly familiar to any post World War II mathematician. To be sure, theories have gotten more complex and proofs have gotten longer, but today's research agendas would seem familiar, and styles of argumentation are substantially the same. What Gray does in this book is chart the growth of mathematical modernism, from its first tentative steps in the nineteenth century, through its coming of age, to the point where it attained the mature character that is recognized as a hallmark of the subject today.

After a brief introduction, Chapter 1, "Modernism and Mathematics" sets the stage for the narrative to follow, setting out some of the themes and issues that will play a role. The next three chapters follow a generally chronological order. Chapter 2, "Before modernism," describes some of the first glimmerings of modern ideas, in geometry, analysis, algebra, and philosophy. This includes the development of projective geometry, the rigorization of analysis, the appearance of algebraic number theory and algebraic logic, and the neo-Kantian positions of Herbart and Fries. Chapter 3, "Mathematical modernism arrives," shows modernism attaining its mature form. We get descriptions of Klein's Erlangen program; Poincaré's work on non-Euclidean geometry; Riemann, Helmholtz, and Lie on the possible geometries of space; Cantor's theory of the transfinite; modern theories of the continuum; elements of Dedekind's structuralism; and Frege's philosophy of logic, among the many topics discussed. In Chapter 4, "Modernism avowed," modern developments are solidified and firmly anchored, making it possible for the subject to settle down to business in the new mold. We get axiomatic geometry à la Hilbert, the modern French analysis, modern set theory, modern algebra, twentieth century philosophy of mathematics (including discussions of Russell, Poincaré, Hilbert, Brouwer, Weyl, and Cassirer), and the crisis of foundations. Having traced the arc of the development in broad terms, Gray then devotes the last three chapters to exploring some aspects of the transformation in greater depth. Chapter 5, "Faces of mathematics," explores the relationship between mathematics and physics (with views of Riemann, Duhem, Poincaré, Hertz, Hilbert, Minkowski, and Einstein); theories of measurement, the continuum, and infinitesimals; and historical and popular presentations of the subject throughout the transformation. Chapter 6, "Mathematics, language, and psychology" brings linguistics and psychology into the picture. Finally, Chapter 7, "After the war," considers the modern transformation in hindsight, focusing on postwar foundational and philosophical understandings of modern mathematics.

Gray has a knack for making the mathematical ideas broadly accessible, say, to anyone with an undergraduate background in mathematics. For all that it does, his treatment is not exhaustive, and a more focused and less ambitious approach to the subject might have covered a number of topics in greater depth. Nor is it fully balanced; for example, the contents are biased towards geometry over algebra and analysis, that being a particular specialization of Gray's. But the evolving views as to the nature of geometry and its relationship to the empirical world play a very important role in the transition to the modern view, so the extra attention given to the subject is justified.

The book tells a number of stories that may not be familiar to contemporary mathematical readers, but should be, given the impact that they have had on the broader history of ideas. For example, many of us today know that Gottlob Frege famously railed against "psychologism," that is, the view that the task of logic was to describe the psychological processes that underlie the laws of thought. The very notion sounds odd today, so much so that it is hard to imagine why such a view ever seemed attractive. For that reason it is especially interesting to see people like Helmholtz and Wundt grapple with the question as to how best to study the nature of human language, knowledge, and thought. Gray manages admirably to give us a sense of the constellation of ideas at play.

Some of the works Gray describes are further out of the mainstream, with lesser impact on the modern tradition, but interesting nonetheless. For example, he provides a brief account of a little-known work of 1907, *Psychologie du nombre*, by a Frenchman, S. Santerre, which undertakes to axiomatize "facts of consciousness" and use that to ground our knowledge of arithmetic. But even the familiar stories in the history of mathematics take on new life in Gray's hands. For example, even readers who are tired of hearing of the early twentieth century "crisis of foundations" are likely to be moved by the drama of Zermelo's well-ordering proof, and the reactions found in the "five letters" of Baire, Borel, Lebesgue, and Hadamard.

One of the interesting features of the book is that Gray not only gathers all the data, but valiantly tries to impose some kind of coherent order to the sprawling assemblage of ideas. But this is not an easy task, and some of the biggest and most obvious questions are ultimately left unanswered. For example, what is the connection between mathematical modernism and the aesthetic modernist movement? Did the latter influence the former, or can they be traced to a common cause? On page 8, Gray suggests that the biological model of "convergent evolution," whereby unrelated species develop similar features in response to similar environmental pressures, may be appropriate; but, he admits, "the common features in the present case are hard to discern." Is mathematical change driven by internal values and problems, or a broader philosophical understanding of the nature and goals the practice? The book offers evidence of both. What, in the end, are we to make of the modernist transformation? Was it ultimately a good thing? The tone of the book is generally positive — the transformation generally comes across as a march towards progress rather than a descent into meaningless abstraction — but the history itself, full of heated debates and opposing viewpoints, reminds us that with any change there are both gains and losses, and that the issues are not so clear cut.

By the end of the book, one begins to suspect that there are no easy answers to the big questions. At best, we can make some progress by delimiting their scope, and the answers we get will be sensitive to how this is done. While Gray is to be commended for not oversimplifying, this conclusion might come as a bit of a let-down to those of us who like to see all our plot lines tied up in a neat package at the end, as in a murder mystery or a Victorian novel. But there is a wealth of valuable data here which, if not fully processed and pigeonholed, is at least tagged and cataloged in a helpful way. *Plato's ghost* provides an insightful and informative resource for anyone doing mathematics today who has wondered how (and perhaps why) the subject has come to possess the features it has today. The book gives us a lot to think about, which is exactly what a good history should do.

By now, we have seen postmodern philosophy and postmodern art. Should we expect to see postmodern mathematics any time soon? Who knows — but if it comes to pass, we can only hope that Gray, or someone of his breadth and insight, will be around to help us make sense of what has occurred.

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