Review of Artemov's *Explicit provability and constructive semantics*[*]

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The meaning of the intuitionistic connectives is often explained, informally, in terms of the Brouwer-Heyting-Kolmogorov interpretation: a proof of $A \wedge B$ consists of a proof of A paired with a proof of B, a proof of $A \rightarrow B$ consists of a procedure for transforming a proof of A into a proof of B, and so on. The simplicity and intuitive appeal of this explanation suggests that there should be a formal semantics lurking underneath. But, after surveying the existing semantics for intuitionistic logic (including Kripke and Beth models, algebraic and topological semantics, realizability, and various syntactic interpretations), Artemov concludes that none of them fit the bill. He then undertakes the challenge of providing one that does.

Artemov's solution involves interpreting intuitionistic propositional logic in a logic LP of propositions and proofs. Consideration of similar interpretations will provide some useful context. Many researchers in constructive logic take the Curry-Howard (or "propositions as types") isomorphism, formally represented by deductive type theories like Martin-Löf's, to offer a formal explication of the BHK interpretation. But Artemov objects that this does not go far enough: such type theories *name* the associated proof objects, but do not supply the linguistic resources to reason about them as proofs per se. As a logic of provability, Gödel's modal logic $S4$ fares better in this respect: for any proposition, p, one can also express the proposition, $\Box p$, that p is provable. Indeed, Gödel's 1933 interpretation of Heyting's propositional calculus in $S4$ shows that intuitionistic logic can be understood in terms of such a notion of provability. But Gödel also pointed out that one cannot interpret the $S4$ box operator in terms of a proof predicate for an arithmetic deductive system, since, under such an interpretation, the $S4$ theorem $\Box(\Box\bot \rightarrow \bot)$ asserts that the system can prove its own consistency. So the challenge amounts to showing how one can understand the $S4$ notion of "provability" in terms of actual "proofs."

Artemov's system LP draws from both the modal and type-theoretic approaches, and, given the constraints he has set up, provides a natural solution. Proof objects are represented by *proof polynomials*, which can be concatenated and applied to one another. The logic is classical, with the usual propositional

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variables and connectives; but whenever t is a proof polynomial and φ is a formula, $t : \varphi$ is a new proposition, intended to denote that t is a proof of φ. To interpret the $S4$ axiom $\Box p \rightarrow \Box\Box p$, Artemov introduces one additional operator: whenever t is a proof of φ, $!t$ is intended to denote a proof that t is a proof of φ. In other words, for each t and φ, LP has axioms of the form $t : \varphi \rightarrow !t : t : \varphi$.

Artemov shows that the system works as advertised. Any derivation in LP can be projected "forgetfully" to a derivation in $S4$, and, conversely, any formula that can be derived in $S4$ has an explicit realization that can be derived in LP. Together with Gödel's interpretation this shows that a formula is derivable in intuitionistic logic if and only if an appropriate realization is derivable in LP. Artemov also shows how to give LP an arithmetic interpretation relative to any standard proof predicate for, say, Peano arithmetic, and he proves soundness and completeness with respect to such interpretations. This in turn yields cut-elimination theorems for suitable sequent formulations of LP.

The historical notes and references are exceptionally thorough, and this paper will serve not only as a standard reference to the various attempts to come to formal terms with the BHK interpretation, but also more generally as a useful source of information for the semantics of intuitionistic and modal logic. Artemov is admirably clear in laying out the motivations and relevant background information, and I have little to add to his exposition.

Because Artemov takes his interpretation to offer a "semantics" for intuitionistic logic, I may, perhaps, indulge in a brief reflection on this notion. I can think of three reasons that one might seek a formal semantics for a deductive system that is already in hand: (1) one might want to explicate the meaning of the deductive formalism in terms that are intuitively prior, or independently interesting; (2) one might want to have a tool for studying the deductive system itself, such as for proving independence or establishing other metamathematical properties; or (3) one may intend the semantics itself (perhaps coupled with the associated deductive system) to have applications to areas outside logic, such as mathematics, linguistics, or computer science.

Artemov's work succeeds with respect to (1). For example, the simple reflective rules of LP (and the use of a fixed-point lemma in obtaining arithmetic interpretations) clarifies the circularity needed to make sense of the $S4$ notion of provability. With respect to (2), it is not clear whether LP can offer anything that cannot be obtained using, say, Kripke structures or cut elimination; in any event, this line is not pursued. With respect to (3), it is possible that LP, viewed as a term calculus, can help provide a foundational framework for reasoning about functional programming languages in which programs are equipped to construct pieces of their own code. Artemov raises the possibility of similar applications in the realm of formal verification. It will be interesting to see if such hopes are borne out.