Review of Arai's *Some results on cut-elimination, provable well-orderings, induction, and reflection*[*]

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The fact that this paper was originally titled "From the Attic" is strong evidence that Arai's attic is more interesting than most. The paper is a collection of results gathered over the course of a decade or so, spanning a wide range of topics in proof theory and tying up a number of loose ends. There are some new results here, but the general emphasis is on providing strengthenings and new proofs of previous results, many of them well known or folklore. Arai is always on the mark: while the paper does not break dramatically new ground, it is full of clever tricks, keen insights, satisfying observations, and nontrivial refinements, presented in a clean and elegant way.

For the most part, each of the eight sections can be read independently. Arai is to be commended for providing detailed references, contextual notes, and explanatory remarks. In the following, therefore, I will provide only a brief synopsis, glossing over many of the details and omitting citations that can be found in the paper itself.

Section 1 addresses the topic of provable well-orderings. Thanks to Gentzen, we know that any elementary recursive ordering that is provably well-ordered in Peano arithmetic has order-type less than ε_0. Takeuti and, independently, Harrington, have shown that if R is an elementary relation that is provably well-ordered in PA, there is, moreover, a $<\varepsilon_0$-recursive comparison map between R and standard notation systems for ε_0. Arai uses a clever coding trick to provide two nice strengthenings: one can, in fact, find a comparison map that is *elementary*, and even under the weaker assumption that R is just provably *well-founded* (not necessarily totally ordered). This analysis carries over to reasonable extensions of PA, but Arai notes that it is open as to whether one can prove the same result for fragments.

In Section 2, Arai shows that one can extract elementary bounds on the increase in length when eliminating cuts from proofs in propositional sequent calculi. This result, which stands in sharp contrast to predicate logic, is well-known, and the standard proofs are not difficult; but Arai uses instead a simple counting argument that highlights the difference between the propositional and

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predicate cases. He also uses these methods to refine some of the results in Beckmann's thesis, involving separations of theories of bounded arithmetic with a predicate variable.

Section 3 strengthens some classic results due to Howard and Kreisel on the relationship between bar induction, comprehension, and reflection. A complementary result due to Friedman on the relationship between bar induction and a principle of ω-model reflection can be found in Stephen Simpson's book, *Subsystems of Second Order Arithmetic* (Springer, 1999).

Section 4 solves a problem posed by Friedman on the equational calculus PRE, i.e. primitive recursive arithmetic minus the induction axioms. If f is a function symbol of PRE, let $Cl(f)$ denote f together with all the functions symbols used in its definition. If e is an equation and E is a set of equations, $Cl(e)$ and $Cl(E)$ are defined analogously to represent the set of function symbols found implicitly in e and E. A proof of e from E in PRE is said to be *direct* if every function symbol used in the proof occurs in $Cl(E) \cup Cl(e)$. Friedman gave two examples of provable entailments in PRE that cannot be established using direct proofs: if S denotes the successor function, $x = y$ follows from $Sx = Sy$ by use of the predecessor function, and $y = z$ follows from $0 = Sx$ using the conditional function. Arai's Theorem 4.1 provides the satisfying conclusion that these are, in a sense, the *only* counterexamples: if one adds these to the system as rules, every provable entailment has a direct proof. Arai uses this to show that the question as to whether an equation is consistent with PRE is decidable, but he notes that the decidability of provability for open formulae is still unresolved.

Buchholz has used a realizability interpretation to show that a certain intuitionistic fixed-point theory, $\widehat{ID}^i_1(strong)$, is conservative over Heyting arithmetic for almost negative formulae. Section 5 provides an elegant use of Goodman's theorem to extend the conservation result to arbitrary arithmetic formulae.

It is a scandal of proof theory that various definitions of a theory's proof-theoretic ordinal need not coincide, though empirically it seems that for "natural" theories they do. Jäger and Primo have come across one of the few real counterexamples, with a second-order theory of weak-fixed point definitions, whose Π^1_1 ordinal is that of Peano arithmetic, i.e. ε_0 (which amounts to saying that ε_0 is the least-upper bound to the theory's provable well-orderings), but whose Π^0_2 ordinal is that of $\Sigma^1_1{-}AC$, i.e. $\varphi_{\varepsilon_0}(0)$ (which amounts to saying that the theory's provably total recursive functions are exactly the ones that are $<\varphi_{\varepsilon_0}(0)$-recursive). While the second fact can be obtained with a direct interpretation, Arai shows that the first fact follows easily from an observation due to Kreisel that adding arbitrary true Σ^1_1 sentences to a theory does not change its Π^1_1 ordinal. This proof is simpler than the one provided by Jäger and Primo, and yields a neat diagnosis of the rogue phenomenon.

In the late 60's, Kreisel and Levy demonstrated a general relationship between reflection principles and induction, and in the late 70's, Schmerl extended the analysis to transfinitely iterated reflection principles and transfinite induction. In Section 7, Arai develops some refinements, obtaining tight connections between transfinitely iterated reflection principles and theories with finitely many applications of a transfinite induction rule. (Related and partially overlapping results have been obtained independently by Beklemishev.)

Interest in the proof theory community in the problem of bounding the lengths of derivations in various rewrite systems began with work by Dieter Hofbauer (see "Termination proofs by multiset path orderings imply primitive recursive derivation lengths," *Theoretical Computer Science* 105 (1992), pages 129–140). In the last section of this paper, Arai uses the results of Section 7 to obtain tight bounds on the length of derivations in rewrite systems compatible with a lexicographic path ordering, refining results due to Buchholz and Weiermann. Along the way, he also draws out connections between such rewrite systems, fast-growing hierarchies, slow-growing hierarchies, and a hierarchy of fragments of arithmetic between $I\Sigma_1$ and $I\Sigma_2$.