

# Notes on $\Pi_1^1$ -conservativity, $\omega$ -submodels, and the collection schema\*

Jeremy Avigad

January 8, 2002

## Abstract

These are some minor notes and observations related to a paper by Cholak, Jockusch, and Slaman [3]. In particular, if  $T_1$  and  $T_2$  are theories in the language of second-order arithmetic and  $T_2$  is  $\Pi_1^1$  conservative over  $T_1$ , it is not necessarily the case that every countable model of  $T_1$  is an  $\omega$ -submodel of a countable model of  $T_2$ ; this answers a question posed in [3]. On the other hand, for  $n \geq 1$ , every countable  $\omega$ -model of  $IS_n$  (resp.  $B\Sigma_{n+1}$ ) is an  $\omega$ -submodel of a countable model of  $WKL_0 + IS_n$  (resp.  $WKL_0 + B\Sigma_{n+1}$ ).

## 1 $\Pi_1^1$ -conservativity and $\omega$ -submodels

If  $T$  is a theory in the language of second-order arithmetic, a Henkin model  $\mathcal{M}$  of  $T$  can be viewed as a structure  $\langle M, S_M, \dots \rangle$ , where first-order variables are taken to range over  $M$ , and second-order variables are taken to range over some subset  $S_M$  of the power set of  $M$ . If  $M = \omega$  and  $\mathcal{M}$  has the standard interpretations of  $+$ ,  $\times$ , etc., then  $\mathcal{M}$  is said to be an  $\omega$ -model. If  $\mathcal{M}_1 = \langle M_1, S_{M_1}, \dots \rangle$  and  $\mathcal{M}_2 = \langle M_2, S_{M_2}, \dots \rangle$  are models, then  $\mathcal{M}_1$  is said to be an  $\omega$ -submodel of  $\mathcal{M}_2$  if  $M_1 = M_2$  and  $S_{M_1} \subseteq S_{M_2}$  (note that  $M_1$  and  $M_2$  need not be  $\omega$ !).

The theories  $RCA_0$ ,  $WKL_0$ ,  $ACA_0$  are fragments of second-order arithmetic in which induction is restricted to  $\Sigma_1^0$  formulae with parameters, and in which comprehension is replaced by recursive comprehension, a weak version of König's lemma, or arithmetic comprehension, respectively. From here on the general reference for subsystems of second-order arithmetic is Simpson [10].

It is not hard to see that if  $T_1$  and  $T_2$  are theories in the language of second-order arithmetic and every countable  $\omega$ -model of  $T_1$  is an  $\omega$ -submodel of a countable model of  $T_2$ , then  $T_2$  is  $\Pi_1^1$ -conservative over  $T_1$ : if  $\psi$  is  $\Pi_1$  and  $T_1$  does not prove  $\psi$ , let  $\mathcal{M}_1$  be a countable model of  $T_1 + \neg\psi$ ; find a model  $\mathcal{M}_2$  of  $T_2$  such that  $\mathcal{M}_1$  is an  $\omega$ -submodel of  $\mathcal{M}_2$ ; then  $\mathcal{M}_2$  is a model of  $T_2 + \neg\psi$ .

---

\*Carnegie Mellon Technical Report CMU-PHIL-125. Section 3 modified slightly, January 8, 2002.

Question 13.2 of [3] asks if the converse holds, i.e. whether the  $\Pi_1^1$  conservation of  $T_2$  over  $T_1$  implies that every countable  $\omega$ -model of  $T_1$  is an  $\omega$ -submodel of a model of  $T_2$ . The following proposition shows that the answer is no.

**Proposition 1.1** *There is a sentence  $\theta$  such that*

- $ACA_0 + \theta$  is  $\Pi_1^1$  conservative over  $ACA_0$ .
- $ACA_0 + \theta$  has no  $\omega$ -model.

*Proof.* In  $ACA_0$  one has a  $\Sigma_1^1$  truth predicate for  $\Sigma_1^1$  sentences. Use this and the fixed-point lemma to construct a sentence  $\psi$  that says

“I am not provable from  $ACA_0$  together with any true  $\Sigma_1^1$  sentence.”

Then  $\neg\psi$  is a false  $\Sigma_1^1$  sentence, and so has no  $\omega$ -model. It suffices to show that  $ACA_0 + \neg\psi$  is  $\Pi_1^1$  conservative over  $ACA_0$ .

Suppose  $ACA_0 + \neg\psi$  proves  $\eta$ , where  $\eta$  is  $\Pi_1^1$ . Then  $ACA_0 + \neg\eta$  proves  $\psi$ . But then  $ACA_0 + \neg\eta$  proves

“ $\neg\eta$  is a true  $\Sigma_1^1$  sentence and there is a proof of  $\psi$  from  $\neg\eta$ .”

In other words,  $ACA_0 + \neg\eta$  proves  $\neg\psi$  as well as  $\psi$ . So  $ACA_0 + \neg\eta$  is inconsistent, and hence  $ACA_0$  proves  $\eta$ .  $\square$

Something like the trick above (or, more precisely, the more refined version in the proof of Proposition 1.2) has been used recently by Arana [1]. The sentence  $\psi$  is equivalent to the assertion “any  $\Pi_1^1$  consequence of  $ACA_0$  together with my negation is true,” and the proof above could equally well have been expressed with this formulation.

In the proof of Proposition 1.1, arithmetic comprehension was used to obtain an adequate definition of truth for  $\Sigma_1^1$  formulae. I do not know whether it is possible to replace  $ACA_0$  by  $RCA_0$  in Proposition 1.1.

Question 13.4 of [3] asks whether the following holds: if  $T_0$  and  $T_1$  are  $\Pi_2^1$  theories which are  $\Pi_1^1$  conservative over a theory  $T$ , then  $T_0 + T_1$  is necessarily  $\Pi_1^1$  conservative over  $T$ . I suspect that answer is no, but the best I can come up with is the following “near miss.”

**Proposition 1.2** *There is a sentence  $\theta$  such that*

- $\theta$  is  $\Pi_2^1$
- $\neg\theta$  can be put in the form  $\exists n \forall X \exists Y \eta$ , where  $\eta$  is arithmetic; in other words,  $\neg\theta$  can be expressed with an existential number quantifier followed by a  $\Pi_2^1$  sentence.
- $ACA_0 + \theta$  is  $\Pi_1^1$  conservative over  $ACA_0$
- $ACA_0 + \neg\theta$  is  $\Pi_1^1$  conservative over  $ACA_0$

*Proof.* Do the Rosser trick: let  $\theta$  say

“If I am provable from  $ACA_0$  plus a true  $\Sigma_1^1$  sentence, then there is a shorter proof of my negation from  $ACA_0$  plus a true  $\Sigma_1^1$  sentence.”

Here “shorter” really means “with smaller Gödel number.” Then  $\theta$  is of the form  $\forall n(\exists X \alpha \rightarrow \exists Y \beta)$  with  $\alpha$  and  $\beta$  arithmetic, and bringing quantifiers to the front in different orders allows one to put  $\theta$  and  $\neg\theta$  in the required forms. An argument similar to the one above shows that both  $ACA_0 + \theta$  and  $ACA_0 + \neg\theta$  are  $\Pi_1^1$  conservative over  $ACA_0$ . For example, suppose  $d$  is a proof of a  $\Pi_1^1$  sentence  $\eta$  in  $ACA_0 + \theta$ . Let  $d'$  be the corresponding proof of  $\neg\theta$  in  $ACA_0 + \neg\eta$ , and let  $e_0, \dots, e_k$  enumerate all proofs of length less than  $d'$  of  $\theta$  from  $\Sigma_1^1$  sentences, say  $\eta_0, \dots, \eta_k$  respectively.

Now argue in  $ACA_0$  to show  $\eta$ . First, if  $\theta$  holds, we are done, using  $d$ . Otherwise, suppose  $\neg\theta$ . Then

There is a proof of  $\theta$  in  $ACA_0$  together with a true  $\Sigma_1^1$  sentence, and no shorter proof of  $\neg\theta$  in  $ACA_0$  together with a true  $\Sigma_1^1$  sentence.

If one of the  $\eta_i$  is true, then the corresponding proof,  $e_i$ , shows that  $\theta$  is true, contradicting the assumption  $\neg\theta$ . So there is no proof of  $\theta$  from a  $ACA_0$  together with a true  $\Sigma_1^1$  sentence with a proof shorter than  $d'$ . By the displayed assertion above,  $d'$  cannot be a proof of  $\neg\theta$  from a true  $\Sigma_1^1$  sentence; in other words,  $\neg\eta$  is false, as required.  $\square$

Note that in the presence of the  $\Sigma_2^1$  axiom of choice, the number quantifier can be moved inwards, and  $\neg\theta$  is equivalent to a  $\Pi_2^1$  sentence. The problem is that  $\Sigma_1^1$ -AC is not a  $\Pi_2^1$  axiom. But if the question above is rephrased so that  $T$  is not required to be  $\Pi_2^1$  (i.e.  $T_0$  and  $T_1$  are required to prove the same  $\Pi_2^1$  sentences as  $T$ , but not necessarily extend  $T$ ), this last observation yields a negative answer.

I do not know the answer to question 13.4 of [3] if one requires the theories involved to be true (in the standard model). In particular, for the  $\theta$  used in the proof of Proposition 1.2, is it the case that every countable model of  $ACA_0$  is an  $\omega$ -submodel of a countable model of  $ACA_0 + \theta$ ?

## 2 $\omega$ -models of Weak König’s Lemma

In the 1980’s, Harvey Friedman proved

**Theorem 2.1** *WKL<sub>0</sub> is conservative over PRA for  $\Pi_2$  sentences.*

Harrington later used a forcing argument, based on Jockusch and Soare’s low basis theorem, to strengthen this to

**Theorem 2.2** *WKL<sub>0</sub> is conservative over RCA<sub>0</sub> for  $\Pi_1^1$  sentences.*

Friedman's theorem follows from this, since  $RCA_0$  is easily interpreted in the fragment of first-order arithmetic  $I\Sigma_1$ , and an old theorem due to Mints, Parsons, and Takeuti independently shows that  $I\Sigma_1$  is  $\Pi_2$  conservative over  $PRA$ .

In [5], Hájek provides the following strengthening:

**Theorem 2.3** *For all  $n \geq 1$ ,  $WKL_0 + I\Sigma_n$  is  $\Pi_1^1$  conservative over  $RCA_0 + I\Sigma_n$ .*

In fact, Hájek obtains an *interpretation* of  $WKL_0 + I\Sigma_n$ , by formalizing the a recursion-theoretic construction of an  $\omega$ -model of  $WKL_0$  in  $I\Sigma_n$ . Avigad [2] independently internalized Harrington's forcing argument to obtain such an interpretation, for  $n = 1$ ; Theorem 2.2 can then be obtained by relativizing the argument to the recursive sets (in which case, for  $n \geq 2$ , forcing for  $\Pi_n^0$  sentences is  $\Pi_n^0$ , and strong forcing for  $\Sigma_n^0$  sentences is  $\Sigma_n^0$ ).

The authors of [3] note that methods of both Hájek [5] or Avigad [2] can be used to strengthen this result as follows:

**Theorem 2.4** *For all  $n \geq 1$ , every countable  $\omega$ -model of  $RCA_0 + I\Sigma_n$  is an  $\omega$ -submodel of a countable  $\omega$ -model of  $WKL_0 + I\Sigma_n$ .*

This is correct, but a few more words of explanation are needed. Roughly, the problem is that Hájek's argument relativizes to a single set, but, on the surface, does not allow one to recapture the entire universe; my argument allows one to recapture the entire universe, but does not immediately work for  $n \geq 1$  unless one restricts set quantifiers to the recursive sets.

The solution is to combine the two arguments. One fairly straightforward way to do this is to use a generic iteration as in [2] to capture all the sets of the original model, but restrict the forcing to recursive subsets of a single set in each step of the iteration. I will describe another method, which provides, in addition, Theorem 2.7 below. The idea is that Hájek's argument would work if one had a function explicitly enumerating the universe of sets; but one can simply force to add such a function.

Note that sequences of sets  $\langle X_0, \dots, X_k \rangle$  can be coded as a single set; I will assume that a reasonable coding has been chosen, so that the length of a sequence is unambiguous, and relevant properties can be verified in  $RCA_0$ . The notation  $F(i) \supseteq F(j)$  in clause 2 of the following proposition should be read as "the sequence coded by  $F(i)$  extends the sequence coded by  $F(j)$ ."

**Proposition 2.5** *Let  $\mathcal{M}$  be any countable structure for the language of second-order arithmetic. Then one can expand  $\mathcal{M}$  to a structure for a language with a new function symbol  $F$  denoting a function from  $M$  to  $S_M$ , such that the expanded structure satisfies the following:*

- $\forall i (\text{length}(F(i)) = i)$
- $\forall i, j (i > j \rightarrow F(i) \supseteq F(j))$
- $\forall X \exists i \exists j < i (F(i)_j = X)$

*Proof.* Take a *condition* to be a finite sequence of sets  $\langle X_0, \dots, X_k \rangle$ , where “stronger than” means “extends, as a sequence.” Then one can read off a function  $F$  from a suitably generic set. The properties that need to be met are: given  $X \in S_M$ , there is a condition in the generic that has  $X$  as an element; and given  $n \in M$ , there is a condition in the generic of length greater than or equal to  $n$ .  $\square$

Note that the expanded model need not satisfy induction or comprehension for formulae involving  $F$ ; so the interaction with  $F$  is mediated solely by the axioms above.

*Proof of Theorem 2.4.* Hájek shows how to construct a formula  $\psi(x, y)$  so that the collection of sets of the form  $S_x = \{x \mid \psi(x, y)\}$ , as  $y$  ranges over  $\mathbb{N}$ , yields an omega model of  $WKL_0 + I\Sigma_n$ , provably in  $I\Sigma_n$ . In other words,  $I\Sigma_n$  proves that each axiom of  $WKL_0 + I\Sigma_n$  holds when one interprets second-order quantifiers as ranging over  $\mathcal{S} = \{S_x \mid x \in \mathbb{N}\}$ . The construction can be relativized to a set parameter  $Z$ .

With only slight modification, one can obtain a formula  $\psi'(x, y, W)$ , such that whenever  $W$  codes a finite sequence of sets,  $\psi'(x, y, W)$  defines an  $\omega$ -model of  $WKL_0 + I\Sigma_n$  containing each of these sets; and with the property that if  $W'$  codes a sequence extending  $W$ , the  $\omega$ -model containing  $W'$  includes the  $\omega$ -model containing  $W$ . In other words, there is a formula  $\psi'(x, y, W)$  such that  $I\Sigma_n$  proves

- each axiom of  $WKL_0 + I\Sigma_n$  holds in the  $\omega$ -model defined by  $\psi'(x, y, W)$ ;
- $\forall \langle X_0, \dots, X_{k-1} \rangle \forall i < k \exists j (\{x \mid \psi'(x, j, \langle X_0, \dots, X_{k-1} \rangle)\} = X_i)$ ;
- $\langle X_0, \dots, X_k \rangle \subseteq \langle Y_0, \dots, Y_l \rangle \rightarrow \forall i \exists j (\{x \mid \psi'(x, i, \langle X_0, \dots, X_{k-1} \rangle)\} = \{x \mid \psi'(x, j, \langle Y_0, \dots, Y_{l-1} \rangle)\}$ .

Now, given a countable model  $\mathcal{M}$  of  $RCA_0 + I\Sigma_n$ , expand it to a model  $\mathcal{M}'$  of the three additional axioms in Proposition 2.5. Define  $\theta(x, y)$  to be the formula

$$\psi'(x, (y)_0, F((y)_1)).$$

Intuitively, this represents the union of the  $\omega$ -models defined by  $\psi'(x, y, F(z))$  as  $z$  ranges over  $\mathbb{N}$ , so it is not hard to see that  $\theta(x, y)$  defines an  $\omega$ -model of  $WKL_0 + I\Sigma_n$  in  $\mathcal{M}'$ . In other words, letting

$$\mathcal{M}'' = \langle \mathcal{M}, \{\{a \in M \mid \mathcal{M}' \models \theta(\bar{a}, \bar{b})\} \mid b \in M\}, \dots \rangle$$

yields a countable model of  $WKL_0 + I\Sigma_n$  with the same first-order part as  $\mathcal{M}$ .  $\square$

In fact, Hájek notes explicitly that for  $n \geq 1$ , the schema of  $\Sigma_n$  collection, denoted  $B\Sigma_n$ , can be used to justify  $\Sigma_n$  collection in the  $\omega$ -model defined by  $\psi$ . As a result, his construction also shows

**Theorem 2.6** *For all  $n \geq 2$ ,  $WKL_0 + B\Sigma_n$  is  $\Pi_1^1$  conservative over  $RCA_0 + B\Sigma_n$ .*

Combining this with the argument above yields:

**Theorem 2.7** *For all  $n \geq 2$ , every countable  $\omega$ -model of  $RCA_0 + B\Sigma_n$  is an  $\omega$ -submodel of a countable  $\omega$ -model of  $WKL_0 + B\Sigma_n$ .*

I should note that I do not know how to obtain these last two theorems using the methods of [2], or any other way; Hájek’s formalization of the recursion-theoretic argument seems essential. I should also note that although it requires more work to obtain the model-theoretic results from the syntactic arguments in [2] and [5], the latter methods have the advantage of providing explicit translations between the theories, with polynomial bounds on increase in proof length. Finally, Simpson and Smith [11] shows that Theorem 2.7 holds for  $n = 1$  if one adds an axiom asserting that exponentiation is total. For related results in the context of bounded arithmetic, see Ferreira [4].

### 3 Separating $\Sigma_{n+1}$ collection and $\Sigma_n$ induction

In the language of first-order arithmetic, the  $\Sigma_n$  collection schema,  $B\Sigma_n$ , is as follows:

$$\forall a, \vec{z} (\forall x < a \exists y \theta(x, y, \vec{z}) \rightarrow \exists b \forall x < a \exists y < b \theta(x, y, \vec{z}))$$

where  $\theta$  is  $\Sigma_n$ . Below  $B\Sigma_n$  is also used to denote the fragment of arithmetic in which induction is replaced by the schema above.

The following theorem is due to Friedman and Paris, independently:

**Theorem 3.1** *For each  $n \geq 0$ ,  $\Sigma_{n+1}$  collection is  $\Pi_{n+2}$  conservative over  $I\Sigma_n$ .*

Paris and Wilkie showed in [8]:

**Theorem 3.2** *For each  $n \geq 0$ ,  $I\Sigma_n$  does not prove  $\Sigma_{n+1}$  collection.*

In [7, page 331], Paris notes that one can extract the following from the proof:

**Theorem 3.3** *For each  $n \geq 0$ , there is a  $\Sigma_{n+2}$  sentence provable from  $B\Sigma_{n+1}$  but not  $I\Sigma_n$ .*

Finding the  $\Sigma_{n+2}$  sentence takes some digging, however, and seems to require a trick (used in Chapter IV of [6]), as follows.

First, note by Gödel’s incompleteness theorem, there is a  $\Sigma_0$  formula  $\psi(x)$  such that  $\exists x \psi(x)$  is false, but consistent with Peano arithmetic. So, in any model, an element  $a$  satisfying  $\psi(a)$  is necessarily nonstandard. We can choose  $\psi$  so that  $I\Sigma_0$  proves  $\forall x, y (\psi(x) \wedge \psi(y) \rightarrow x = y)$ .

Let  $\alpha(e, x)$  say, roughly, “ $e$  is a  $\Sigma_{n+1}$  formula defining  $x$ ,” using a  $\Sigma_{n+1}$  truth predicate, as in the Paris-Wilkie proof. Let  $\alpha(e, x)$  be equivalent to  $\exists u \theta(e, u, x)$ , where  $\theta$  is  $\Pi_n$ .

The sentence I extracted from the Paris-Wilkie proof is

$$\exists b \forall a (\psi(a) \wedge \forall x < (a + 1) \exists e < a \exists u \theta(e, u, x)) \rightarrow \\ \forall x < (a + 1) \exists e < a \exists u < b \theta(e, u, x))$$

Notice that if you move the  $\exists b$  to the consequence of the implication, this just says that a certain instance of collection holds for  $a + 1$ , where  $a$  is the value satisfying  $\psi$ , if there is one. It is false in the Paris-Wilkie model  $K_{n+1}$  of  $I\Sigma_n$ , assuming the nonstandard model has an element satisfying  $\psi$ ; but it is an easy consequence of  $\Sigma_{n+1}$  collection.

Bringing the universal quantifier over  $x$  out of the antecedent yields

$$\exists b \forall a \exists x (\psi(a) \wedge (x < (a + 1) \rightarrow \exists e < a \exists u \theta(e, u, x))) \rightarrow \\ \forall x < (a + 1) \exists e < a \exists u < b \theta(e, u, x))$$

Using the fact that there is at most one  $a$  satisfying  $\psi(a)$ , we can switch the order of the second and third quantifiers:

$$\exists b \exists x \forall a (\psi(a) \wedge (x < (a + 1) \rightarrow \exists e < a \exists u \theta(e, u, x))) \rightarrow \\ \forall x < (a + 1) \exists e < a \exists u < b \theta(e, u, x))$$

Using  $\Sigma_n$  collection,  $I\Sigma_n$  proves that this statement is  $\Sigma_{n+2}$ .

A different construction of a  $\Sigma_{n+2}$  sentence satisfying Theorem 3.2 is described in Section 4.1 of Sieg [9].

## 4 Some thoughts on $RT_2^2$ and $B\Sigma_2$

Let  $RT_2^2$  denote the infinitary Ramsey's theorem for two-colorings of pairs of integers, as described in [3]. Hirst has shown that the infinitary Ramsey's theorem for pairs implies  $B\Sigma_2$ ; on the other hand, [3] shows that the first-order consequences of  $RCA_0 + RT_2^2$  are included in  $I\Sigma_2$ .

It is still open as to what exactly the first-order consequences of  $RCA_0 + RT_2^2$  are, and nothing as of yet precludes the possibility that the first-order consequences of  $RT_2^2$  are *exactly* those of  $B\Sigma_2$ . In other words, one might try to prove that  $RCA_0 + B\Sigma_2 + RT_2^2$  is conservative over  $B\Sigma_2$ , by replacing  $I\Sigma_2$  by  $B\Sigma_2$  in Corollary 8.6, Lemma 9.4, and Lemma 10.4 of [3].

Two observations are encouraging in that regard:

- By the results of the Section 2, every countable model of  $RCA_0 + B\Sigma_2$  is an  $\omega$ -submodel of a countable model of  $WKL_0 + B\Sigma_2$ . (This is the analogue of Corollary 8.6.)
- The use of  $\Sigma_2$  induction in the proof of Lemma 9.9 in [3] is unnecessary; the choice principle it is used to derive in fact follows from  $WKL_0$  alone. (See [10], Lemma VIII.2.4, page 319.) So one use of  $B\Sigma_2$  in the proof of Lemma 9.4 can be eliminated.

But I see no way of modifying the proof of Lemma 9.10 in [3] to obtain the analogue of Lemma 9.4 for  $B\Sigma_2$ , let alone the corresponding version of Lemma 10.4.

## References

- [1] Andrew Arana. Solovay's theorem cannot be simplified. To appear.
- [2] Jeremy Avigad. Formalizing forcing arguments in subsystems of second-order arithmetic. *Annals of Pure and Applied Logic*, 82:165–191, 1996.
- [3] Peter Cholak, Carl Jockusch, and Theodore Slaman. On the strength of ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66:1–55, 2001.
- [4] Fernando Ferreira. A feasible theory for analysis. *Journal of Symbolic Logic*, 59:1001–1011, 1994.
- [5] Petr Hájek. Interpretability and fragments of arithmetic. In Peter Clote and Jan Krajčček, editors, *Arithmetic, Proof Theory, and Computational Complexity*, pages 185–196. Oxford University Press, Oxford, 1993.
- [6] Petr Hájek and Pavel Pudlák. *Metamathematics of first-order arithmetic*. Springer, Berlin, 1993.
- [7] Jeff Paris. A hierarchy of cuts in models of arithmetic. In Pacholski et al., editor, *Model Theory of Algebra and Arithmetic*, Lecture Notes in Mathematics 834, pages 312–337. Springer, 1980.
- [8] Jeff Paris and Robin Kirby.  $\Sigma_n$  collection schemas in arithmetic. In Paris et al., editor, *Logic Colloquium '77*, pages 199–209. North Holland, 1978.
- [9] Wilfried Sieg. Fragments of arithmetic. *Annals of Pure and Applied Logic*, 28:33–72, 1985.
- [10] Stephen Simpson. *Subsystems of Second-Order Arithmetic*. Springer, Berlin, 1998.
- [11] Stephen Simpson and Rick Smith. Factorization of polynomials and  $\Sigma_1^0$  induction. *Annals of Pure and Applied Logic*, 31:289–306, 1986.