The mathematical context behind Frege’s analysis of function and object*

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Abstract

In 1837, Dirichlet proved that there are infinitely many primes in any arithmetic progression in which the terms do not all share a common factor. Presentations of Dirichlet’s proof over the next century show a very gradual transition to a modern perspective, whereby certain functions that appear in the proof assume the status of ordinary mathematical objects. In prior work [3, 4] we studied this transition in detail, and explored the methodological and philosophical ramifications. Here we use these considerations to illuminate the treatment of functions and objects in Frege’s work on the foundations of logic, and we argue that his philosophical and logical decisions were influenced by many of the same factors that shaped the mathematical developments.

1 Introduction

In 1837, Dirichlet proved that there are infinitely many primes in any arithmetic progression in which the terms do not all share a common factor. Contemporary presentations of the proof are explicitly higher-order, in that they involve quantifying over and summing over Dirichlet characters, which are certain types of functions. This way of thinking about characters, however, was alien to the early nineteenth-century modes of thought, and, indeed, there is no notion of a “character” in Dirichlet’s original proof. Rather, there are calculations with certain symbolic expressions that we, today, recognize as the values of the characters.

In two prior papers [3, 4], we traced a long history of presentation of Dirichlet’s theorem from 1837 to 1927. We enumerated a number of the ways that characters are treated as ordinary mathematical objects, on par with the natural numbers, in contemporary proofs. For example, certain functions, \( L(s, \chi) \),

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take characters as arguments; sequences are indexed by characters; one forms
sums \( \sum_x \) that range over characters; and, for each natural number \( m \geq 1 \),
the characters “modulo \( m \)” are seen to have a certain group structure, under
the operation of pointwise multiplication. We noted that Dirichlet’s nineteenth
century followers were slow to adopt such modern notational and conceptual
innovations, and we carefully traced the long and very gradual transition to the
modern point of view.

The history of Dirichlet’s theorem is part and parcel of the broader nine-
teenth century transition to the use of set-theoretic language and methods,
and we took the history of Dirichlet’s theorem to provide unique insights into
how and why these methods entered the mathematical canon. In particular,
we paid close attention to the ways that the modern view of functions as or-
dinary mathematical objects served to simplify Dirichlet’s proof and support
subsequent generalizations. One important benefit of the modern viewpoint,
we argued, is the uniformity it brings: common constructions involving ordinary
mathematical objects could all of a sudden be applied to collections of functions,
and general properties of these constructions and methods for reasoning about
them could be carried over.

Given the patent benefits involved, it may then seem strange that the tran-
sition took so long. But our analysis of the history also enabled us to highlight
some of the difficulties involved in adopting such fundamental changes in math-
ematical method and language. There were a number of concerns that had to be
grappled with: What does it mean to sum over a collection of functions, taken
abstractly? What does it mean to send a function as an argument to another
function? What are the rules governing such a treatment of functions? Are they
coherent, and even consistent? Is the use of such methods appropriate in the
proof of a number-theoretic statement like Dirichlet’s theorem? Does it answer
the questions that we wanted to answer, and does it provide the information
that a mathematical proof is supposed to provide? These questions were not,
by and large, debated explicitly. Rather, the answers emerged organically, in
the way that individual authors adopted the changes in their work, and shared
that work with others. Paying close attention to the history thus helped us
understand and assess some of the reasons that we treat functions as objects in
current mathematical practice.

In fact, the modern practice of treating characters as objects in the proof of
Dirichlet’s theorem can be factored into two components, neither one a prereq-
usite for the other: first, the recognition of the notion of a “character” as an
instance of the more general notion of “function”; and second, the treatment
of functions as objects in the various senses alluded to above. In [3, Section
2.1], we observed that for most of the nineteenth century, the word “function”
(and its Latin, French, and German cognates) was used exclusively for functions
defined on the real or complex numbers. As far as we can tell, it was not until
1879 that the word was used in anything like its modern sense of a correspon-
dence between two arbitrary domains. This in fact occurred in two disparate
sources from that year. One was the third edition of Dedekind’s supplements to
Dirichlet’s lectures on number theory [12], in which Dedekind described certain
types of characters, related to the ones discussed above, as “functions.” The
other was Frege’s Begriffsschrift [17]. It is striking that in these works both
authors not only introduced a general notion of “function,” but also stretched
the boundaries that govern the treatment of its various instances.

Frege’s development of formal logic was designed to represent mathemati-
cal language and methods of reasoning, and offer clear recommendations as to
proper usage. Our goal here is to use the historical perspective on the mathemati-
cal treatment of functions as objects described above to illuminate the
constraints he faced, and explain what we see as a rather conflicted attitude in
his treatment of functions as objects. In particular, we argue that key choices
in the design of his formal system were motivated by the same sorts of consider-
erations that were faced by nineteenth-century mathematicians. This is not to
say that Frege’s logico-philosophical concerns should be seen as properly math-
ematical, or vice versa. Rather, they both stem from the need to balance two
key desiderata: the desire, on the one hand, for flexible and uniform ways of
dealing with higher-order entities in the many guises in which they appear, and
the desire, on the other hand, to make sure that the methods of doing so are
clear, coherent, and meaningful.

2 Frege’s view of functions and objects

In 1940, Alonzo Church presented a formulation of type theory [9], now known
as “simple type theory.” Simple type theory can serve as a foundation for a
significant portion of mathematics, and, indeed, is the axiomatic foundation of
choice for a number of computational interactive theorem provers today [24, 25,
29]. One starts with some basic types, say, a type \( \mathbb{B} \) of Boolean truth values
and a type \( \mathbb{N} \) of natural numbers, and one forms more complex types \( \sigma \times \tau \) and
\( \sigma \rightarrow \tau \) from any two types \( \sigma \) and \( \tau \). Intuitively, elements of type \( \sigma \times \tau \) are
ordered pairs, consisting of an element of type \( \sigma \) and an element of type \( \tau \), and
elements of type \( \sigma \rightarrow \tau \) are functions from \( \sigma \) to \( \tau \). In a type-theoretic approach
to the foundations of mathematics, one identifies sets of natural numbers with
predicates, which is to say, elements of type \( \mathbb{N} \rightarrow \mathbb{B} \). Binary relations on the
natural numbers are then elements of type \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{B} \), and sequences of natural
numbers are elements of type \( \mathbb{N} \rightarrow \mathbb{N} \). Objects at this level are called type 1
elements, because they require one essential use of the function space arrow.
Integers can be identified as pairs of natural numbers and rationals can be
identified as pairs of integers in the usual ways. Real numbers are then Cauchy
sequences of rationals (elements of type 1), or equivalence classes of such, which
puts them at type 2. Functions from the reals to the reals and sets of reals
are then elements of type 3, and sets of functions from the reals to reals or
collections of sets of real numbers are then elements of type 4. For example, the
collection of Borel sets of real numbers is an element of type 4, as is Lebesgue
measure, which maps certain sets of real numbers to the real numbers. A set of
measures on the Borel sets of the real numbers is an element of type 5. And so
on up the hierarchy.
Simple type theory can be viewed as a descendant of the ramified type theory of Russell and Whitehead’s *Principia Mathematica* [36], which, in turn, was inspired by the formal system of Frege’s *Grundgesetze der Arithmetik* [21]. Starting with a basic type of individuals, Frege’s system also has variables ranging over higher-type functionals, and so can be seen as an incipient form of modern type theory. For that reason, it may come as a surprise to logicians familiar with the modern type-theoretic understanding that the foundational outlook just described is not at all the image of mathematics that Frege had in mind. It is this image that we wish to explore here.

Frege took concepts to be instances of functions; for example, in “Function and concept” he wrote that “a concept is a function whose value is always a truth value” [19, p. 139]. And, throughout his career, he was insistent that functions are not objects. The third “fundamental principle” in his *Grundlagen der Arithmetik* of 1884 was “never to lose sight of the distinction between concept and object” [18, Introduction], and he later asserted that “it will not do to call a general concept word the name of a thing” [18, § 51]. The distinction features prominently in his essays “Function and concept,” “Comments on Sinn and Bedeutung” and “Concept and object” of 1891, 1891/2, and 1892, respectively.

According to Frege, the proper distinction is tracked by linguistic usage: objects are denoted by words and phrases that can fill the subject role in a grammatical sentence, whereas concepts are denoted by words and phrases that can play the role of a predicate. In “Concept and object” he wrote:

> We may say in brief, taking “subject” and “predicate” in the linguistic sense: a concept is the *Bedeutung* of a predicate; an object is something that can never be the whole *Bedeutung* of a predicate, but can be the *Bedeutung* of a subject.4

And:

> A concept—as I understand the word—is predicative. On the other hand, a name of an object, a proper name, is quite incapable of being used as a grammatical predicate.5

In the sentence, “Frege is a philosopher,” the word “Frege” denotes an object, and the phrase “is a philosopher” denotes a concept. Frege clarified the distinction by explaining that functional expressions, including concept expressions,

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1 We should note that in this section we will focus on his views from 1884 onwards. Prior to this, he seems to have held a different view of concepts, though he still maintained that they are not objects; see [27, p. 136].

2 “...der Unterschied zwischen Begriff und Gegenstand ist in Auge zu behalten.”

3 “...ist es unpassend, ein allgemeines Begriffwort Namen eines Dinges zu nennen.”

4 “Wir können kurz sagen, indem wir “Prädikat” und “Subjekt” im sprachlichen Sinne verstehen: Begriff ist Bedeutung eines Prädikates, Gegenstand ist, was nie die ganze Bedeutung Prädikates, wohl aber Bedeutung eines Subjekts sein kann.” The word *Bedeutung* is often translated as “reference” or “denotation.” But for difficulties in the translation, see §4 of the introduction to Beaney [5].

5 “Der Begriff—wie ich das Wort verstehe—is prädikativ. Ein Gegenstandsname hingegen, ein Eigennname ist durchaus unfähig, als grammatisches Prädikat gebraucht zu werden.”
are “unsaturated,” or incomplete. These stand in contrast to signs that are used to denote objects, which are complete in and of themselves. For example, in the sentence “Frege is a philosopher,” the expression “Frege” is saturated, and succeeds in picking out an object. In contrast, the expression “. . . is a philosopher” contains a gap, and fails to name an object until one fills in the ellipsis, at which point the expression denotes a truth value.

Having distinguished between concepts and objects in such a way, Frege had to deal with objections, such as the one he attributed to Benno Kerry in “Concept and object.” In the sentence “The concept ‘horse’ is a concept easily attained” the concept denoted by “horse” does fill the subject role. Frege’s surprising answer was to deny that the phrase “the concept ‘horse’ ” denotes a concept. He conceded that this sounds strange:

It must indeed be recognized that we are confronted by an awkwardness of language…if we say that the concept horse is not a concept…7 [20, pp. 196–197]

Yet, he insisted, this is what we must do. He was already clear about this in the Grundlagen:

The business of a general concept word is precisely to signify a concept. Only when conjoined with the definite article or a demonstrative pronoun can it be counted as the proper name of a thing, but in that case it ceases to count as a concept word. The name of a thing is a proper name.8 [18, §51]

And so, in “Concept and object,” he reminded us:

If we keep it in mind that in my way of speaking expressions like “the concept F” designate not concepts but objects, most of Kerry’s objections already collapse.9 [20, pp. 198–199]

He similarly urged us to reconstrue expressions like “all mammals have red blood” as “whatever is a mammal has red blood” so as to avoid the impression that the predicate “has red blood” is being applied to an object, “mammal.”

6While the distinction between saturated and unsaturated expressions is cast as a distinction between linguistic signs, in his 1904 essay “What is a Function?” Frege made it clear that the dichotomy extends to functions and objects themselves: “The peculiarity of functional signs, which we here called ‘unsaturatedness’, naturally has something answering to it in the functions themselves. They too may be called ‘unsaturated’…” (“Der Eigentümlichkeit der Fuktionszeichen, die wir Ungesättigtheit genannt haben, entspricht natürlich etwas an den Funktionen selbst. Auch diese können wir ungesättigt nennen…” ) [22, p. 665].

7“Es kann ja nicht verkannt werden, daß hier eine freilich unvermeidbare sprachliche Härte vorliegt, wenn wir behaupten: der Begriff Pferd ist kein Begriff…”


9“Wenn wir festhalten, daß in meiner Redeweise Ausdrücke wie “der Begriff F” nicht Begriffe, sondern Gegenstände bezeichnen, so werden die Einwendungen Kerrys schon größtenteils hinfällig.”
Although these examples deal with concepts, Frege’s analysis makes it clear that he intended the linguistic separation to remain operant for other kinds of functions as well.

At the same time, Frege was equally dogmatic in insisting that what we commonly take to be mathematical objects really are mathematical objects as such. The introduction to his *Grundlagen* begins as follows:

> When we ask someone what the number one is, or what the symbol 1 means, we get as a rule the answer “Why, a thing.”

The claim is so curious as to give one pause. The fact that Frege used such a brazen rhetorical flourish to frame the whole project makes it clear just how central the issue is to his analysis. Once again, he took the distinction to be tracked by linguistic use. For example, because the number 7 plays the role of a subject in the statement “7 is odd,” 7 must be an object. But, once again, Frege had to deal with sentences where the syntactic role of a number is murkier. For example, he considered uses of number terms in language that are attributive and do not occur prefixed by the definite article, for example, “Jupiter has four moons” [18, §57]. He wrote

> “. . . our concern here is to arrive at a concept of number usable for the purposes of science; we should not, therefore, be deterred by the fact that in the language of everyday life number appears also in attributive constructions. That can always be got round.”

Specifically, it can be got round by writing an attributive statement such as “Jupiter has four moons” as “the number of Jupiter’s moons is the number 4, or 4” [18, §57], thereby eliminating the attributive usage.

So, for Frege, functions are not objects, but numbers are, because they play the subject role in mathematical statements and can be used with the definite article. There is clearly a difficulty lurking nearby. At least from a modern standpoint, we tend to view functions, sequences, sets, and structures as objects, and certainly in Frege’s time locutions such as “the function f” and “the series s” were common. Frege’s response was similar to his response to Kerry’s objection, namely, to deny that that expressions like these denote functions. To understand how this works, consider the fact that Frege’s logical system includes an operator which takes any function \( f \) from objects to objects and returns an object, \( \varepsilon f(\varepsilon) \), intended to denote its “course-of-values” or “value range.” If \( f \) is a concept, which is to say, a function which for each object
return a truth value, the course-of-values of $f$ is called the “extension” of the concept. Frege’s Basic Law V asserts that two functions which are extensionally equal—that is, which return equal output values for every input—have the same courses-of-values.

Frege used these courses-of-values and extensions as object-proxies for functions and concepts. This is how he analyzed the concept of a cardinal number. Let $F$, for example, be a second-level concept, such that $F$ holds of a first-level concept $f$ if and only if $f$ holds of exactly one object. Frege took the number one to be the extension of $F$, thereby achieving the goal of making the number one, well, a thing. But this “pushing down” trick is central to the methodology of the Grundgesetze: whenever the formal analysis of common mathematical objects seems to suggest identifying such objects as functions or concepts, Frege avoided doing so by replacing the function or concept with its extension. For example, in the Grundgesetze he circumvented the need to define mathematical operations on sequences and relations construed as functions, defining the operations rather on the associated courses-of-values.\(^{13}\) Referring to Frege’s concepts as “attributes” and their extensions as “classes,” Quine described the difference as follows:

Frege treated of attributes of classes without looking upon such discourse as somehow reducible to a more fundamental form treating of attributes of attributes. Thus, whereas he spoke of attributes of attributes as second-level attributes, he rated the attributes of classes as of first level; for he took all classes as rock-bottom objects on par with individuals. [32, p. 147]

Frege never got so far as developing mathematical analysis in his system, and we cannot say with certainty how he would have developed, for example, ordinary calculus on the real numbers. But there is a strong hint that here, too, he would have taken, for example, operations like integration and differentiation to operate on extensions, rather than functions, in his system. He touched on the history of analysis in his *Function and concept* of 1891, and noted that, for example, differentiation can be understood as a higher-type functionals.

Now at this point people had particular second-level functions, but lacked the conception of what we have called second-level functions. By forming that, we make the next step forwards. One might think that this would go on. But probably this last step is not so rich in consequences as the earlier ones; for instead of second-level functions one can deal, in further advances, with first-level functions—as shall be shown elsewhere.\(^{14}\) [19, p. 31]

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\(^{13}\)In fact, the definition of the number one earlier in the paragraph describes, more precisely, the construction in the *Grundlagen*. In the Grundgesetze, he took $F$ to be a first-level concept that holds of classes, i.e. extensions of concepts, that contain one element. This is a nice illustration of how the “pushing down” trick can be used repeatedly to avoid the use of higher types. See Reck [33, Section 5] for a discussion of the two definitions, and Burgess [7] for an overview of Frege’s methodology.

\(^{14}\)Damit hatte man nun einzelne Funktionen zweiter Stufe, ohne jedoch das zu erfassen, was
Presumably, he had the method of replacing functions by their extensions in mind.

Notice, incidentally, that Frege’s method of representing mathematical functions as courses-of-values has the effect that mathematical functions are treated extensionally. For example, defining the integral as an operation that applies to a course-of-values means that integration cannot distinguish mathematical functions that are extensionally equal, since any two descriptions of a function that satisfy extensional equality have the same course-of-values, by Basic Law V.

There are other interesting features of Frege’s treatment of functions that push us away from identifying them with the functions of ordinary mathematics. For example, for Frege, every function has to be defined on the entire domain of individuals; even if one is interested in the exponential function on the real numbers, one has to specify a particular (but arbitrary) value of this function for every object in existence.\(^{15}\) And the separation of functions and objects has other effects on the system. There is only one basic type, so, for example, truth values live alongside everything else. There is no notion of identity between higher-type objects—the equality symbol can only be applied to equality between objects—even though Frege pointed out that one can define an extensional notion of “sameness” of functions and concept, for example, saying two functions from individuals to individuals are the “same” if their values are identical at each input. Frege’s system, of course, includes the axiom of universal instantiation. In contemporary notation, this would be expressed as \(\forall x \varphi(x) \rightarrow \varphi(a)\) where \(x\) is a variable ranging over individuals and \(a\) is any individual term. It also includes the corresponding axiom \(\forall F \varphi(F) \rightarrow \varphi(A)\), where \(F\) ranges over functions from objects to objects. Notably, however, the system does not include analogous axioms for elements of the higher types: the “pushing down trick” obviates the need for these.

All things considered, Frege’s foundational treatment of mathematics seems closer to modern set-theoretic treatments, where there is one homogeneous universe of individuals. Truth values are individuals, numbers are individuals, mathematical sequences and series are individuals—all bona-fide mathematical objects are individuals. Functions are special sorts of entities that our partial expressions refer to when we make statements about objects, but they are not objects in their own right. As Marco Panza puts it:

\[\ldots\text{ according to Frege, appealing to functions is indispensable in order to fix the way his formal language is to run, but functions are not as such actual components of the language. More generally, func-}\]

\[^{15}\text{However, Patricia Blanchette has argued [6] that Frege intended theories presented in his formal system to treat objects in the domain of a particular subject, in which case “every object in existence” really means “every object in the theory’s intended domain.”}\]
tions manifest themselves in our referring to objects—either concrete or abstract—and making statements about them, but they are not as such actual inhabitants of some world of concreta and abstracta.

[30, p. 14]

This is not to say that functions are any less “real” or objective than mathematical objects like numbers, only that they play a distinct role: they allow us to define objects, say things about objects, and reason about objects, but they are not objects themselves.

3 Frege’s foundational concerns

We have seen that a curious tension lies at the core of Frege’s formal representation of mathematics. On the one hand, Frege asserted, repeatedly, that functions, in the logical and linguistic sense, are not objects. On the other hand, when it comes to formalizing mathematical constructions, he clearly felt that functions, in the mathematical sense, have to be objects. His course-of-values operator, together with his Basic Law V, allowed him to have his cake and eat it too, maintaining clear borders between the two realms while passing between them freely. But Frege is often taken to task for failing to realize that this strategy opens the door to Russell’s paradox. Indeed, the strategy feels like a hack, a desperate attempt to satisfy the two central constraints. Why was he so committed to them? The goal of this section is to suggest that the concerns Frege was trying to address with the design of the logic of the Grundgesetze parallel some of the informal mathematical concerns we were able to discern in the nineteenth century treatment of characters.

When one speculates as to the philosophical and logical considerations that influenced the design of Frege’s logic, two possibilities come to mind. One is that Frege determined that functions and objects should be separate on broad metaphysical grounds, and then designed the logic accordingly. The other is that he designed the logic, determined it worked out best with a separation of individuals and functions, and read off the metaphysical stance from that. But, in fact, there is no clear distinction between these two descriptions. Frege designed his logic to try to model scientific practice at its best, and account for and support its successes while combating and eliminating confusions. The examples in the previous section show that Frege had no qualms about reinterpreting ordinary locutions and reconstruing everyday language, so he was by no means slave to naive metaphysical intuitions. But even when doing so he appealed to intuitions to convince us that the reconstruals are reasonable. Thus “doing metaphysics” meant analyzing the practice, sorting out intuitions, and trying to regiment and codify them in a coherent and effective way. From the other direction, “getting the logic to work” meant being able to account for the informal practice effectively and efficiently, and supporting our intuitions to the extent that they can be fashioned into a coherent system. So it is not a question as to whether the metaphysics or the logic comes first; working out the
metaphysics and designing the logic are part and parcel of the same enterprise. The following questions therefore seem more appropriate:

1. What considerations pushed Frege to maintain the sharp distinction between function and object?

2. What considerations pushed Frege to identify mathematical entities, including ordinary mathematical functions, as objects?

Let us consider each in turn.

It seems to us that the answer to the first question is simply that Frege felt that failure to respect the distinction results in linguistic confusion.

If it were correct to take “one man” in the same way as “wise man,” we should be able to use “one” also as a grammatical predicate, and to be able to say “Solon was one” just as much as “Solon was wise.” It is true that “Solon was one” can actually occur, but not in a way to make it intelligible on its own in isolation. It may, for example, mean “Solon was a wise man,” if “wise man” can be supplied from the context. In isolation, however, it seems that “one” cannot be a predicate. This is even clearer if we take the plural. Whereas we can combine “Solon was wise” and “Thales was wise” into “Solon and Thales were wise,” we cannot say “Solon and Thales were one.” But it is hard to see why this should be impossible, if “one” were a property both of Solon and of Thales in the same way that “wise” is.\footnote{Wenn ‘Ein Mensch’ ähnlich wie ‘weiser Mensch’ aufzufassen wäre, so sollte man denken, dass, ‘Ein’ auch als Praedicat gebraucht werden könne, sodass man wie ‘Solon war weise’ auch sagen könnte ‘Solon war Ein’ oder ‘Solon war Einer’. Wenn nun der letzte Ausdruck auch vorkommen kann, so ist er doch dür sich allein nicht verständlich. Er kann z.B. heissen: Solon war ein Weiser, wenn ‘Weiser’ aus dem Zusammenhange zu ergänzen ist. Aber allein scheint ‘Ein’ nicht Praedicat sein zu können. Noch deutlicher zeigt sich dies beim Plural. Während man ‘Solon war weise’ und ‘Thales war weise’ zusammenziehen kann in ‘Solon und Thales waren weise,’ kann man nicht sagen ‘Solen und Thales waren Ein.’ Hiervon wäre die Unmöglichkeit nicht einzusehen, wenn ‘Ein’ sowie ‘weise’ eine Eigenschaft sowohl des Solon als auch des Thales wäre.”} [18, §29]

In other words, even though in some contexts an object word like “one” can appear to be used as a predicate, and in other contexts a concept can appear to be used as a subject, closer inspection shows that these uses do not conform to the rules that govern the use of prototypical subjects and predicates, and so should not be categorized in the naive way.

One of Frege’s favorite pastimes was to show that assertions made by philosophical and mathematical colleagues degenerate into utter nonsense when they fail to maintain sufficient linguistic hygiene. For example, in his 1904 essay, “What is a Function,” Frege was critical of conventional mathematical accounts of variables and functions. It is a mistake, he said, to think of a variable as being an object that varies:

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a number does not vary; for we have nothing of which we could predicate the variation. A cube never turns into a prime number; an irrational number never becomes rational.\footnote{\[22, p. 658\]}

He took the mathematician Emanuel Czuber to task for giving such a sloppy account of variables and functions in an introductory mathematical text. For example, he criticized Czuber’s terminology “a variable assumes a number” \footnote{\[22, 288\]} as being incomprehensible. On Czuber’s account, a variable is an “indefinite number,” so the terminology can be rephrased “an indefinite number assumes a (definite) number”; but where we may talk about an object assuming a property, what can it mean for an object to assume another object?

In other connections, indeed, we say that an object assumes a property, here the number must play both parts; as an object it is called a variable or a variable magnitude, and as a property it is called a value. That is why people prefer the word “magnitude” to the word “number”; they have to deceive themselves about the fact that the variable magnitude and the value it is said to assume are essentially the same thing, that in this case we have not got an object assuming different properties in succession, and that therefore there can be no question of a variation.\footnote{\[22, p. 660–661\]}

The essay closes with the following assessment:

The endeavor to be brief has introduced many inexact expressions into mathematical language, and these have reacted by obscuring thought and producing faulty definitions. Mathematics ought properly to be a model of logical clarity. In actual fact there are perhaps no scientific works where you will find more wrong expressions, and consequently wrong thoughts, than in mathematical ones. Logical correctness should never be sacrificed to brevity of expression. It is therefore highly important to devise a mathematical language that combines the most rigorous accuracy with the greatest possible brevity. To this end a symbolic language would be best adapted, by means of which we could directly express thoughts in written or printed symbols without the intervention of spoken language.\footnote{\[22, p. 665\]}
Frege aimed to give a clear account of the rules that govern proper logical reasoning. Although, in ordinary language, the line between concepts and objects is sometimes blurry, failure to diagnose and manage the blurriness opens the door to nonsensical reasoning. Even though words like “one” and “horse” sometimes seem to denote both concepts and objects, conflating the two causes problems. For Frege, the only viable solution was to analyze and regiment such uses in a way that cordons off problematic instances. He found that the best way to do this is to maintain a clear separation of concept and object, and then supplement the analysis with an explanation as to how some words seem to cross the divide in certain contexts.

Now let us turn to the second question: why was Frege so dogged in his insistence that mathematical entities like numbers have to be treated as objects, and so persistent, in practice, in pushing mathematical constructions down to that realm? We believe that the answer lies in an observation that we found in Heck [26]: Frege wanted his numbers to be able to count all sorts of entities, and the only way he could make that work was by treating all these entities as inhabitants of the same type. Consider the following statements:

- There are two truth values.
- There are two natural numbers strictly between 5 and 8.
- There are two constant functions taking values among the truth values.
- There are two characters on \((\mathbb{Z}/4\mathbb{Z})^*\).
- There are two subsets of a singleton set.

Frege would have insisted that the word “two” in each of these statements refers to the same object. We would like to say that the number of truth values is equal to the number of natural numbers between 5 and 8, but if truth values and numbers were different types of entities, his analysis of number would not do that: even if Frege had a notion of identity for each type, we would have to define a different notion of two for each such type. In other words, for each type \(\sigma\), we would have to define a concept \(\text{Two}_\sigma\) that holds of concepts of arguments of type \(\sigma\) under which two elements fall.20 Taking extensions according to Frege’s construction would yield an object \(2_\sigma\) for each \(\sigma\). But this results in a proliferation of twos, and since \(2_\sigma\) and \(2_\tau\) are not guaranteed to be the same object, one would have to exercise great care when reasoning about the relationship between them. This is clearly unworkable.


20In Frege’s system, in which the equality symbol can only be used with objects, this would have to be expressed instead in terms of a “sameness” relation for elements of type \(\sigma\), for any \(\sigma\) other than the type of objects.
Instead, Frege designed his numbers to count objects, *simpliciter*: 2 is the extension of the concept of being a concept of objects under which exactly two objects fall. But this means that if you want to count a collection of things, those things have to be elements of the type of objects. This, in turn, provides a strong motivation to locate mathematical entities of all kinds among the type of objects.

Our analysis of Dirichlet’s theorem [3, 4] demonstrates that what holds true of counting holds true of other mathematical operations, relations, and constructions as well. Contemporary proofs of Dirichlet’s theorem have us sum over finite sets of characters just as we sum over finite sets of objects. We view the general operation here as summation over a finite set of objects, viewing both characters and numbers as such. Contemporary proofs also have us consider groups of characters, just as we consider groups of residues. Once again, we consider these as instances of the general group concept, with the understanding that a group’s underlying set can be any set of objects. This allows us to speak of a homomorphism between any two groups, without requiring a different notion of “homomorphism” depending on the type of objects of the groups’ carriers.

Characters were not the only mathematical entities studied in the latter half of the nineteenth century that encouraged set-theoretic reification. Gauss’ genera of quadratic forms, discussed briefly in “Concept,” also bear a group-theoretic structure, and these are sets of quadratic forms. Dedekind developed his theory of ideals in order to supplement rings of algebraic integers with “ideal divisors,” extending the unique factorization property of the ordinary integers to these more general domains. Dedekind found that these ideal divisors could be identified with sets of elements in the original ring, now known as “ideals.” Like the characters, the ideals of a ring of algebraic integers bear an algebraic structure, and Dedekind was adamant that they should be treated as bona-fide objects. Similarly, Dedekind constructed the real numbers by identifying each of them with a pair of sets of rational numbers [10]. By the end of the century, it was common to view a quotient group as a group whose elements are equivalence classes, or cosets.

The reasons given above to treat mathematical functions and sets as objects also speak in favor of treating them extensionally. The statements that “there are two characters on \( \mathbb{Z}/2\mathbb{Z}^* \)” and “there are \( \varphi(m) \) characters on \( (\mathbb{Z}/m\mathbb{Z})^* \)” are false if we take characters to be representations, as there are many different representations of the same character. We could, of course, develop notions of “counting up to equivalence.” In the early days of finite group theory, Camille Jordan described quotient groups as systems just like ordinary groups except that equality is replaced by an appropriate equivalence relation. But, if we do

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21 See footnote 13.
22 See Avigad [1], especially page 172, and Edwards [13].
23 See the detailed discussion in Schlimm [37, Section 3]. Other nice examples of pieces of nineteenth century mathematics that push in favor of set-theoretic abstraction are discussed in Wilson [41, 42].
24 Again, see Schlimm [37, Section 3].
that, mathematical statements become “relativized” to the appropriate equivalence relations, which constitute additional information that needs to be carried along and managed. The alternative is to extensionalize: then the only equivalence relation one has to worry about is equality.

We do not know the extent to which Frege was familiar with examples like these. But Wilson [41, 42] calls attention to an important example of abstraction with which Frege was quite well acquainted. Frege was trained as a geometer, and studied under Ernst Schering in Göttingen. His dissertation, completed in 1873, was titled “Über eine geometrische Darstellung der imaginären Gebilde in der Ebene” (“On a geometric representation of imaginary forms in the plane”). Early nineteenth century geometers found great explanatory value and simplification in extending the usual Euclidean plane with various ideal objects, like “points at infinity” and “imaginary” points of intersection. One of the few motivating examples that Frege provided in the *Grundlagen* (§64–§68) is the fact that one can identify the “direction” of a line $a$ in the plane with the extension of the concept “parallel to $a$.” As Wilson points out (though Frege does not), these “directions” are exactly what is needed to serve as points at infinity, enabling one to embed the Euclidean plane in the larger projective plane, which has a number of pleasing properties. In the projective plane, all points have equal standing, and so it stands to reason that the concept-extensions used to introduce the new entities should be given the same ontological rights as the Euclidean points and lines used in their construction. Wilson characterizes such strategies for expansion as forms of “relative logicism,” since they provide a powerful means of relating the newly-minted objects to the more familiar ones.25

When it comes down to the nitty-gritty details, however, the only sustained formal development we have from Frege is his treatment of arithmetic. But even in this particular case, many of the issues we have raised come to the fore. In the *Grundgesetze*, Frege defined a number of general operations and relations on tuples, sequences, functions, and relations. All of these now can be viewed as general set-theoretic constructions. What gives these constructions universal validity is that they can be applied to any domain of objects, and we now have great latitude in creating objects, as they are needed, to populate these domains. It is precisely the ability to bring a wide variety of mathematical constructions into the realm of objects, and the ability to define predicates and operations uniformly on this realm, that renders Frege’s logic so powerful—too powerful, alas. But given Frege’s goals, it should be clear why the extension operator held so much appeal.26

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25See also Tappenden [38] for other ways that nineteenth century mathematics seems to have influenced his Frege’s philosophical views.

26The same uniformity is achieved in set theory by having a large universe of sets, and incorporating set-forming operations which return new elements of that universe. Russell introduced the notion of *typical ambiguity* [35, 15] to allow “polymorphic” operations defined uniformly across types, and modern interactive theorem provers based on simple type theory follow such a strategy to obtain the necessary uniformities. For example, most such systems have operations $\text{card}_\sigma$ which maps a finite set of elements of type $\sigma$ to its cardinality, a natural number. The systems include mechanisms that allow one to define this family of operations
To sum up, we have traced a central tension in Frege’s work to the need to balance two competing desiderata:

1. the need for flexible but rigorous ways of talking about higher-type entities, like functions, predicates, and relations, without falling prey to incoherence; and

2. the need for ways of dealing with mathematical objects uniformly, since mathematical constructions and operations have to be applied to many sorts of objects, many of which cannot be foreseen in advance.

When one compares this to the considerations that nineteenth century mathematicians faced (see especially the analysis in [3, Section 7] and [4, Section 7]), the similarities are striking.

At the end of an essay on Frege’s treatment of concepts and objects, Thomas Ricketts briefly discusses aspects of nineteenth-century mathematical practice that may have had an influence on Frege. Agreeing with Wilson’s assessment of the importance of being able to construct ideal elements in projective geometry, Ricketts writes:

Throughout his career, Frege is concerned with the introduction of new domains in mathematics, with the ‘creation’ of new mathematical concepts. He vigorously polemicizes against formalist account of this practice and aims to develop an alternative to it. Frege’s own approach here shines forth in a comment on Dedekind’s account of the real numbers:

The most important thing for an arithmetician who recognizes in general the possibility of creation [of mathematical objects] will be to develop in an illuminating way [in einleuchender Weise] the laws governing this in order to prove in advance of each individual creative act that the laws allow it. Otherwise, everything will be imprecise, and proofs will degenerate to a mere appearance, to a good-willed self-delusion. [21, Vol. 2, §140].

The desired foundation will be provided by formulating a logical law that, in the context of other logical laws, will yield as a theorem the existence of the desired new objects. [34, pp. 217–218]

What we have aimed to do here is to explain in greater detail why it is mathematically important to treat certain sorts of things as objects, and what, exactly, that amounts to. Not just Frege’s work in geometry, but also his construction of the natural number system, would have impressed upon him the importance uniformly, once and for all, treating $\sigma$ as a parameter. One can then write $\text{card} \ A$, and let the system infer the relevant type parameter from the type of $A$. This provides one means of coping with the nonuniformities that arise from a type-theoretic compartmentalization of the mathematical universe, but the difficulties that accrue to taking simple type theory as a mathematical foundation are complex; see, for example, [2].
of having uniform operations and constructions on higher-order entities, and having a uniform way of making general assertions about these operations and constructions.

In other words, Frege, like the various mathematical authors we have considered, was responding to methodological pressures that are inherent in the nature of the mathematical enterprise. As Ricketts emphasizes, Frege’s entire foundational project was designed to address the important mathematical need of introducing clear means of expression, and developing general consensus as to the rules of use, while ensuring that the expressions and rules are meaningful, reliable, and consistent. While mathematicians from Dirichlet to Landau were focused on extending the edifice of mathematical knowledge, Frege’s goal was to shore up the foundations. This difference translates to differences in perspective, focus, and method, but the distinctions are not sharp. Working from different ends of the spectrum, both Frege and his mathematical counterparts were working to clarify and extend mathematical method in powerful ways. In doing so, they addressed similar mathematical goals, and responded to similar mathematical constraints.

Frege is often faulted for failing to recognize the simple inconsistency that arises from the formal means he introduced to resolve the tension between the two concerns enumerated above. Nonetheless, it is worth highlighting the extent to which these two concerns were central to the subsequent development of logic and foundations. Russell’s paradox shows that Frege was perfectly right to worry that an overly naive treatment of functions, concepts, and objects would lead to problems in the most fundamental use of our language and methods of reasoning. And, going into the twentieth century, developments in all branches of mathematics called for liberal means of constructing new mathematical domains and structures, as well as uniform ways of reasoning about their essential properties. The most fruitful and appropriate means of satisfying these needs was by no means clear at the turn of the twentieth century. Indeed, these issues were at the heart of the tumultuous foundational debates that were looming on the horizon.

References


[38] Jamie Tappenden. The Riemannian background to Frege’s philosophy. In [16], pages 97–132.


