A variant of the double-negation translation*

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Abstract

An efficient variant of the double-negation translation explains the relationship between Shoenfield's and Gödel's versions of the Dialectica interpretation.

Fix a classical first-order language, based on the connectives $\vee$, $\wedge$, $\neg$, and $\forall$. We will define a translation to intuitionistic (even minimal) logic, based on the usual connectives. The translation maps each formula φ to the formula $\varphi^* = \neg\varphi_*$, so φ_* is supposed to represent an intuitionistic version of the *negation* of φ. The map from φ to φ_* is defined recursively, as follows:

$$\begin{array}{rcl} \varphi_* & = & \neg\varphi, \quad \text{when } \varphi \text{ is atomic} \\ (\neg\varphi)_* & = & \neg\varphi_* \\ (\varphi \vee \psi)_* & = & \varphi_* \wedge \psi_* \\ (\varphi \wedge \psi)_* & = & \varphi_* \vee \psi_* \\ (\forall x\, \varphi)_* & = & \exists x\, \varphi_* \end{array}$$

Note that we can eliminate either $\vee$ or $\wedge$ and retain a complete set of connectives. If Γ is the set of classical formulas $\{\varphi_1, \ldots, \varphi_k\}$, let Γ^* denote the set of formulas $\{\varphi_1^*, \ldots, \varphi_k^*\}$. The main theorem of this note is the following:

Theorem 0.1

1. *Classical logic proves $\varphi \leftrightarrow \varphi^*$.*
2. *If φ is provable from Γ in classical logic, then φ^* is provable from Γ^* in minimal logic.*

Note that both these claims hold for the usual Gödel-Gentzen translation $\varphi \mapsto \varphi^N$. Thus the theorem is a consequence of the following lemma:

Lemma 0.2 *For every φ, minimal logic proves $\varphi^* \leftrightarrow \varphi^N$.*

*Carnegie Mellon Technical Report CMU-PHIL 179. This note was written in response to a query from Grigori Mints. After circulating a draft, I learned that Thomas Streicher and Ulrich Kohlenbach had hit upon the same solution [5], and that the version of the double-negation translation described below is due to Streicher and Reus [6], inspired by a similar translation by Jean-Louis Krivine. These results now appear as exercises in [3]. Since this variant of the double-negation translation and its application to the Dialectica translation are not well known, however, posting this note seemed worthwhile. Similar double-negation translations, for formulas in negation-normal form, can be found in [1, 2]. July 6, 2007: I am grateful to Jaime Gaspar for pointing out an error in the statement of Proposition 0.4, which has now been corrected.

Proof. By induction on φ. The cases where φ is atomic or a negation are immediate. For $\vee$, we have

$$(\varphi \vee \psi)^* = \neg(\varphi_* \wedge \psi_*) \equiv \neg(\neg\neg\varphi_* \wedge \neg\neg\psi_*) \equiv \neg(\neg\varphi^N \wedge \neg\psi^N) = (\varphi \vee \psi)^N.$$

For $\wedge$, we have

$$(\varphi \wedge \psi)^* = \neg(\varphi_* \vee \psi_*) \equiv \neg\varphi_* \wedge \neg\psi_* \equiv \varphi^N \wedge \psi^N = (\varphi \wedge \psi)^N.$$

For $\forall$, we have

$$(\forall x\ \varphi)^* = \neg\exists x\ \varphi_* \equiv \forall x\ \neg\varphi_* \equiv \forall x\ \varphi^N = (\forall x\ \varphi)^N.$$

This concludes the proof. □

In his textbook [4], Shoenfield defines a version of the Dialectica translation for the language of arithmetic based on the connectives $\vee$, $\neg$, and $\forall$. Each formula φ is mapped to a formula φ^S of the form $\forall a\ \exists b\ \varphi_S(a, b)$, where a and b are sequences of variables. Assuming φ^S is as above and ψ^S is $\forall c\ \exists d\ \psi_S(c, d)$, the translation is defined recursively, as follows:

$$\begin{aligned} \theta^S &= \theta, \quad \text{when } \theta \text{ is atomic} \\ (\neg\varphi)^S &= \forall B\ \exists a\ \varphi_S(a, B(a)) \\ (\varphi \vee \psi)^S &= \forall a, c\ \exists b, d\ (\varphi_S(a, b) \vee \psi_s(c, d)) \\ (\forall x\ \varphi)^S &= \forall x, a\ \exists b\ \varphi_S(a, b) \end{aligned}$$

Shoenfield's main result is this:

Theorem 0.3 *If φ is provable in classical arithmetic, there are terms B such that $\varphi_S(a, B(a))$ is provable in Gödel's theory T.*

If η is a formula in the language of intuitionistic logic, let η^D denote the usual Dialectica translation. It is straightforward to verify the following by recursion on formulas:

Proposition 0.4 *Suppose φ^S is $\forall a\ \exists b\ \varphi_S(a, b)$. Then $(\varphi^*)^D$ is of the form $\exists B\ \forall a\ \hat{\varphi}(a, B(a))$, where $\hat{\varphi}$ is intuitionistically equivalent to φ_S.*

Thus Shoenfield's result is just a corollary of Gödel's, together with the $*$ mapping of classical to intuitionistic arithmetic.[1]

As I have presented it, the $*$ translation is remarkably parsimonious in adding negations to a formula. It fares slightly worse on the connectives $\rightarrow$ and $\forall$:

$$\begin{aligned} (\varphi \rightarrow \psi)_* &= \neg\varphi_* \wedge \psi_* \\ (\exists x\ \varphi)_* &= \forall x\ \neg\neg\varphi_*. \end{aligned}$$

Thus it adds a negation for each $\rightarrow$, and *two* negations for each $\exists$. This is reminiscent of the Kuroda translation, which adds two negations after each universal quantifier, and two at the beginning of the formula. (Note, however, that

[1]Streicher and Kohlenbach point out that Theorem 0.3 still holds if one defines $(\varphi \wedge \psi)^S$ to be $\forall a, c\ \exists b, d\ (\varphi_S(a, b) \wedge \psi_S(c, d))$. But if one wants Proposition 0.4 to hold as stated, one has to define $(\varphi \wedge \psi)^S$ to be the intuitionistically equivalent formula $\forall z, a, c\ \exists b, d\ ((z = 0 \rightarrow \varphi_S(a, b) \wedge (z \neq 0 \rightarrow \psi_S(c, d)))$.

verifying the Kuroda translation of a classical theorem requires *intuitionistic* logic, not just minimal logic.)

The nice thing is that when translating formulas from classical to intuitionstic logic, one can use the Kuroda and the $*$ translations interchangeably, since the resulting formulas are equivalent. When carrying out the Dialectica interpretation of a classical theorem, the $*$-based Shoenfield translation is often more convenient.

References

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[5] T. Streicher and U. Kohlenbach. Shoenfield = Gödel $\circ$ Krivine. Manuscript available online, http://www.mathematik.tu-darmstadt.de/~streicher/GKS.pdf.gz

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