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Revision proofness

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Abstract

We analyze an equilibrium concept called *revision-proofness* for infinite-horizon games played by a dynasty of players. Revision-proofness requires strategies to be robust to joint deviations by multiple players and is a refinement of sub-game perfection. Sub-game perfect paths that can only be sustained by reversion to paths with payoffs below those of an alternative path are not revision-proof. However, for the important class of quasi-recursive games careful construction of off-equilibrium play can render many, and in some cases all, sub-game perfect paths revision-proof.

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1. Introduction

We consider infinite horizon repeated and dynamic games played by a sequence or dynasty of strategic players. These players may be interpreted as distinct individuals or as the selves of a single individual with time inconsistent preferences. Following Kocherlakota [10], they may be interpreted as policymakers in a reduced form representation of a macroeconomic policy game. Infinite horizon dynastic games often admit large sets of sub-game perfect equilibria. Moreover, some sub-game perfect equilibria are unappealing severe and potentially vulnerable to coordinated reforms that make all participants weakly better off and some strictly so. In this paper

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we explore an equilibrium refinement called *revision-proofness* that requires robustness to such reforms. A strategy is revision-proof if there is no alternative strategy that weakly raises the payoffs of all players in a sub-game and strictly raises the payoffs of some. A path of actions is revision-proof if it can be implemented with a revision-proof strategy. This concept was previously proposed by Asheim [3]¹ and, in finite games, by Caplin and Leahy [7].² However, as yet limited characterization of the concept has been given. Our goal is to fill this gap. Our main result is that in the important class of quasi-recursive dynastic games many and sometimes all sub-game perfect equilibrium paths can be supported by revision-proof strategies. Consequently, to the extent that our assumptions are satisfied, concerns about vulnerability of sub-game perfect outcomes to coordinated reform are not warranted.

Sub-game perfection in dynastic games requires that a strategy be robust to a rather limited set of “revisions”, namely those that involve a single player altering her action and taking the response of her successors to this alteration as given. Since players do not fully internalize the effect of their actions on other players’ payoffs, sub-game perfection often permits strategies that all players agree are weakly inferior and some think are strictly inferior to alternatives. The latter, however, can only be reached via coordinated reforms and these are not possible. Revision-proofness requires that a strategy be robust to a much larger set of alternatives. Now, all feasible paths of actions are candidate revisions. However, revision-proofness permits a path to disrupt a strategy only if *every* player along the path weakly prefers continuing with it to returning to the strategy and at least one player strictly prefers this.

The extent to which revision-proofness refines sub-game perfection, depends upon the structure of player preferences. We show that if (i) a player strictly prefers path *A* to path *B* and (ii) players are deterred from joining *A* only if their successors receive payoffs below those implied by *A*, then *B* is not revision-proof. We give two implications of this result. First, if in every sub-game there is a path whose continuation is optimal for the current and later players, then a strategy is revision-proof if (and only if) it attains such an optimum in every sub-game. Sub-game perfection does not ensure this.³ Second, if a sub-game perfect strategy uses an exploding sequence of punishments to deter players from unilaterally joining a revision, then it is not revision-proof. If sub-game perfect implementation of a path requires such a sequence, then the path is not revision-proof.

In the remainder of the paper, we focus on quasi-recursive dynastic games. In these games a player and her successor may have differing preferences over the successor’s choices, but conditional on the successor’s choice, identical preferences over future play. We give examples including a pension game of Hammond [9], a macro-policy game of Kocherlakota [10] and a savings game with quasi-hyperbolic discounting. We provide conditions that ensure many, and in some cases all, action paths in quasi-recursive games are revision-proof. Consequently, to the extent that the conditions we give are satisfied, sub-game perfect paths and outcomes are robust to coordinated mutually beneficial deviations.⁴ Our procedure for showing that a given

¹ Although formulated quite differently.

² Hammond [9] gives an earlier variation on the definition. Alternative renegotiation-proof concepts for repeated games have been provided by, inter alia, Kocherlakota [10], Pearce [12] and Farrell and Maskin [8]. We discuss these briefly below.

³ It does so under the stronger condition that player preferences are time consistent and satisfy a “continuity-at-infinity” condition.

⁴ Although this implies revision-proofness is a weak refinement with respect to paths and outcomes, it remains restrictive with respect to off-equilibrium play.

path is revision-proof (i.e. is implemented by at least one revision-proof strategy) is constructive. We build a strategy that implements the path and confronts all sufficiently long revisions with a “blocking” player. For this player there is no alternative continuation path that raises her predecessors’ payoffs without reducing her own or her successors’ payoffs. Consequently, the blocking player (or her successors) will reject any attempt to revise the strategy towards a continuation path that her predecessors prefer; any successful revision must leave the path following the blocking player intact. The strategy is then constructed to ensure that any revision which does this either leaves all players’ payoffs unaltered or makes some prior player worse off. Consequently, the strategy is revision proof.

The revision-proofness concept presupposes that intertemporally separated players can engage in mutually beneficial coordination. We do not explicitly model how this coordination takes place. One possibility is that earlier players can communicate and recommend mutually beneficial courses of action to later ones. Thus, a player who is made strictly better off by the adoption of an alternative path of actions that leaves no future agent worse off can leave a recommendation to future agents to follow the path (as an explicit message or a memory if the players are selves of a common individual). Future agents would have no reason not to follow the recommendation and the original strategy from which the defection occurs is undone. One might consider weaker refinements of sub-game perfection that involve a more limited ability of earlier players to organize a multilateral deviation or that require stronger incentives for future agents to participate (for example, that all agents who alter their actions receive a strict increase in payoff). However, our main result is that many sub-game perfect paths in quasi-recursive games are robust to the stronger form of revision-proofness we adopt. They are therefore robust to these weaker forms.

The paper proceeds as follows. After a brief review of the literature, a general dynastic game is outlined in Section 2. The formulation allows players to care about past histories of actions in arbitrary ways and so accommodates both repeated games (in which players do not care about the past) and dynamic games (in which they care about a state variable inherited from the past). Section 3 defines sub-game perfection and revision-proofness respectively and introduces some preliminary results. It also provides conditions for a path of actions not to be revision-proof and considers the implications of these conditions for games featuring weak agreement over optima and (potentially) exploding punishment paths. Sections 4 and 5 give conditions for revision-proofness in quasi-recursive games first without and then with state variables. Section 6 relates revision-proofness to Asheim’s revision-proofness concept and Kocherlakota’s reconsideration-proof concept. Section 7 concludes. [Appendices A–E](#) give proofs and supplementary results.

1.1. Literature

The revision-proofness concept was introduced by Asheim [3]. Our formulation of the concept is more intuitive and accessible than his. We prove the equivalence of the two formulations in Section 6. Asheim [3] relates revision-proofness to the coalition-proof Nash concept of Bernheim, Peleg, and Whinston [5] and to the consistent planning concept of Strotz [13]. It is well known that consistent plans need not exist. By extension, revision-proof strategies need not exist. Caplin and Leahy [7] address the existence problem in finite horizon settings. In Hammond [9] a strategy is defined to be a *dynamic equilibrium* if there exists no alternative strategy that strictly raises the payoffs of players whenever it prescribes different actions for those players. Thus, Hammond’s refinement is weaker than that proposed here.

Kocherlakota [10] gives a refinement for dynastic games which he calls *reconsideration-proofness*. A strategy is reconsideration-proof if it is best amongst payoff stationary

sub-game perfect strategies. Revision-proofness allows a player to compare two strategies across histories and coordinate her successors onto the dominating one. In contrast, reconsideration-proofness supposes that a player compares the continuations of a given strategy and selects the dominant one subject to the constraint that her successors will do likewise. In the absence of state variables or history dependence, only payoff stationary equilibria survive this internal reconsideration process. Kocherlakota [10] further assumes that players coordinate onto a best payoff stationary equilibrium. Reconsideration-proofness can be interpreted as a specialization of the renegotiation concept of Farrell and Maskin [8] to dynastic games. Kocherlakota [10] develops reconsideration-proofness only for repeated games. To date no extension of the concept to dynamic games has been provided.

Pearce [12] supplies an alternative notion of renegotiation-proofness for repeated games played by a finite number of infinitely-lived players. He assumes that a strategy will be revised if there is a history and an alternative strategy that gives higher payoffs at all subsequent histories relative to those obtained under the original strategy and at the original history. Again a difficulty with this renegotiation-proof concept is that its extension to dynamic games is unclear.

2. The environment

Let (\mathcal{P}, ρ) denote a metric space of actions and \mathbb{N} the natural numbers. A dynasty of one period-liver players, $t \in \mathbb{N}$, choose actions p_t from \mathcal{P} . Players may be interpreted as different individuals or different selves of the same individual. Let $P = \{p_t\}_{t=1}^{\infty}$ be the complete path of actions chosen by agents and for $T \in \mathbb{N}$, let $P^T = \{p_t\}_{t=1}^T$ be a T -period history of actions and $P_{T+1} = \{p_t\}_{t=T+1}^{\infty}$ a period $T + 1$ continuation path. The sets $\mathcal{P}^{\infty} = \mathcal{P} \times \mathcal{P} \times \dots$ and $\mathcal{P}^T := \mathcal{P} \times \dots \times \mathcal{P}$, containing paths and T -period histories respectively, are endowed with the associated product topologies. \mathcal{P}^0 is set equal to \emptyset and p_0 and P^0 to null elements.

The objectives of the players are given by a payoff function $U : \mathbb{N} \times \mathcal{P}^{\infty} \rightarrow \bar{\mathbb{R}}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, with $U(t, \cdot)$ the objective of the t -th player. $U(t, \cdot)$ is defined over the entire action path so that the t -th player's payoffs depend upon her ancestors', her own and her descendants' actions. Depending on the setting, this dependence can occur because of preference interactions across individuals (e.g. altruism or envy), or selves (e.g. intra-personal discounting, habit formation). In addition, earlier players may affect the choice sets of later ones by, for example, the accumulation of capital. To economize on notation we encode such effects into player objectives by assigning $-\infty$ payoff values to paths that past play renders infeasible. This becomes a further channel via which players can affect the payoffs of successors.

Definition 1. The set of *feasible continuation paths* for player t at P^{t-1} is given by $\Pi_t(P^{t-1}) = \{P \in \mathcal{P}^{\infty} \mid U(t, P^{t-1}, P) > -\infty\}$. $P = (P^{t-1}, P_t)$ is *feasible for player t* if $P_t \in \Pi_t(P^{t-1})$. The set of *feasible actions* for player t at P^{t-1} is $\Gamma_t(P^{t-1}) = \{p \mid \exists P \text{ with } (p, P) \in \Pi_t(P^{t-1})\}$.

We impose the following condition.

Assumption 1. Let U satisfy (i) for all $t \in \mathbb{N}$ and $P^{t-1} \in \mathcal{P}^{t-1}$, $\Pi_t(P^{t-1}) \neq \emptyset$ and (ii) for all $t \in \mathbb{N}$ and $P^{t-1} \in \mathcal{P}^{t-1}$, $\Pi_t(P^{t-1}) = \{(p, P) \mid p \in \Gamma_t(P^{t-1}), P \in \Pi_{t+1}(P^{t-1}, p)\}$.

Assumption 1 (i) ensures that every player at every history has a feasible continuation path. **Assumption 1** (ii) ensures that if a path is feasible for a player, then it is feasible for all of the player's successors. Conversely, a feasible action choice for the current player combined with any

feasible continuation path for the next player forms a feasible continuation path for the current player. Together the set \mathcal{P} and the payoff function U defines a dynastic game, $\mathcal{G}(U)$. Each sub-game of $\mathcal{G}(U)$ is itself a dynastic game. The sub-game following P^t is denoted $\mathcal{G}(U; P^t)$.

A strategy $\sigma = \{\sigma_t\}_{t=1}^\infty$, $\sigma_t : \mathcal{P}^{t-1} \rightarrow \mathcal{P}$, describes player behavior after every history. Let $(\sigma|P^t) = \{\sigma_{t+r}(P^t, \cdot)\}_{r=1}^\infty$ denote the continuation of σ after some history P^t and let $\Phi(\sigma)$ denote the action path induced by strategy σ . We require strategies to be feasible for players in the following sense, for all t and P^{t-1} ⁵:

$$U(t, P^{t-1}, \Phi(\sigma|P^{t-1})) > -\infty. \tag{1}$$

Let $\mathcal{S}(U)$ (resp. $\mathcal{S}(U; P^t)$) denote the set of feasible strategies for $\mathcal{G}(U)$ (resp. $\mathcal{G}(U; P^t)$). By [Assumption 1](#), $\mathcal{S}(U) \neq \emptyset$. Of course, if $\sigma \in \mathcal{S}(U)$, then $(\sigma|P^t) \in \mathcal{S}(U; P^t)$. In the remainder of the paper the term strategy will refer to a feasible strategy.

The function U implies indirect payoff functions over players, histories and strategies, $V_t : \bigcup_s \mathcal{P}^s \times \mathcal{S} \rightarrow \mathbb{R}$, where:

$$\forall P^r \in \bigcup_s \mathcal{P}^s, \quad V_t(P^r, \sigma) := U(t, P^r, \Phi(\sigma|P^r)).$$

$V_t(P^r, \sigma)$ gives the t -th player’s evaluation of history P^r and the continuation path induced by σ after P^r .

3. Revision-proofness

This section introduces revision-proofness in the general dynastic setting of [Section 2](#).

3.1. Definitions

We define sub-game perfection, then revision-proofness.

Definition 2. Given a game $\mathcal{G}(U)$, $\sigma \in \mathcal{S}(U)$ is *sub-game perfect* if there is no $t \in \mathbb{N}$, history $P^{t-1} \in \mathcal{P}^{t-1}$ and action $p \in \mathcal{P}$ such that⁶:

$$V_t(P^{t-1}, p, \sigma) > V_t(P^{t-1}, \sigma).$$

$P \in \mathcal{P}^\infty$ is a *sub-game perfect path* if $P = \Phi(\sigma)$ for some sub-game perfect strategy σ .

Definition 3. Given a game $\mathcal{G}(U)$, $\sigma \in \mathcal{S}(U)$ is *revision-proof* if there is no $t \in \mathbb{N}$, history $P^{t-1} \in \mathcal{P}^{t-1}$ and alternative strategy σ' such that for all $r \in \mathbb{N}$ and $P^r \in \mathcal{P}^r$:

$$V_{t+r}(P^{t-1}, P^r, \sigma') \geq V_{t+r}(P^{t-1}, P^r, \sigma), \tag{2}$$

with the inequality (2) strict for at least one P^r . A path $P \in \mathcal{P}^\infty$ is *revision-proof* if $P = \Phi(\sigma)$ for some revision-proof strategy σ . P' is a *successful revision path* for σ at P^{t-1} if for all $r \in \mathbb{N}$,

$$U(t-1+r, P^{t-1}, P') \geq V_{t+r}(P^{t-1}, P^r, \sigma), \tag{3}$$

with strict inequality for at least one r .

⁵ By [Assumption 1](#), the period t feasibility condition in (1) does not restrict the future actions of players at $t+r$, $r \in \mathbb{N}$, beyond requiring that they too are feasible.

⁶ The restriction to feasible strategies eliminates uninteresting cases in which each player takes an action yielding an $-\infty$ payoff to its predecessor.

3.2. Basic properties

Sub-game perfection rules out strictly profitable unilateral deviations; revision-proofness rules out multilateral deviations that are strictly profitable for some and weakly profitable for all players in a sub-game. Consequently, revision-proofness is a refinement of sub-game perfection, a fact we record below.

Proposition 1. *If σ is revision-proof for $\mathcal{G}(U)$, then it is sub-game perfect for $\mathcal{G}(U)$.*

A proof is given in [Appendix A](#); see also Asheim [[3](#), [Proposition 3](#)]. The next proposition gives recursivity properties of sub-game perfect and revision-proof strategies.

Proposition 2. *σ is sub-game perfect for $\mathcal{G}(U)$ if and only if (i) σ_1 is optimal for the initial player given the continuations $(\sigma|p)$, $p \in \mathcal{P}$, and (ii) each continuation $(\sigma|p)$ is sub-game perfect for $\mathcal{G}(U; p)$. σ is revision-proof for $\mathcal{G}(U)$ if and only if (i) $\Phi(\sigma) \in \operatorname{argmax}\{U(1, P): \forall t = 2, 3, \dots, U(t, P) = V_t(P^{t-1}, \sigma)\}$ and (ii) for each $p \in \mathcal{P}$, $(\sigma|p)$ is revision-proof for $\mathcal{G}(U; p)$.*

A proof of [Proposition 2](#) is given in [Appendix A](#). The recursive characterization of sub-game perfect strategies in [Proposition 2](#) is well known. Following Abreu, Pearce, and Stacchetti [[1](#)] and with additional structure on preferences such as the quasi-recursive structure assumed below, this recursivity may be exploited to show that the set of sub-game perfect payoffs is the largest fixed point of a time invariant Bellman-like operator. Revision-proof strategies satisfy a stronger property: the continuation of a revision-proof strategy is an infinite-horizon refinement of a revision-proof strategy. Consequently, recursive methods are much less useful in characterizing revision-proof strategies and we are compelled to use non-recursive methods.⁷

3.3. Revision-proofness as a strict refinement

In this section, we explore the extent to which revision-proofness can refine the set of strategies and paths. To fix ideas, we begin with two simple examples. The first is similar to Example 2 of Asheim [[3](#)].

Example 1. Suppose two actions A and B . Players derive utility from their own and their successor's choices. Their preferences are given by $U(t, P) = u(p_t) + v(p_{t+1})$, where $u(A) = u(B)$ and $v(A) > v(B)$. Thus, each player is indifferent between taking A or B , but strictly prefers that her successor takes A . All players agree that $A^\infty = (A, A, A, \dots)$ is optimal. The strategy σ^1 with for all t and P^{t-1} , $\sigma_t^1(P^{t-1}) = A$ delivers the maximal payoff of $u(A) + v(A)$ to all players and is clearly revision-proof.

Consider strategy σ^2 , with for all t and P^{t-1} , $\sigma_t^2(P^{t-1}) = B$. Under σ^2 each player is taking a maximal action given continuation play and σ^2 is sub-game perfect. But it is not revision-proof. From any history P^{t-1} , the path A^∞ is a successful revision: it raises the payoff to all players

⁷ Under the quasi-recursive restriction of later sections, the set of revision-proof payoffs is a fixed point of an extended Abreu, Pearce, and Stacchetti [[1](#)]-operator. However, it need not be the largest, bounded fixed point. Ales and Sleet [[2](#)] show that this gives the set of payoffs from finite revision-proof strategies, i.e. strategies that are robust to finite revisions of arbitrary length. All revision-proof strategies are finite revision-proof, but the converse is not true. In [Example 2](#) below all sub-game perfect strategies are finite revision-proof, but only one is revision-proof.

$t - 1 + r, r \in \mathbb{N}$, at histories (P^{t-1}, A^r) , by breaking the indifference of players in favor of predecessors.

In [Example 1](#), revision-proofness breaks indifferences of later players in favor of earlier ones and, hence, refines the set of sub-game perfect strategies. Caplin and Leahy [7] emphasize this aspect of the concept in finite horizon games. In infinite horizon games, revision-proofness may additionally exclude strategies in which some or all players strictly prefer to take an action that lowers their current payoff given their successors' prescribed responses. In these cases, there may be no indifferences to break (and no successful finite-length revision), but an infinite-length revision may still improve all players' payoffs relative to reversion to the strategy. The next example illustrates.

Example 2. Suppose three actions A, B and C . A player derives utility from her own and her successor's action choices. Her preferences are given by: $U(t, P) = u(p_t) + v(p_{t+1})$, where $u(A) > u(B) \geq u(C), v(A) > v(C) > v(B)$, and $u(C) + v(C) > u(A) + v(B)$. All players agree that A is a weakly optimal action in all periods (and a strictly optimal one during their own lifetime). However, they would rather play C and have their successor do likewise than play A and have their successor play B . It is clear that the strategy σ^1 with for all t and $P^{t-1}, \sigma_t^1(P^{t-1}) = A$ is sub-game perfect and delivers the maximal payoff of $u(A) + v(A)$ to all players. Consequently, it is also revision-proof.

A second sub-game perfect strategy is defined by σ^2 , with $\sigma_1^2 = C$ and, for $t > 1$,

$$\sigma_t^2(P^{t-1}) = \begin{cases} C & \text{if } p_{t-1} = \sigma_{t-1}^2(P^{t-2}) \\ B & \text{otherwise.} \end{cases}$$

In this case, the threat of a future play of B deters players from choosing their optimal action A . No player would wish to be the last in a revision consisting of a finite number of plays of A followed by reversion to σ^2 . Thus, all revision paths $(A^r, \Phi(\sigma^2|A^r))$ are unsuccessful. But A^∞ is a successful revision path: it raises the payoff to players at all sub-histories A^r . σ^1 is the only revision-proof strategy in this game. Any other strategy, σ^2 for example, delivers a weakly lower payoff at all histories and a strictly lower payoff at some.

A common element in the previous examples is the existence of a “dominating path” $\tilde{P} = A^\infty$ and a pair of “dominated continuation path sets”, $\mathcal{D}_i, i = 0, 1$. In [Example 2](#) these are given by $\mathcal{D}_0 = \mathcal{P}^\infty \setminus \{A^\infty\}$ and $\mathcal{D}_1 = \mathcal{P}^\infty$.⁸ The triple $(\tilde{P}, \mathcal{D}_0, \mathcal{D}_1)$ satisfies two properties. First, for any t and P^{t-1} , the t -th player strictly prefers \tilde{P} to any path in \mathcal{D}_0 and players $t + r, r \in \mathbb{N}$, weakly prefer the continuation of \tilde{P} to any path in \mathcal{D}_1 . Second, to deter the $t - 1 + r$ -th player from defecting away from a path in \mathcal{D}_0 (if $r = 1$) or \mathcal{D}_1 (if $r > 1$) and playing A , a continuation path in \mathcal{D}_1 must be played. This second property is rather trivial in the example since $\mathcal{D}_1 = \mathcal{P}^\infty$. However, in more general settings it is not. These properties ensure that no path in \mathcal{D}_0 is revision-proof. Essentially, deterring successive players $t + r$ from being the last to join a finite revision \tilde{P}^r “traps” continuation play into sets that are payoff dominated by \tilde{P} .

The following proposition extends the intuition of the previous paragraph to settings where the set \mathcal{D}_1 might change for every successive player. Starting from t and a prior history P^{t-1} , [Proposition 3](#) supposes a dominating path \tilde{P} and a sequence of dominated continuation path

⁸ In [Example 1](#), $\mathcal{D}_0 = \mathcal{P}^\infty \setminus \{A^\infty, (B, A^\infty)\}$.

sets $\mathcal{D}_r, r \in \mathbb{N}$. These satisfy two properties analogous to those above. First, each player $t - 1 + r, r \in \mathbb{N}$, weakly prefers (strictly for $r = 1$) the continuation of \tilde{P} to the paths in \mathcal{D}_r -th continuation path set. Second, to deter the $t - 1 + r$ -th player from defecting away from a path in \mathcal{D}_r and playing \tilde{p}_r , a continuation path in \mathcal{D}_{r+1} must be played. Proposition 3 shows that this ensures no path in \mathcal{D}_0 is revision-proof.

Proposition 3. Let Assumption 1 hold. Suppose that at P^{t-1} , there is a path \tilde{P} and a family of path sets $\{\mathcal{D}_r\}_{r=0}^\infty, \mathcal{D}_r \subseteq \mathcal{P}^\infty$, such that for each $r \in \{0\} \cup \mathbb{N}$ the following conditions hold:

- (i) if $P \in \mathcal{D}_r$, then $U(t + r, P^{t-1}, \tilde{P}) \geq U(t + r, P^{t-1}, \tilde{P}^r, P) > -\infty$ with the first inequality strict if $r = 0$;
- (ii) if $P \in \mathcal{D}_r$ and $U(t + r, P^{t-1}, \tilde{P}^r, P) \geq U(t + r, P^{t-1}, \tilde{P}^{r+1}, P')$, then $P' \in \mathcal{D}_{r+1}$.

No path in \mathcal{D}_0 is revision-proof.

Proof. See Appendix A. \square

We give two applications of Proposition 3. Let $\mathcal{P}_t^*(P^{t-1}) := \operatorname{argmax}_{\mathcal{P}^\infty} U(t, P^{t-1}, P)$ be the set of optimal plans for the t -th player in the sub-game $\mathcal{G}(U; P^{t-1})$. Let $\bar{U}_t(P^{t-1}) := \max_{P \in \mathcal{P}^\infty} U(t, P^{t-1}, P)$ be the associated optimal payoff.

Definition 4. U exhibits weak agreement over an optimum at t and P^{t-1} , if there is a P^* such that for all $r \in \mathbb{N}, P_r^* \in \mathcal{P}_{t-1+r}^*(P^{t-1}, P^{*r-1})$. U exhibits weak agreement over optima if it exhibits weak agreement over an optimum at all t and P^{t-1} .

Weak agreement over (some) optima does not exclude disagreement over others. It permits optima for the t -th player whose continuations are not optimal for later players or continuations that are optimal for later players but not for the t -th player. Weak agreement over optima is much weaker than time consistency which requires successive players to have identical preference orderings over continuation paths.

It is clear that weak agreement over optima is necessary for a revision-proof strategy to attain the optimal payoffs $\bar{U}_t(P^{t-1})$ in all sub-games.⁹ Our first application of Proposition 3 shows that it is also sufficient for all revision-proof strategies to attain optima in all sub-games.¹⁰

Corollary 1. Assume that U exhibits weak agreement over optima. σ is revision-proof for $\mathcal{G}(U)$ if and only if for all $t \in \mathbb{N}$ and $P^{t-1} \in \mathcal{P}^{t-1}, \Phi(\sigma | P^{t-1})$ is in $\mathcal{P}_t^*(P^{t-1})$.

Proof. See Appendix A. \square

⁹ If weak agreement did not hold, there would be at least one sub-game in which the attainment of an optimal payoff $\bar{U}_t(P^{t-1})$ by a player precludes its attainment by a later player.

¹⁰ A similar result is proved in Asheim [3, Proposition 1(ii)]. Here we derive it as a consequence of the more general Proposition 3.

Examples 1 and 2 show that weak agreement over optima is *not* sufficient to ensure that all sub-game perfect strategies attain optima in all sub-games.¹¹

As a second application of Proposition 3, we give an example in which revision-proofness excludes paths whose sub-game perfect implementations require “explosively” bad sequences of penalties to deter repeated deviations.

Example 3. Let $\mathcal{P} = \mathbb{R}$ and for all t and P , let:

$$U(t, P) = W(P_t) := (1 - \beta)u(p_t) + \sum_{r=1}^{\infty} \beta^r (1 - \beta)v(p_{t+r}),$$

where $P_t = \{p_{t-1+r}\}_{r=1}^{\infty}$. Assume that $u, v : \mathcal{P} \rightarrow \mathbb{R}$ are not equal, bounded above and unbounded below. Let $\hat{p} \in \operatorname{argmax}_{\mathcal{P}} u(p)$ and $\underline{p} \in \operatorname{argmin}_{\mathcal{P}} v(p) - u(p)$, so that \hat{p} maximizes a player’s current payoff and \underline{p} minimizes the difference between a predecessor’s continuation payoff v and a player’s current payoff u . For each $r \in \{0\} \cup \mathbb{N}$, $\mathcal{D}_r = \mathcal{D} := \{P \mid u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})] > W(P)\}$. This set is non-empty since u and v are unbounded below. We show that *no* path in \mathcal{D} is revision-proof.

Let $\hat{P} = (\hat{p}, \hat{p}, \dots)$. By simple algebra, $(1 - \beta)u(\hat{p}) + \beta v(\hat{p}) > u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})]$ and so $\mathcal{D} \subset \{P \mid W(\hat{P}) > W(P)\}$. Thus, the sequence of sets $\{\mathcal{D}_r\}$, each $\mathcal{D}_r = \mathcal{D}$, satisfies (i) in Proposition 3. It remains to check that (ii) in Proposition 3 holds. To this end, suppose $P \in \mathcal{D}$ and $W(P) \geq W(\hat{p}, P')$. Thus, $(\hat{p}, P') \in \mathcal{D}$ as well and so:

$$u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})] > (1 - \beta)u(\hat{p}) + \sum_{s=1}^{\infty} \beta^s (1 - \beta)v(p'_s).$$

Rearranging the last expression and using $v(p'_1) - u(p'_1) \geq v(\underline{p}) - u(\underline{p})$ gives:

$$u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})] > (1 - \beta)u(p'_1) + \sum_{s=1}^{\infty} \beta^s (1 - \beta)v(p'_{s+1}),$$

i.e. $P' \in \mathcal{D}$. It then follows from Proposition 3 that for all histories, no continuation path in \mathcal{D} is revision-proof. However, it is easy to check that \mathcal{D} *does* contain paths that are sub-game perfect. These paths are sustained by reversion to ever more severe paths within \mathcal{D} . For example, if a player receives the payoff w with $u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})] - \varepsilon > w$, then defection from w can only be sustained by reversion to a path P' that delivers a payoff of w' to the subsequent player where:

$$\begin{aligned} u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})] - \varepsilon > w &\geq (1 - \beta)u(\hat{p}) + (1 - \beta)\beta(v(p'_1) - u(p'_1)) + \beta w' \\ &\geq (1 - \beta)\{u(\hat{p}) + \beta(v(\underline{p}) - u(\underline{p}))\} + \beta w' \end{aligned}$$

so $w - \frac{(1-\beta)}{\beta}\varepsilon > w'$. Consequently, in a sub-game perfect equilibrium, an exploding sequence of punishments is necessary to deter long sequences of deviations from a path with initial payoff below $u(\hat{p}) + \beta[v(\underline{p}) - u(\underline{p})]$. However, such paths cannot be implemented by a revision-proof strategy.

¹¹ A sufficient condition for this is time consistency (U is time consistent if for all $t \in \mathbb{N}$, (P^t, P) and $(P^t, P') \in \mathcal{P}^{\infty}$, $U(t, P^t, P) > U(t, P^t, P') \Leftrightarrow U(t + 1, P^t, P) > U(t + 1, P^t, P')$) and continuity of each $U(t, P^{t-1}, \cdot)$ in the relative product topology on $\Pi_t(P^{t-1})$.

4. Quasi-recursive games

In quasi-recursive games a player and her successor have differing preferences over the successor’s current action, but conditional on this action identical preferences over the successor’s future action path. Many dynamic games have such a structure, see, inter alia, the diverse contributions of Bernheim, Ray, and Ş. Yeltekin [6], Leininger [11], Asheim [3] and the examples below drawn from Kocherlakota [10] and Hammond [9]. This section explores the implications of revision-proofness for quasi-recursive games without state variables. In the next section, state variables are introduced.

A quasi-recursive game (without state variables) is defined by a tuple (\mathcal{P}, u, Q, R) , where $u : \mathcal{P} \rightarrow \mathbb{R}$ is a current payoff function and $Q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $R : \mathcal{P} \times \mathbb{R} \rightarrow \mathbb{R}$ are a pair of intertemporal payoff aggregators. We impose the following assumption on (\mathcal{P}, u, Q, R) .

Assumption 2. (i) \mathcal{P} is a compact, convex subset of a normed space $(\mathcal{P}_0, \|\cdot\|)$. \mathcal{P}^∞ is equipped with the associated relative product topology. (ii) $Q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is increasing, concave and continuous. $u : \mathcal{P} \rightarrow \mathbb{R}$ is concave and continuous. (iii) $R : \mathcal{P} \times \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing in its second argument, concave and continuous on $\mathcal{P} \times \mathbb{R}$. Also, there is a $\beta \in [0, 1)$ such that for all $y, y' \in \mathbb{R}$,

$$\sup_{\mathcal{P}} |R[p, y] - R[p, y']| \leq \beta |y - y'|.$$

Assumption 2 ensures that there is a unique bounded, continuous and concave function $Y : \mathcal{P}^\infty \rightarrow \mathbb{R}$ satisfying for all (p, P) , $Y(p, P) = R[p, Y(P)]$, see Appendix B. We refer to Y as a continuation payoff function and define the bounded, continuous and concave payoff function $W : \mathcal{P}^\infty \rightarrow \mathbb{R}$ according to for all (p, P) , $W(p, P) = Q[u(p), Y(P)]$. In a quasi-recursive game, for all $t \in \mathbb{N}$ and $P \in \mathcal{P}^\infty$:

$$U(t, P) = W(P_t).$$

Quasi-recursiveity implies that given a history P^{t+r} both the t -th and $t + r$ -th players have preferences over continuation paths P_{t+r+1} described by Y , but given a history P^{t+r-1} different preferences over continuation paths P_{t+r} described by Y and W respectively. The quasi-recursive structure accommodates many types of time inconsistency considered in the literature. We give two applications below.

Example 4 (*Hammond’s overlapping generations pension game*). (See Hammond [9].) A two period-lived agent is born at each $t \in \mathbb{N}$; a one period-lived old agent lives at $t = 1$. Agents have an endowment of 1 when young and 0 when old. A young agent may share her endowment with a contemporaneous old agent. Let $u^y, u^o : \mathbb{R}_+ \rightarrow \mathbb{R}$ denote the utility from consumption of young and old agents. Players are identified with two period lived agents and are indexed by the date $t \in \mathbb{N}$ at which they are born and are young. Let p_t denote the consumption of the player (young agent) at t . The t -th player’s preferences over paths (of young agent consumption) are given by $U(t, P) = u^y(p_t) + u^o(1 - p_{t+1})$. Thus, $u = u^y$, $Q[u(p), y] = u(p) + y$ and $R[p, y] = u^o(1 - p)$.

Example 5 (*Kocherlakota’s policy game*). (See Kocherlakota [10].) At $t = 1$, a one period-lived household derives utility $\frac{1}{2}g_1$ from public goods. In each period $t \in \mathbb{N}$ a two period-lived household is born and derives lifetime utility $u_t(c_t, g_{t+1}) = c_t + \frac{\beta}{2}g_{t+1}$, $\beta \in (0, 1)$, from private

consumption c_t and public consumption g_{t+1} . Households receive an endowment of income q when old and must borrow against this to finance private consumption when young. They can borrow on an international loan market at rate i . At each date $t \in \mathbb{N}$, a government taxes the income of the old at rate p_t to finance public consumption. The young household at $t \in \mathbb{N}$ borrows the maximal amount subject to being able to pay her taxes, i.e. borrows $(1 - p_{t+1})q/(1 + i)$. The goal of a date t government is to maximize the discounted utility of current and future generations of households including the current old generation, i.e. to maximize:

$$\sum_{r=0}^{\infty} \beta^r \left(c_{t+r} + \frac{1}{2} g_{t+r} \right). \tag{4}$$

By substituting optimal household choices and the government budget constraints $g_{t+r} = p_{t+r}q$ into (4), the objective of the government at t can be re-expressed as a function of the continuation tax path P_t :

$$W(P_t) = \sum_{r=0}^{\infty} \beta^r q \left(\frac{1}{2} p_{t+r} + \frac{(1 - p_{t+r+1})}{(1 + i)} \right). \tag{5}$$

Under the assumption that $i < 1$, the preferred path for the government at t is to set $p_t = 1$ (and, hence, force the old to default), and $p_{t+r} = 0, r \in \mathbb{N}$, to enable high private consumption of current and future young. Rearranging (5) gives:

$$W(P_t) = \frac{1}{2} p_t q + \frac{q}{1 + i} \frac{1}{1 - \beta} - \left(\frac{1}{1 + i} - \frac{\beta}{2} \right) q \sum_{r=1}^{\infty} \beta^{r-1} p_{t+r}.$$

By setting $u(p) = \frac{1}{2} p q + \frac{q}{1+i} \frac{1}{1-\beta}$, $Q[u(p), y] = u(p) + \beta y$, $v(p) = -(\frac{1}{1+i} - \frac{\beta}{2}) q p$ and $R[p, y] = v(p) + \beta y$, a quasi-recursive game is obtained.

4.1. Sub-game perfection in quasi-recursive games

Let \mathcal{Y} denote the set of sub-game perfect continuation payoffs,

$$\mathcal{Y} := \{ y \mid y = Y(\Phi(\sigma)), \sigma \text{ is sub-game perfect} \}.$$

Proposition 9 in Appendix B shows that under Assumption 2, \mathcal{Y} is a compact interval, $[\underline{y}, \bar{y}]$. Lemma 3 in the same appendix characterizes the endpoints of this interval. It shows that there is a stationary sub-game perfect action path $\bar{P} = (\bar{p}, \bar{p}, \dots)$ that attains \bar{y} and satisfies $\bar{y} = R[\bar{p}, \bar{y}]$. It shows that for the worst sub-game perfect continuation payoff \underline{y} there are two possibilities. In the first, \underline{y} is obtained by a stationary sub-game perfect action path $(\underline{p}, \underline{p}, \dots)$, $\underline{y} = R[\underline{p}, \underline{y}]$ and $u(\underline{p}) = u^* := \max_{\mathcal{P}} u(p)$. In this case, action \underline{p} is best for a current player and, amongst actions consistent with sub-game perfection, worst for her predecessor. For the second case, there is no stationary sub-game perfect path that attains the worst sub-game perfect continuation payoff. Instead this payoff is attained by path that begins with a severe action \underline{p} satisfying $\underline{y} > Y(\underline{p}, \underline{p}, \dots)$. The subsequent period's continuation payoff $y'(\underline{p})$ satisfies $\underline{y} = R[\underline{p}, y'(\underline{p})] < y'(\underline{p})$.

The previous remarks relate to sub-game perfect continuation payoffs. The best and worst sub-game perfect payoffs for a player are given by $Q[u^*, \bar{y}]$ and $Q[u^*, \underline{y}]$.

Example 4 (Hammond's pension game, cont.). The action $p_t = 1$ is both the best current action for the t -th player, since it implies she consumes the entire endowment when young, and the

worst continuation action for the $t - 1$ -th player since it implies she gets nothing when old. Consequently, the “no-sharing path” $\underline{p} = (1, 1, 1, \dots)$ gives both the worst sub-game perfect payoff $Q[u^*, \underline{y}]$ and the worst sub-game perfect continuation payoff \underline{y} to all players. The best sub-game perfect continuation payoff \bar{y} is attained by a path on which a player’s successor consumes the minimal amount consistent with sub-game perfection, i.e. consumes \bar{p} , where:

$$\bar{p} = \min_{(p,p') \in \mathcal{P}^2} \{p \mid u^y(p) + u^o(1 - p') \geq u^y(1) + u^o(0), p' \in [\bar{p}, 1]\}. \tag{6}$$

There are two possibilities. In the first, $\bar{p} = 0$ and $u^y(0) + u^o(1) \geq u^y(1) + u^o(0)$, i.e. the player’s successor consumes nothing when young and gives everything to her predecessor. She in turn receives a large amount (possibly the entire endowment) when old. In the second case, $\bar{p} \in (0, 1]$ and $u^y(\bar{p}) + u^o(1 - \bar{p}) = u^y(1) + u^o(0)$. In this case, the player’s successor gives the maximal amount $1 - \underline{p}$ consistent with sub-game perfection when young and is motivated to do so by a gift of this amount when old. $(1, \bar{p}, \bar{p}, \dots)$ is the unique best sub-game perfect path for a player.

4.2. Revision-proofness in quasi-recursive games

To build intuition consider a finite horizon game lasting T periods. Suppose that the T -th player has an objective $W_T : \mathcal{P} \rightarrow \mathbb{R}$ and the $T - 1$ -th player a continuation objective $Y_T : \mathcal{P} \rightarrow \mathbb{R}$. Further suppose that prior players, $t = 1, \dots, T - 1$, have objectives and continuation objectives $W_t : \mathcal{P}^{T+1-t} \rightarrow \mathbb{R}$ and $Y_t : \mathcal{P}^{T+1-t} \rightarrow \mathbb{R}$ generated by increasing aggregators Q and R and current payoff functions $u, v : \mathcal{P} \rightarrow \mathbb{R}$ according to $W_t(P^{T+1-t}) = Q[u(p_t), Y_{t+1}(P^{T-t})]$ and $Y_t(P^{T+1-t}) = R[v(p_t), Y_{t+1}(P^{T-t})]$.¹² In this setting a revision-proof strategy σ can be derived by a backwards induction argument. Specifically, for all histories P^{T-1} , let σ_T be such that:

$$\sigma_T(P^{T-1}) \in \operatorname{argmax}_{\mathcal{U}_T^*} Y_T(p), \quad \mathcal{U}_T^* = \operatorname{argmax}_{\mathcal{P}} W_T(p).$$

Thus, σ_T maximizes the $T - 1$ -th player’s continuation payoff Y_T subject to maximizing the T -th player’s payoff W_T . For prior dates t and histories P^{t-1} , let σ_t be such that:

$$\sigma_t(P^{t-1}) \in \operatorname{argmax}_{\mathcal{U}^*} v(p), \quad \mathcal{U}^* = \operatorname{argmax}_{\mathcal{P}} u(p). \tag{7}$$

Then each $\sigma_t(P^{t-1})$ maximizes the continuation payoff v of the $t - 1$ -th player subject to maximizing the current payoff u of the t -th player. A strategy constructed in this way prescribes optimal actions and breaks indifferences over a current player’s optimal actions in favor of prior players. Consequently, it is not possible to raise the payoff of an earlier player without reducing the payoff of a later one. The strategy is revision-proof.

The above construction relies on a terminal period in which the last player obtains her optimal payoff. The argument cannot be applied to infinite horizon games, but it suggests a procedure that can. In finite horizon games, a revision-proof strategy prescribes an action for the terminal player that is best for her and, subject to this, best for her predecessors. Rather trivially, this player “blocks” attempts by predecessors to revise play in the terminal period. For infinite horizon games, we design revision-proof strategies that endogenously create “blocking players” in all

¹² Here, in a slight abuse of notation, we simplify by assuming that R aggregates a current payoff $v(p_t)$ rather than a current action p_t with a continuation payoff. Hence, R is fully separable.

sub-games. These players are analogous to terminal players in finite horizon games. They receive a best sub-game perfect payoff and take actions that are maximal for predecessors subject to being maximal for themselves. The payoffs of a blocking player or her successors are reduced by any modification to their play that raises predecessor payoffs. Thus, blocking players create obstacles to revision. We use Hammond’s pension game to show how revision-proof strategies that feature blocking players can be constructed. In this particular game, *all* sub-game perfect paths can be implemented with such strategies and so all are revision-proof.

Example 4. (*Hammond’s pension game, cont.*) Let \tilde{P} be an arbitrary sub-game perfect path. Consider the strategy σ defined as follows. First, $\Phi(\sigma) = \tilde{P}$. Second, for all t and $P^t \neq (P^{t-1}, \sigma_t(P^{t-1}))$, $\Phi(\sigma|P^t) = (\hat{p}, \bar{p}, \bar{p}, \dots)$, where $\hat{p} = 1$ and \bar{p} is defined in (6). Thus, the player following a defection takes her unique best current action and obtains her best sub-game perfect payoff. In the language of the previous paragraph, this player is a blocking player.

It is easy to verify that σ is sub-game perfect; we check that it is revision-proof. We focus on the simplest case in which $\bar{p} = 0$, i.e. the case in which it is possible to sustain 0 consumption when young as part of a sub-game perfect strategy, because players sufficiently value consumption when old.¹³ Let P^t be a history and P'' a successful revision path at P^t . Then, each player $t + r$, $r \in \mathbb{N}$, receives a weakly higher payoff and some receive a strictly higher payoff from P'' than from reversion to the strategy. Thus, for all $r \in \mathbb{N}$, $u^y(p''_r) + u^o(1 - p''_{r+1}) \geq W(\Phi(\sigma|P^t, P''^{r-1}))$ with the inequality strict for some r . Let $t + r_0$ be the first date such that $u^y(p''_{r_0}) + u^o(1 - p''_{r_0+1}) > W(\Phi(\sigma|P^t, P''^{r_0-1}))$. Then either p''_{r_0} is above or p''_{r_0+1} is below the corresponding actions prescribed by σ . Suppose that $p''_{r_0} \neq \sigma_{t+r_0}(P^t, P''^{r_0-1})$. Then $\Phi(\sigma|P^t, P''^{r_0}) = (1, 0, 0, \dots)$. This continuation path is not just the best sub-game perfect path for the $t + r_0 + 1$ -th player, it is a best possible path for this player: she consumes the maximal amount of 1 in both periods of her life. The player at $t + r_0 + 1$ will “block” revisions that fail to match this. Thus, if $p''_{r_0} \neq \sigma_{t+r_0}(P^t, P''^{r_0-1})$, then p''_{r_0+1} must equal 1 and p''_{r_0+2} must equal 0. Hence, the $t + r_0$ -th player receives a payoff of $u^y(p''_{r_0}) + u^o(1 - p''_{r_0+1}) = u^y(p''_{r_0}) + u^o(0) \leq u^y(1) + u^o(0)$. But $u^y(p''_{r_0}) + u^o(1 - p''_{r_0+1}) > W(\Phi(\sigma|P^t, P''^{r_0-1})) \geq u^y(1) + u^o(0)$, where the first inequality is by assumption and the second stems from the fact that σ is sub-game perfect and $u^y(1) + u^o(0)$ is the lowest possible sub-game perfect payoff. However, this leads to a contradiction. So, p''_{r_0} must equal and p''_{r_0+1} must be lower than the actions prescribed by the strategy. In particular, $p''_{r_0+1} < 1$. But exactly the same argument applied at $t + r_0 + 1$ shows that $p''_{r_0+1} = 1 \geq \sigma_{t+r_0+1}(P^t, P''^{r_0})$, another contradiction. We conclude that there are no successful revision paths at any history. Thus, by Lemma 2 in Appendix A, σ and, hence, \tilde{P} is revision-proof. Since \tilde{P} was an arbitrary sub-game perfect path, it follows that all such paths are revision-proof in this game.

The pension game has two features that greatly simplify the analysis. First, players care only about actions in the two periods of their life and, second, a player and her immediate successor have strictly conflicting preferences over the successor’s action. Consequently, the unique optimal action for a successor gives the worst possible continuation payoff for a predecessor. This enables a blocking player to be placed in every sub-game immediately following a devi-

¹³ Our analysis extends to situations in which $\bar{p} > 0$ and this case is covered by our general result, Theorem 1. However, to maximize transparency, we do not detail it here.

ation. **Theorem 1** shows that the essential logic of the example generalizes. In more general settings, a blocking player who attains her best sub-game perfect payoff may not inflict a very severe penalty on her immediate predecessors. Hence, to sustain a given path, it is often necessary to embed the blocking player deeper into the sub-game following a defection. We construct (revision-proof) strategies that do this. These strategies prescribe repeated play of an action \hat{p} that maximizes a player’s current payoff function u and, given this, maximizes the continuation payoff of predecessors. Thus, \hat{p} solves a problem similar to (7) in the finite horizon setting. By making the string of \hat{p} plays long enough a continuation payoff very nearly equal to $\hat{y} = R[\hat{p}, \hat{y}]$, $\hat{y} = Y(\hat{P})$, $\hat{P} = (\hat{p}, \hat{p}, \dots)$ is attained. This is more severe than $Y(\hat{p}, \bar{P}) = R[\hat{p}, \bar{y}]$, the continuation payoff obtained if a blocking player immediately follows a defector, and, hence, sustains more paths. As time passes and the blocking player is approached player continuation payoffs are built up towards \bar{y} . The blocking player or her successors are made worse off by any revision to their play that benefits predecessors. Given this, the player immediately prior to the blocking player cannot revise future play in her favor and by playing \hat{p} obtains her best possible current payoff. Further any change to her play that benefits prior players makes her worse off. This logic extends back through the game tree until the initial defection that triggered play of the \hat{p} actions is reached. If the initial defecting player is better off adhering to the strategy than triggering this play, then the strategy is proof against any revision that benefits her and, conditional on her defection, players between her and the blocking player. This leaves open the possibility of a successful revision for players who move after the blocking player. Such a possibility is eliminated if the continuation of a best sub-game perfect path can itself be sustained by reversion to play involving a later blocking player.

Remark 1. Analysis of the pension game was simplified by the fact that a player and her successor had conflicting objectives over the later player’s action and that players cared about actions in only two periods. Kocherlakota’s [10] policy game has the first of these properties, but not the second. Pursuing the logic sketched above and formalized in **Theorem 1** below, as in the pension game, all sub-game perfect paths may be shown to be revision-proof in this game.¹⁴

Definition 5. $\mathcal{U}^* = \operatorname{argmax}_{\mathcal{P}} u(p)$ is the best current action set; $\tilde{\mathcal{R}}^*(y) = \operatorname{argmax}_{\mathcal{U}^*} R[p, y]$ is the set of best current actions that are best for predecessors given y .

By **Assumption 2** \mathcal{U}^* and each $\tilde{\mathcal{R}}^*(y)$ are non-empty and compact. The following (weak) separability restriction is also useful.

Assumption 3. The correspondence $\tilde{\mathcal{R}}^* : Y(\mathcal{P}^\infty) \rightarrow 2^{\mathcal{P}}$ has a constant value \mathcal{R}^* .

Assumption 3 holds if the restriction of R to $\mathcal{U}^* \times \mathcal{Y}$ has the form $R[p, y] = R'[v(p), y]$ with $v : \mathcal{U}^* \rightarrow \mathbb{R}$ and each $R'[\cdot, y]$ increasing on $v(\mathcal{U}^*)$. This in turn holds trivially if \mathcal{U}^* is single-valued or if R has the form $R'[v(p), y]$ on all of $\mathcal{P} \times \mathcal{Y}$. Let \hat{p} belong to \mathcal{R}^* and define $\hat{P} = (\hat{p}, \hat{p}, \dots)$, $\hat{w} := Q[u(\hat{p}), Y(\hat{P})]$ and $\hat{y} := Y(\hat{P})$. In quasi-recursive games with weak agreement over optima \hat{w} is the best sub-game perfect payoff for a player; in Hammond’s pension game, Kocherlakota’s policy game and games in which all best current actions are worst for predecessors \hat{w} is the worst sub-game perfect payoff. We impose a second assumption.

¹⁴ More precisely, **Theorem 1** applies to the reduced form quasi-recursive game that we associated with Kocherlakota’s policy game.

Assumption 4. $Q(u(\bar{p}), \bar{y}) \geq \hat{w} := Q[u(\hat{p}), \hat{y}]$. If $Q[u(\bar{p}), \bar{y}] = \hat{w} := Q[u(\hat{p}), \hat{y}]$, then $R[u(\bar{p}), \bar{y}] \geq \hat{y} := R[u(\hat{p}), \hat{y}]$.

This assumption implies that a player weakly prefers implementing the continuation of a predecessor's preferred sub-game path \bar{P} (and obtaining her own best continuation payoff \bar{y}) to taking a best current action that is best for predecessors \hat{p} and having all successors do the same. Further, if a player is indifferent between these paths, then her predecessor weakly prefers the former. We use this assumption to ensure that neither players nor, if they are indifferent, their predecessors prefer deviating from the best sub-game perfect path used to reward a blocking player to paths with payoffs approximately equal to \hat{w} that feature a later blocking player. **Assumption 4** holds automatically in Hammond's pension game and Kocherlakota's policy game since in these games \hat{P} is a worst sub-game perfect path and, hence, gives a payoff \hat{w} below that from the sub-game perfect path \bar{P} . It also holds in games with weak agreement over optima in which paths of the form \hat{P} are simultaneously best and best continuation sub-game perfect paths.

We now state our first main result for quasi-recursive games.

Theorem 1. Let **Assumptions 2 to 4** hold. Let \tilde{P} be any path such that (i) for all t , $W(\tilde{P}_t) \geq \hat{w}$ and (ii) if $t > 1$ and $W(\tilde{P}_t) = \hat{w}$, then $Y(\tilde{P}_t) \geq \hat{y}$. There is a revision-proof strategy σ that implements the path \tilde{P} .

Proof. See [Appendix C](#). \square

Corollary 2 below strengthens condition (ii) of **Theorem 1** to hold in all periods including the first and obtains a stronger result. It is now not possible to raise the *first period* continuation payoff $Y(\tilde{P})$ implied by the strategy without reducing the payoff of a later player. This stronger conclusion is not needed to establish revision-proofness since there is no 0-th period player to receive a continuation payoff of $Y(\tilde{P})$. However, it is useful in deriving later results.¹⁵

Corollary 2. Let **Assumptions 2 to 4** hold. Let \tilde{P} be any path such that (i) for all t , $W(\tilde{P}_t) \geq \hat{w}$ and (ii) for all t , if $W(\tilde{P}_t) = \hat{w}$, then $Y(\tilde{P}_t) \geq \hat{y}$. There is a revision-proof strategy σ that implements the path \tilde{P} and there is no alternative σ' that delivers the same payoffs to all players as σ and satisfies $Y(\Phi(\sigma')) > Y(\tilde{P})$.

Proof. See [Appendix C](#). \square

In Hammond's pension game and Kocherlakota's policy game the worst sub-game perfect path involves repeated play of the unique best current action $\hat{p} = 1$. **Theorem 1** implies that in such games *all* sub-game perfect paths are revision proof. However, it is easy to construct quasi-recursive games in which repeated play of a best current action is *not* a worst sub-game perfect path. In such games, there are sub-game perfect paths with payoffs below \hat{w} at some or all dates. **Theorem 1** is silent on the revision-proofness of these.

We now show that with some strengthening of the conditions of **Theorem 1**, *all* paths delivering payoffs strictly above the worst sub-game perfect payoff, \underline{w} , are revision-proof. To the extent

¹⁵ Specifically, it is used in the proof of **Theorem 2**, where a revision-proof strategy is constructed from revision-proof continuation strategies.

that these conditions hold, the revision-proof concept is quite permissive with respect to paths and outcomes. The argument underpinning the result is quite long, but the idea is simple. Recall that \underline{p} is the first action of a sub-game perfect path giving the lowest sub-game perfect continuation payoff \underline{y} , see Subsection 4.1. Given a path \tilde{P} that delivers payoffs $W(\tilde{P}_t) > \underline{w}$ to successive players, a strategy σ is constructed that implements this path and prescribes \underline{p} following a defection (from \tilde{P}) until player payoffs are driven above \hat{w} . Irrespective of post-defection play \hat{w} is achieved within a finite number of periods. Once \hat{w} is reached a revision-proof continuation strategy defined as in Corollary 2 is pursued.

The construction of σ implies sub-games in which revision-proof continuation strategies are played. By definition, it is not possible to revise these so as to raise the payoffs of a player in the sub-game without reducing the payoff of some other player in the sub-game. Revisions that begin and end before such a revision-proof continuation strategy is reached (and player payoffs have been driven above \hat{w}) involve a last deviator whose payoff is reduced by construction. Revisions that begin before a revision-proof continuation strategy is reached, but continue on into a sub-game in which such a continuation strategy is played are more complicated, but these too turn out to leave all players' payoffs unaltered or they reduce the payoffs to some.

Our argument requires two further assumptions. These are used to ensure that \underline{p} prescriptions are consistent with the building up of payoffs to values above \hat{w} . Assumption 5 asserts that a player is better off playing \underline{p} and receiving a best sub-game perfect continuation payoff \bar{y} , than taking a best current action \hat{p} that is best for predecessors and having all her successors do the same. It combines concern with the future with disagreement between successive players about the later player's play (i.e. play of \underline{p} by a successor is not too bad for the successor, but is bad for a predecessor; play of \hat{p} is optimal for a successor, but bad for a predecessor).

Assumption 5. $Q[u(\underline{p}), \bar{y}] > Q[u(\hat{p}), \hat{y}] = \hat{w}$.

Assumption 6 restricts the slope of Q in its second argument.

Assumption 6. For all $y > y'$ and p , there is a value $\kappa > 0$ such that

$$\beta\kappa(y - y') \leq Q[u(p), y] - Q[u(p), y'] < \kappa(y - y').$$

This assumption is satisfied if, in addition to Assumption 2, Q and R are quasi-linear. For example, if $Q[u(p), y] = u(p) + \delta y$ and $R[p, y] = v(p) + \beta y$ with $\delta \in (0, \infty)$ and $\beta \in (0, 1)$, then $Q[u(p), y] - Q[u(p), y'] = \delta(y - y')$ and Assumption 6 holds with $\kappa = \delta/\beta$. We now state the second main result of this section.

Theorem 2. Suppose that Assumptions 2 to 6 hold. Let \tilde{P} be any path such that for all t , $W(\tilde{P}_t) > \underline{w}$, then \tilde{P} can be supported by a revision-proof strategy.

Proof. In Appendix D. \square

We give a simple example of a game that satisfies these assumptions.

Example 6 (Overlapping generations game with partial agreement). Overlapping generations of players $t \in \mathbb{N}$ live for two periods. Let $\mathcal{S} = [0, 1]$. The objective of the t -th player, who is young in period t and old in $t + 1$ is given by:

$$U(t, P) = u^y(p_t) + u^o(p_{t+1}),$$

where $u^y(p) = \frac{1}{2}p(1 - p)$ and $u^o(p) = 1 - p$. Hence, $u = u^y$, $Q[u(p), y] = u(p) + y$ and $R[p, y] = u^o(p)$. The function u^y is maximized by $\hat{p} = \frac{1}{2}$, while u^o is minimized by $\underline{p} = 1$ and maximized by $\bar{p} = 0$. It is easily checked that $1 = Q[u(\bar{p}), \bar{y}] = Q[u(\underline{p}), \bar{y}] > \hat{w} = Q[u(\hat{p}), \hat{y}] = \frac{5}{8} > \underline{w} = Q[u(\hat{p}), \underline{y}] = \frac{1}{8}$. Thus, Assumptions 4 and 5 are satisfied and the quasi-linearity of Q and R ensure that Assumption 6 holds. By Theorem 2 all paths with payoffs that remain strictly above $\frac{1}{8}$ are revision-proof.

5. Quasi-recursive games with state variables

We briefly describe an extension of the quasi-recursive environment of Section 4 that incorporates state variables. Let $\mathcal{K} \subseteq \mathbb{R}^m$ denote a set of feasible states and $\Lambda : \mathbb{R}^m \times \mathcal{P} \rightarrow \mathbb{R}^m$ a law of motion for states with for all $k \in \mathcal{K}$, $\Gamma(k) = \{p \in \mathcal{P} \mid \Lambda(k, p) \in \mathcal{K}\} \neq \emptyset$. Denote the iterates of Λ by $\Lambda^1 = \Lambda$ and, for $t \geq 2$, $\Lambda^t : \mathcal{K} \times \mathcal{P}^t \rightarrow \mathbb{R}^m$, where:

$$\Lambda^t(k, P^{t-1}, p_t) = \Lambda(\Lambda^{t-1}(k, P^{t-1}), p_t). \tag{8}$$

The set of feasible paths (i.e. paths consistent with states in \mathcal{K}) at initial state $k \in \mathcal{K}$ is:

$$\Pi(k) = \{P \in \mathcal{P}^\infty \mid \forall t, \Lambda^t(k, P^t) \in \mathcal{K}\}. \tag{9}$$

Let $u : \mathcal{K} \times \mathcal{P} \rightarrow \mathbb{R}$ and $R : \mathcal{K} \times \mathcal{P} \times \mathbb{R} \rightarrow \mathbb{R}$ be extensions of the analogous functions from the preceding section and $Q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be as before. We impose the following analogue of Assumption 2.

Assumption 2'. (i) \mathcal{P} is a closed, convex subset of a normed space \mathcal{P}_0 and \mathcal{K} a closed, convex subset of \mathbb{R}^m . (ii) Λ is continuous and concave and $\Gamma : \mathcal{K} \rightarrow 2^\mathcal{P} \setminus \emptyset$ is compact-valued and continuous. (iii) $u : \text{Graph } \Gamma \rightarrow \mathbb{R}$ is continuous. $Q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is increasing, continuous and concave. (iv) $R : \text{Graph } \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and for each (k, p) , $R[k, p, \cdot]$ is non-decreasing. There is a $\beta \in (0, 1)$, an $\alpha \in (0, \beta^{-1})$ and a continuous function $\psi : \mathcal{K} \rightarrow \mathbb{R}_{++}$ such that for all $(k, p) \in \text{Graph } \Gamma$ and $y, y' \in \mathbb{R}$

$$|R[k, p, y] - R[k, p, y']| \leq \beta |y - y'|$$

and

$$\sup_{(k,p) \in \text{Graph } \Gamma} \left| \frac{\psi(\Lambda(k, p))}{\psi(k)} \right| < \alpha \quad \text{and} \quad \sup_{(k,p) \in \text{Graph } \Gamma} \left| \frac{R(k, p, 0)}{\psi(k)} \right| < \infty.$$

Proposition 10 in Appendix E establishes that under Assumption 2' there is a unique, continuous continuation payoff function $Y : \text{Graph } \Pi \rightarrow \mathbb{R}$ satisfying:

$$Y(k, p, P) = R[k, p, Y(\Lambda(k, p), P)].$$

Define $W : \mathcal{K} \times \mathcal{P}^\infty \rightarrow \mathbb{R} \cup \{-\infty\}$,

$$W(k, p, P) = \begin{cases} Q[u(k, p), Y(\Lambda(k, p), P)] & P \in \Pi(k) \\ -\infty & \text{otherwise.} \end{cases}$$

Given a fixed initial k_1 , a dynastic game $\mathcal{G}(U)$ with payoffs $U : \mathbb{N} \times \mathcal{P}^\infty \rightarrow \mathbb{R} \cup \{-\infty\}$,

$$U(t, P) = W(\Lambda^t(k_1, P^{t-1}), P_t), \tag{10}$$

may then be recovered. We call games defined in this way quasi-recursive games with state variables. From (10), the t -th player’s objective depends on histories P^{t-1} through their effect on the current state variable $k_t = \Lambda^{t-1}(k_1, P^{t-1})$. This formulation accommodates psychological state variables such as habit formation or altruism for predecessors as well as physical state variables via the assigning of $-\infty$ payoffs to infeasible choices. Since quasi-recursive games with state variables are a special case of the general games in Section 2, the definition of revision-proofness in Section 3 is immediately applicable.

The key to extending our earlier results to quasi-recursive games with state variables is the construction of an analogue to the \hat{p} sequence (i.e. best-for-past agents given best-for-current agents sequence) of the previous section. We turn to this now. Let $\bar{\mu}(k) = \sup_{\Pi(k)} Y(k, P)$ and $\underline{\mu}(k) = \inf_{\Pi(k)} Y(k, P)$. Given continuity of R in its third argument, it is easily seen that:

$$\bar{\mu}(k) = \sup_{p \in \Gamma(k)} R[k, p, \bar{\mu}(\Lambda(k, p))] \quad \text{and} \quad \underline{\mu}(k) = \inf_{p \in \Gamma(k)} R[k, p, \underline{\mu}(\Lambda(k, p))].$$

Define the sub-game perfect continuation payoff correspondence $\mathcal{Y} : \mathcal{K} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$,

$$\mathcal{Y}(k) = \{y \mid y = Y(k, \Phi(\sigma)), \sigma \text{ is sub-game perfect}\} \subseteq \mathcal{Y}_0(k) := [\underline{\mu}(k), \bar{\mu}(k)].$$

By Proposition 11 in Appendix E, \mathcal{Y} is compact-valued. Hence, there are maximal and minimal sub-game perfect continuation payoff functions: $\bar{y}(k) = \max \mathcal{Y}(k)$ and $\underline{y}(k) = \min \mathcal{Y}(k)$. We can further show that:

$$\begin{aligned} \bar{y}(k) &= \max R[k, p, \bar{y}(\Lambda(k, p))] \\ \text{s.t. } p &\in \Gamma(k) \text{ and } Q[u(k, p), \bar{y}(\Lambda(k, p))] \geq \sup_{p' \in \Gamma(k)} Q[u(k, p'), \underline{y}(\Lambda(k, p'))] \end{aligned} \tag{11}$$

and that the best possible sub-game perfect payoff at each $k \in \mathcal{K}$ is given by:

$$\bar{w}(k) = \sup_{\Gamma(k)} Q[u(k, p), \bar{y}(\Lambda(k, p))]. \tag{12}$$

Let $\bar{p} : \mathcal{K} \rightarrow \mathcal{P}$ be a selection from the optimal policy correspondence for (11) and let:

$$S(y)(k) = \max_{\mathcal{U}^*(y)(k)} R[k, p, y(\Lambda(k, p))] \quad \text{and} \quad \tilde{\mathcal{H}}^*(y)(k) = \operatorname{argmax}_{\mathcal{W}^*(y)(k)} R[k, p, y(\Lambda(k, p))],$$

where $\mathcal{U}^*(y)(k) = \operatorname{argmax}_{\Gamma(k)} Q[u(k, p), y(\Lambda(k, p))]$. $S(y)(k)$ is the best possible continuation payoff for a preceding player at k given that the current player makes an optimal choice and is faced with the continuation payoff function y and $\tilde{\mathcal{H}}^*(y)(k)$ is the corresponding set of maximizers. Let $y_0 = \bar{y}$, $y_n = S(y_{n-1})$ and $\mathcal{R}_n^* = \tilde{\mathcal{H}}^*(y_n)$, $n \in \mathbb{N}$. The continuation value function y_n is induced by a sequence of n players taking a best action for predecessors amongst their best action sets and the n -th player obtaining her best possible payoff. Thus, for some initial $k_0 \in \mathcal{K}$ and $T \in \mathbb{N}$, the policy correspondences \mathcal{R}_n^* , generate a sequence of actions analogous to the \hat{p} actions used in the constructions of the previous section. Without state variables, paths consisting of repeated play of \hat{p} followed by an optimal path for a “blocking player” were used to construct revision-proof strategies. With state variables the situation is similar except that the policy correspondences \mathcal{R}_n^* are used to construct play leading up to the blocking player. Let $w_{n+1}(k) = \max_{\Gamma(k)} Q[u(k, p), y_n(\Lambda(k, p))]$ and let $w_\infty(k) = \liminf_{n \rightarrow \infty} w_n(k)$. For an appropriate choice of n a payoff arbitrarily close to $w_\infty(k)$ can be attained by the first player in a sequence of the sort described above.

We impose the following analogue of Assumption 4. It implies that players along the continuation of a best path (for a blocking player) are strictly better off than if a path with payoff close to $w_\infty(k)$ is pursued.

Assumption 4’. For each $k \in \mathcal{K}$, $Q[u(k, \bar{p}(k)), \bar{y}(k)] > w_\infty(k)$.

We can now state an analogue of Theorem 1 for quasi-recursive games with state variables. The proof follows the same logic as the proof of Theorem 1 and is available upon request.

Proposition 4. Let Assumptions 2’ and 4’ hold. Given $k_1 \in \mathcal{K}$, let $\tilde{P} \in \Pi(k_1)$ be any path such that for all t , $W(\Lambda^t(k_1, \tilde{P}^{t-1}), \tilde{P}_t) > w_\infty(\Lambda^t(k_1, \tilde{P}^{t-1}))$. There is a revision-proof strategy σ that implements the path \tilde{P} at k_1 .

The following example illustrates.

5.1. Example: saving with quasi-hyperbolic preferences

A decision-maker has an initial endowment of capital k_1 and access to a linear technology for producing output from capital $q = Ak$, $A > 0$. In each period t she divides current output Ak_t into consumption p_t and future capital k_{t+1} . Let \mathcal{K} , $\mathcal{P} = \mathbb{R}_+$ denote sets of capitals and consumptions. Capital evolves according to a function $\Lambda : \mathbb{R} \times \mathcal{P} \rightarrow \mathbb{R}$, where:

$$\Lambda(k, p) = Ak - p.$$

For $k \in \mathcal{K}$, define $\Gamma(k) = \{p \mid p \in [0, Ak]\}$. Let $\Lambda^1 = \Lambda$ and, for $t \geq 2$, define $\Lambda^t : \mathcal{K} \times \mathcal{P}^t \rightarrow \mathbb{R}$ as in (8). $\Lambda^t(k, P^t)$ gives capital for use in $t + 1$ given an initial capital k and prior consumption choices P^t . The set of feasible consumption paths given capital k , $\Pi(k)$, is then defined as in (9). Let $\beta \in (0, \min(1, A^{\gamma-1}))$ and $\gamma \in (0, 1)$. Each player’s continuation payoff function is identified with, $P \in \Pi(k)$,

$$Y(k, P) = \sum_{t=1}^{\infty} \beta^{t-1} \frac{P_t^{1-\gamma}}{1-\gamma},$$

where $0 \leq Y(k, P) \leq \bar{b}k^{1-\gamma}$, $\bar{b} = \frac{A^{1-\gamma}}{1-\gamma} (1 - (\beta A^{1-\gamma})^{\frac{1}{\gamma}})^{-\gamma}$. Y satisfies the recursion:

$$Y(k, p, P') = R[k, p, Y(\Lambda(k, p), P')],$$

where $R[k, p, y] = u(k, p) + \beta y$ and $u(k, p) = \frac{p^{1-\gamma}}{1-\gamma}$ for $p \in \Gamma(k)$. The decision-maker’s payoff in each period is:

$$W(k, p, P') = \begin{cases} Q[u(k, p), Y(\Lambda(k, p), P')] & (p, P') \in \Gamma(k) \\ -\infty & \text{otherwise,} \end{cases}$$

where $Q[u(k, p), y] = u(k, p) + \beta \delta y$, $\delta \in (0, 1)$. The following proposition characterizes the set of sub-game perfect continuation payoffs in this case.

Proposition 5. The sub-game perfect continuation payoff correspondence \mathcal{Y} is given by:

$$\mathcal{Y}(k) = [\underline{b}k^{1-\gamma}, \bar{b}k^{1-\gamma}].$$

The parameter of the minimal sub-game perfect continuation payoff function $\underline{y}(k) = \underline{b}k^{1-\gamma}$ is the unique value satisfying:

$$\underline{b} = \frac{1}{1-\gamma} \frac{(A - \theta(k))^{1-\gamma}}{1 - \beta\theta(k)^{1-\gamma}}, \quad \text{where } \theta(k) = \operatorname{argmax}_{[0,A]} \frac{(A - \theta')^{1-\gamma}}{1 - \gamma} + \beta\delta\underline{b}\theta'^{1-\gamma},$$

i.e. \underline{y} is the unique Markov perfect value function.

The proof of the proposition is quite long and in the interests of space is omitted. It is available from the authors on request. In addition, defining $b_1 = \bar{b}$ and for each $n \in \mathbb{N}$,

$$b_{n+1}k^{1-\gamma} = \frac{(Ak - k_{n+1}(k))^{1-\gamma}}{1 - \gamma} + \beta b_n k_{n+1}(k)^{1-\gamma},$$

where $k_{n+1}(k) = \operatorname{argmax}_{[0,Ak]} \frac{(Ak - k')^{1-\gamma}}{1-\gamma} + \beta\delta b_n k'^{1-\gamma}$, we have that $b_n \downarrow \underline{b}$. Thus, the value function obtained by having a terminal player obtain her optimal payoff and a sequence of preceding players choosing their best current payoff converges to the Markov (worst sub-game perfect) payoff function. Application of Proposition 4 implies that all paths with payoffs above the Markov (worst sub-game perfect) value function are revision-proof.

6. Revision-proofness and the literature

We relate revision-proofness to similar concepts in the literature.

6.1. Asheim’s notion of revision-proofness

Asheim [3] introduces an equilibrium concept he calls *revision-proofness* and we will call *A-revision-proofness*. We show that despite apparent differences in their formulations, the concepts are identical. Following Asheim, a strategy correspondence S associates each history P^{t-1} with a set of strategies, i.e. $S : \bigcup_{t \in \mathbb{N}} \mathcal{P}^{t-1} \rightarrow 2^{\mathcal{S}}$.

Definition 6 (Asheim). A strategy correspondence is *internally stable* if for all $t \in \mathbb{N}$, $P^{t-1} \in \mathcal{P}^{t-1}$ and $\sigma \in S(P^{t-1})$, there is no $P^{t+r-1} = (P^{t-1}, P^r)$ and $\sigma' \in S(P^{t+r-1})$ such that $V_{t+r}(P^{t+r-1}, \sigma') > V_{t+r}(P^{t+r-1}, \sigma)$. It is *externally stable* if for all $t \in \mathbb{N}$, $P^{t-1} \in \mathcal{P}^{t-1}$, $\sigma' \in \mathcal{S} \setminus S(P^{t-1})$, there is some $P^{t+r-1} = (P^{t-1}, P^r)$ and $\sigma \in S(P^{t+r-1})$ such that $V_{t+r}(P^{t+r-1}, \sigma) > V_{t+r}(P^{t+r-1}, \sigma')$. It is *stable* if it is both internally and externally stable.

Definition 7 (Asheim). σ is *A-revision-proof* at P^{t-1} if it belongs to $S(P^{t-1})$, where S is stable strategy correspondence.

Lemma 1.

- (1) Let S be an internally stable strategy correspondence, then there is a function $\Psi : \bigcup_{t \in \mathbb{N}} \mathcal{P}^{t-1} \rightarrow \mathbb{R}$ such that for each P^{t-1} and all $\sigma \in S(P^{t-1})$, $V_t(P^{t-1}, \sigma) = \Psi(P^{t-1})$.
- (2) If, in addition, S is an externally stable strategy correspondence and $\sigma \in S(P^{t-1})$, then for all $P^{t+r-1} = (P^{t-1}, P^r)$, $\sigma \in S(P^{t+r-1})$ and $V_{t+r}(P^{t+r-1}, \sigma) = \Psi(P^{t+r-1})$.

Proof. Part (1) If $\sigma, \sigma' \in S(P^{t-1})$, then immediately from the definition of internal stability $V_t(P^{t-1}, \sigma) \geq V_t(P^{t-1}, \sigma') \geq V_t(P^{t-1}, \sigma)$. Set $\Psi(P^{t-1})$ to this common value. Part (2) Let $P^{t+r-1} = (P^{t-1}, P^r)$. If $\sigma \in S(P^{t-1})$, then, either $\sigma \in S(P^{t+r-1})$, in which case, by part (1), $V_{t+r}(P^{t+r-1}, \sigma) = \Psi(P^{t+r-1})$ or $\sigma \notin S(P^{t+r-1})$. In the latter case, by external stability of S , there is a successor history $P^{t+r+s-1} = (P^{t-1}, P^r, P^s) = (P^{t-1}, P^{r+s})$ such that $\Psi(P^{t+r+s-1}) > V_{t+r+s}(P^{t+r+s-1}, \sigma)$. But since $\sigma \in S(P^{t-1})$, internal stability implies that $V_{t+r+s}(P^{t+r+s-1}, \sigma) \geq \Psi(P^{t+r+s-1})$. We deduce that, in fact, if $\sigma \in S(P^{t-1})$, then $\sigma \in S(P^{t+r-1})$ and $V_{t+r}(P^{t+r-1}, \sigma) = \Psi(P^{t+r-1})$. \square

The preceding lemma implies that if σ is A-revision-proof at P^{t-1} , then it is A-revision-proof at all successor histories P^{t+r-1} . We will say that σ is A-revision-proof if it is A-revision-proof at P^0 (and, hence, at all successor histories). If σ is A-revision proof, then all other strategies σ' must either deliver the same payoffs as σ after each history (and, hence, can be taken to belong to the same stable correspondence as σ) or they must deliver a strictly lower payoff than σ after some history. Consequently, we have the following result.

Proposition 6. *σ is revision-proof if and only if it is A-revision-proof.*

Proof. (If) Suppose that σ is A-revision-proof and that there is a history P^{t-1} and an alternative strategy σ' such that $\forall P^r, V_{t+r}(P^{t+r-1}, \sigma') \geq V_{t+r}(P^{t+r-1}, \sigma)$ with the inequality strict for at least one P^r . The A-revision-proofness of σ implies that there is a stable strategy correspondence S such that $\sigma \in S(P^0)$. Now, σ' cannot be in $S(P^{t-1})$ as well, since then S would fail to be internally stable. On the other hand if $\sigma' \notin S(P^{t-1})$, then S fails to be externally stable since for all $P^{t+r-1}, V_{t+r}(P^{t+r-1}, \sigma') \geq V_{t+r}(P^{t+r-1}, \sigma) = \Psi(P^{t+r-1})$, where Ψ is the payoff function associated with S and the equality stems from the previous lemma. Hence, if σ is A-revision-proof, it is revision-proof.

(Only If) Suppose that σ is revision-proof. For each P^{t-1} , define $S(P^{t-1})$ to be the set of strategies that give the same payoffs at P^{t-1} and all successor histories as σ . Each $S(P^{t-1})$ is non-empty since it contains σ . Since for all P^{t-1} , the members of $S(P^{t-1})$ give the same payoffs at P^{t-1} and all successor histories, S is internally stable. Revision-proofness and the definition of S imply that for all P^{t-1} any $\sigma' \notin S(P^{t-1})$ must give a strictly lower payoff after some successor history. Thus, S is externally stable. Since $\sigma \in S(P^0)$, σ is A-revision-proof. \square

6.2. Kocherlakota’s reconsideration-proofness

Kocherlakota [10] specializes Farrell and Maskin’s [8] definition of renegotiation-proofness to games played by dynasties. He considers only repeated games without history dependence. Player preferences are defined by a function $W : \mathcal{P}^\infty \rightarrow \mathbb{R}$ according to, for all t and $P, U(t, P) = W(P_t)$. The payoffs induced by a strategy σ are given by: $V_t(P^{t-1}, \sigma) = W(\Phi(\sigma|P^{t-1}))$.

Definition 8 (Kocherlakota). In a repeated game, a strategy σ is *reconsideration-proof* if (1) it is sub-game perfect, (2) it is payoff stationary, i.e. for all $P^{t-1}, V_t(P^{t-1}, \sigma) = \bar{V}$ for some number \bar{V} , (3) there is no other sub-game perfect equilibrium σ' such that for all $P^{t-1}, V_t(P^{t-1}, \sigma') = \bar{V}' > \bar{V}$.

Proposition 7. *If, in a repeated game, σ is revision-proof and payoff stationary, then it is reconsideration-proof.*

Proof. If σ is revision-proof, then it is also sub-game perfect and, hence, satisfies (1) of the reconsideration-proof definition. Payoff stationarity implies that σ satisfies (2) of the reconsideration-proof definition. Additionally, there is no alternative sub-game perfect equilibrium such that for all P^{t-1} , $V_t(P^{t-1}, \sigma') \geq V_t(P^{t-1}, \sigma) = V$ with strict inequality after some P^{t-1} . In particular (3) of the reconsideration-proof definition is satisfied. \square

We re-emphasize that revision-proofness is applicable to dynamic games, reconsideration-proofness is not.

7. Conclusion

What outcomes can we expect to emerge when players with conflicting objectives sequentially take actions that affect their predecessors and, possibly, their successors? To address this question we consider revision-proof strategies that are robust to joint deviations of successive players. Although this concept is not new, the literature contains very little prior characterization.

The main message of this paper is that if it is possible to support a path with off-equilibrium play that deters unilateral defection and blocks every infinite revision by eventually giving some player more than she can be obtain from joining the revision, then the path is revision-proof. In games where there is some consensus about how to play, this may only be possible for a small set of paths and the concept may be quite selective. In quasi-recursive games where optimal actions for successors are quite costly for predecessors, but optimal actions for predecessors are not too costly for successors, the concept is quite permissive with respect to paths.

We have given applications of our results to games played by overlapping generations of players, to savings games played by an agent with quasi-hyperbolic discounting preferences and to a simple macroeconomic policy game. In the latter case, the macroeconomic policy game mapped directly into a reduced form quasi-recursive game. For macroeconomic policy games in which private households take actions that influence the physical state of a later policymaker, a mapping into a quasi-recursive game exists, but it is more complicated. In these settings, the associated reduced form quasi-recursive game explicitly incorporates private households' beliefs about future policy actions, as well as the actions themselves. The revision-proofness concept can be extended to these games and we conjecture that analogues of our results are available for them. We also conjecture that our procedure for designing revision-proof strategies, with off-equilibrium payoff build-ups culminating in blocking players, has application to a broader range of quasi-recursive and non-quasi-recursive games. We leave these extensions to future work.

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Appendix A. Proofs for Section 3

Proof of Proposition 1. If σ is not sub-game perfect for $\mathcal{G}(U)$, then there is some $P^t = (P^{t-1}, p_t)$ such that $V_t(P^t, \sigma) > V_t(P^{t-1}, \sigma)$. Let σ' equal σ for all histories except P^{t-1} . Set

$\sigma'_t(P^{t-1}) = p_t$. Then $V_t(P^{t-1}, \sigma') > V_t(P^{t-1}, \sigma)$ and for all $r \in \mathbb{N}$ and P^r , $V_{t+r}(P^t, P^r, \sigma') = V_{t+r}(P^t, P^r, \sigma)$. Thus, σ is not revision-proof for $\mathcal{G}(U)$. \square

Proof of Proposition 2. The characterization of sub-game perfection given in the proposition is well known. Necessity of conditions (i) and (ii) in the proposition for revision-proofness is a straightforward consequence of the definition. For sufficiency, by condition (ii) there is no P^{t-1} , $t = 2, 3, \dots$ and σ' satisfying (2) at all P^r and strictly at some P^r . Hence, any path P either lowers the payoff of a player at some $t > 1$ relative to reversion to the strategy or leaves all such players' payoffs unaltered. It only remains to rule out the possibility that a path P and, hence, a strategy σ' strictly raises the first player's payoff while leaving all subsequent players' payoffs unchanged. This follows from condition (i). \square

The following pathwise characterization of revision-proofness is used repeatedly in subsequent proofs. The proof is elementary and is omitted.

Lemma 2. Let Assumption 1 hold. σ is revision-proof for $\mathcal{G}(U)$ if only if there is no history P^{t-1} and path P' such that P' is a successful revision path for σ at P^{t-1} .

Proof of Proposition 3. Let \tilde{P} and a family of sets $\{\mathcal{D}_r\}$ satisfy the conditions in the proposition at some P^{t-1} . Suppose that σ is a revision-proof strategy and that $\Phi(\sigma|P^{t-1}) \in \mathcal{D}_0$. Let $\Phi(\sigma|P^{t-1}, \tilde{P}^r) \in \mathcal{D}_r$. Since σ is revision-proof, it is also sub-game perfect (Proposition 1) and so:

$$U(t+r, P^{t-1}, \tilde{P}^r, \Phi(\sigma|P^{t-1}, \tilde{P}^r)) \geq U(t+r, P^{t-1}, \tilde{P}^{r+1}, \Phi(\sigma|P^{t-1}, \tilde{P}^{r+1})).$$

Thus, by condition (ii), $\Phi(\sigma|P^{t-1}, \tilde{P}^{r+1}) \in \mathcal{D}_{r+1}$. Hence, by induction, for all $r \in \{0\} \cup \mathbb{N}$, $\Phi(\sigma|P^{t-1}, \tilde{P}^r) \in \mathcal{D}_r$. Then, by condition (i), for all $r \in \{0\} \cup \mathbb{N}$, $U(t+r, P^{t-1}, \tilde{P}) \geq U(t+r, P^{t-1}, \tilde{P}^r, \Phi(\sigma|P^{t-1}, \tilde{P}^r))$, with strict inequality at $r = 0$. By Lemma 2, this contradicts revision-proofness of σ . \square

Proof of Corollary 1. If for all $t \in \mathbb{N}$ and $P^{t-1} \in \mathcal{P}^{t-1}$, $\Phi(\sigma|P^{t-1})$ is in $\mathcal{P}_t^*(P^{t-1})$, then evidently σ is revision-proof since then it is not possible to find any t , P^{t-1} and P' such that $U(t, P^{t-1}, P') > V_t(P^{t-1}, \sigma)$. Conversely, suppose that σ is revision-proof and that there is some t and P^{t-1} such that $\Phi(\sigma|P^{t-1}) \notin \mathcal{P}_t^*(P^{t-1})$. Since U exhibits weak agreement over optima, there is a path $P^* \in \mathcal{P}_t^*(P^{t-1})$. Thus, $\Phi(\sigma|P^{t-1}) \in \mathcal{D}_0 := \{P \mid U(t, P^{t-1}, P^*) > U(t, P^{t-1}, P)\}$. Weak agreement over optima implies that for all $r \in \mathbb{N}$, $\mathcal{D}_r := \{P \mid U(t+r, P^{t-1}, P^*) \geq U(t+r, P^{t-1}, P^*, P)\} = \mathcal{P}$. Thus, the conditions of Proposition 3 hold and σ is not revision-proof. \square

Appendix B. Sub-game perfection with quasi-recursive payoffs

Let \mathcal{C} denote the set of bounded, continuous functions $f : \mathcal{P}^\infty \rightarrow \mathbb{R}$ equipped with the sup-norm $\|\cdot\| : \mathcal{C} \rightarrow \mathbb{R}_+$, $\|f\| = \sup_{\mathcal{P}^\infty} |f(P)|$, and the partial order \geq , $f \geq f'$ if $\forall P \in \mathcal{P}^\infty$, $f(P) \geq f'(P)$. Let $\mathbf{0}$ denote the zero function: $\mathbf{0} : \mathcal{P}^\infty \rightarrow 0$.

Proposition 8. Suppose Assumption 2 holds. Then there is a unique function $Y \in \mathcal{C}$ such that $Y(p, P') = R(p, Y(P'))$. Also, $\lim \|T^n(\mathbf{0}) - Y\| \rightarrow 0$.

Proof. Let $T : \mathcal{C} \rightarrow \mathbb{R}^{\mathcal{P}}$ be given by $T(f)(p, P) = R[p, f(P)]$. By the continuity of R , $T(f)$ is continuous for each $f \in \mathcal{C}$. Since R is non-decreasing in its second argument, T is non-decreasing. By the continuity of R and compactness of \mathcal{P} , $\|T(\mathbf{0})\| = \max_{\mathcal{P}} |R[p, 0]| < \infty$. Finally, by the discounting property of R in Assumption 2, $T(f + b)(p, P) = R[p, f(P) + b] \leq R[p, f(P)] + \beta b = T(f)(p, P) + \beta b$. This verifies the assumptions of the contraction mapping theorem of Becker and Boyd [4, p. 49]. Hence, T has a unique fixed point in \mathcal{C} . Also, for any P , $|Y(P) - T^n(\mathbf{0})(P)| \leq \beta^n |Y(P_n)| \leq \beta^n \|Y\|$. Since the last term converges to 0 as n converges to ∞ , we have $\lim \|T^n(\mathbf{0}) - Y\| \rightarrow 0$. \square

Evidently, $Y(\mathcal{P}^\infty) \subset \mathcal{Y}_0 := [-\|Y\|, \|Y\|]$. By Assumption 2, $W(p, P) = Q[u(p), Y(P)]$ is also continuous with respect to the relative product topology on \mathcal{P}^∞ . Define the \mathcal{B} -operator as follows.

Definition 9. Let $\mathcal{B} : 2^{\mathbb{R}} \rightarrow 2^{\mathbb{R}}$ be given, for all $\mathcal{Y}' \subset \mathbb{R}$, by

$$\mathcal{B}(\mathcal{Y}') = \{y \mid \exists (p, y') \in \mathcal{P} \times \mathcal{Y}', \text{ with } y = R[p, y'], Q[u(p), y'] \geq \sup_{\mathcal{P}} \inf_{\mathcal{Y}'} Q[u(p''), y'']\}.$$

Let $\mathcal{Y} = \{y \mid y = Y(\Phi(\sigma)) \text{ with } \sigma \text{ sub-game perfect}\}$.

Proposition 9. (i) $\mathcal{Y} = \mathcal{B}(\mathcal{Y})$. (ii) If $\mathcal{W} \subset \mathbb{R}$ satisfies $\mathcal{W} \subseteq \mathcal{B}(\mathcal{W})$, then $\mathcal{W} \subseteq \mathcal{Y}$. (iii) \mathcal{Y} is a compact interval, $[y, \bar{y}]$.

Proof. (i) and (ii) are immediate applications of arguments in Abreu, Pearce, and Stacchetti [1] and are omitted. Endow $2^{\mathbb{R}}$ with the set inclusion ordering. \mathcal{B} is monotone in the set inclusion ordering and since $\mathcal{B}(\mathcal{Y}_0) \subset \mathcal{Y}_0$, the sequence $\{\mathcal{B}^n(\mathcal{Y}_0)\}$ is monotone decreasing. Let y_1, y_2 belong to $\mathcal{B}(\mathcal{W})$ with \mathcal{W} a compact interval. Then for $k = 1, 2$ there exists $(p_k, y'_k) \in \mathcal{P} \times \mathcal{W}$ with $y_k = R[p_k, y'_k]$ and $Q[u(p_k), y'_k] \geq \sup_{\mathcal{P}} \inf_{\mathcal{W}} Q[u(p), y]$. By the convexity of \mathcal{P} and \mathcal{W} , the function $H : [0, 1] \rightarrow \mathbb{R}$, $H(\psi) := R[\psi p_1 + (1 - \psi)p_2, \psi y'_1 + (1 - \psi)y'_2]$, is well defined. Let $\lambda \in [0, 1]$, then by continuity of R and, hence, H , there is a ψ such that $(1 - \lambda)y_1 + \lambda y_2 = H(\psi)$. By concavity of u and Q , $Q[u(\psi p_1 + (1 - \psi)p_2), \psi y'_1 + (1 - \psi)y'_2] \geq \psi Q[u(p_1), y'_1] + (1 - \psi)Q[u(p_2), y'_2] \geq \sup_{\mathcal{P}} \inf_{\mathcal{W}} Q[u(p), y]$. Hence, $\lambda y_1 + (1 - \lambda)y_2 \in B(\mathcal{W})$ and $B(\mathcal{W})$ is an interval. Let $y_\infty \in \text{cl}(B(\mathcal{W}))$. Then there is a sequence $\{y_k\}$ with each $y_k \in B(\mathcal{W})$ and $\lim_{k \rightarrow \infty} y_k = y_\infty$. For each y_k there is a $(p_k, y'_k) \in \mathcal{P} \times \mathcal{W}$ such that $y_k = R[p_k, y'_k]$ and $Q[u(p_k), y'_k] \geq \sup_{\mathcal{P}} \inf_{\mathcal{W}} Q[u(p), y]$. Since \mathcal{P} and \mathcal{W} are compact, the sequence $\{p_k, y'_k\}$ admits a convergent subsequence (p_{k_n}, y'_{k_n}) with limit (p_∞, y'_∞) . By the continuity of R , u and Q , $y_\infty = \lim R[p_{k_n}, y'_{k_n}] = R[p_\infty, y'_\infty]$ and each $Q[u(p_{k_n}), y'_{k_n}] \geq \sup_{\mathcal{P}} \inf_{\mathcal{W}} Q[u(p), y]$ so $Q[u(p_\infty), y'_\infty] \geq \sup_{\mathcal{P}} \inf_{\mathcal{W}} Q[u(p), y]$. Hence, $y_\infty \in B(\mathcal{W})$ and so $B(\mathcal{W})$ is closed. Since \mathcal{P} and \mathcal{W} are compact and R is continuous, $B(\mathcal{W})$ is bounded. Combining results, $\mathcal{B}(\mathcal{W})$ is a compact interval.

Hence, since \mathcal{Y}_0 is a compact interval, the $\{\mathcal{B}^n(\mathcal{Y}_0)\}$ is a decreasing sequence of compact intervals. Thus, $\mathcal{Y}_\infty := \lim_{n \rightarrow \infty} \mathcal{B}^n(\mathcal{Y}_0) = \bigcap_{n=0}^\infty [y_n, \bar{y}_n] = [y, \bar{y}]$, where $[y_n, \bar{y}_n] = \mathcal{B}^n(\mathcal{Y}_0)$, $y = \sup_n y_n$ and $\bar{y} = \inf_n \bar{y}_n$. Since $\mathcal{Y} \subseteq \mathcal{Y}_0$, $\mathcal{B}(\mathcal{Y}_0) \subseteq \mathcal{Y}_0$, $\mathcal{Y} = \mathcal{B}(\mathcal{Y})$ and \mathcal{B} is monotone, we have for all n , $\mathcal{Y} \subseteq \mathcal{B}^n(\mathcal{Y}_0)$. Thus, $\mathcal{Y} \subseteq \mathcal{Y}_\infty$. For the reverse inclusion see Abreu, Pearce, and Stacchetti [1]. This proves (iii). \square

By Proposition 9 (iii), it is sufficient to characterize the endpoints \underline{y} and \bar{y} of \mathcal{Y} . By Proposition 9 (i), these endpoint payoffs satisfy:

$$\underline{y} = \min\{R[p, y'] \mid \text{s.t. } (p, y') \in \mathcal{P} \times [\underline{y}, \bar{y}] \text{ and } Q[u(p), y'] \geq Q[u^*, \underline{y}]\} \tag{MIN}$$

and

$$\bar{y} = \max\{R[p, y'] \mid \text{s.t. } (p, y') \in \mathcal{P} \times [\underline{y}, \bar{y}] \text{ and } Q[u(p), y'] \geq Q[u^*, \underline{y}]\}, \tag{MAX}$$

where $u^* = \max_{\mathcal{P}} u(p)$. In addition, by Proposition 9 (ii) if \underline{y}' and \bar{y}' satisfy the above conditions, then $\underline{y} \leq \underline{y}' \leq \bar{y}' \leq \bar{y}$. Let $\mathcal{X} = \{p \in \mathcal{P} \mid Q[u(p), \bar{y}] \geq Q[u^*, \underline{y}]\}$.

Lemma 3.

1. The solution to (MIN) is attained by a pair $(\underline{p}, y'(\underline{p}))$, where: $\underline{p} \in \arg \min_{\mathcal{X}} R[p, y'(p)]$ and for each $p \in \mathcal{X}$, $y'(p)$ is the unique element of \mathcal{Y} satisfying $Q[u(p), y'(p)] = Q[u^*, \underline{y}]$. Either $y'(p) = \underline{y}$ in which case $\underline{y} = R[\underline{p}, \underline{y}]$ and $u(\underline{p}) = u^*$ or $y'(p) > \underline{y}$ in which case $y > Y(\underline{p}, \underline{p}, \dots)$.
2. There is a $\bar{p} \in \mathcal{X}$ such that $\bar{y} = R[\bar{p}, \bar{y}]$. If $Q[u(\bar{p}), \bar{y}] > Q[u^*, \underline{y}]$, then $\bar{y} = Y(\bar{p}, \bar{p}, \dots) = \max_{\mathcal{P}^\infty} Y(P)$.

Proof. Let $(p, y') \in \mathcal{P} \times \mathcal{Y}$ be a solution to (MIN). Feasibility of (p, y') implies $Q[u(p), y'] \geq \sup_{\mathcal{P}} Q[u(p'), \underline{y}]$. Suppose that $y' > \underline{y}$ and $Q[u(p), y'] > Q[u^*, \underline{y}]$. Then, by the monotonicity of $R[p, \cdot]$ and the continuity of $Q[u(p), \cdot]$, the payoff $R[p, y']$ can be weakly reduced by lowering y' . Thus, there must be a solution to (MIN) with either $Q[u(p), y'] = Q[u^*, \underline{y}]$ or $y' = \underline{y}$. However, if $y' = \underline{y}$, then $Q[u^*, \underline{y}] \geq Q[u(p), y']$ and feasibility of (p, y') implies that $Q[u(p), y'] = Q[u^*, \underline{y}]$ for this case as well. For each p in \mathcal{X} , by the continuity and monotonicity of $Q[u(p), \cdot]$, there is a unique $y'(p) \in \mathcal{Y}$ such that $Q[u(p), y'(p)] = Q[u^*, \underline{y}]$. Thus, wlg (MIN) may be replaced with the restricted problem: $\min_{\mathcal{X}} R[p, y'(p)]$, which has a solution since (MIN) does. Let \underline{p} denote such a solution. If $y'(\underline{p}) = \underline{y}$, then we have immediately that: $\underline{y} = R[\underline{p}, \underline{y}]$ and $Q[u(\underline{p}), \underline{y}] = Q[u^*, \underline{y}]$. Hence, $u(\underline{p}) = u^*$ and the minimal payoff and minimal continuation payoff are attained by repetition of \underline{p} . On the other hand, if $y'(\underline{p}) > \underline{y}$, then

$$\underline{y} = R[\underline{p}, y'(\underline{p})] > R[\underline{p}, \underline{y}] = R[\underline{p}, R[\underline{p}, y'(\underline{p})]] > R[\underline{p}, R[\underline{p}, \underline{y}]] > \dots > Y(\underline{p}, \underline{p}, \dots)$$

and the minimal continuation payoff exceeds that from repetition of \underline{p} .

For part (2) of the lemma, let (\bar{p}, \bar{y}) be a solution to (MAX). Since R is non-decreasing in its second argument, (\bar{p}, \bar{y}) is a solution as well and $\bar{y} = R[\bar{p}, \bar{y}]$. Since \mathcal{P} is compact, \mathcal{P}^∞ is compact in the product topology. By Proposition 8, Y is continuous in this topology and so there is a $P' = (p', P'')$ that maximizes Y on \mathcal{P}^∞ and attains a maximal payoff $\bar{\mu}$. Since R is non-decreasing in its second argument it follows that P'' attains $\bar{\mu}$ as well, i.e. $\bar{\mu} = R[p', \bar{\mu}]$. Suppose that $Q[u(\bar{p}), \bar{y}] > Q[u^*, \underline{y}]$ and $\bar{\mu} = R[p', \bar{\mu}] > \bar{y} = R[\bar{p}, \bar{y}]$. Let $\lambda \in [0, 1]$, $p^\lambda = \lambda \bar{p} + (1 - \lambda)p'$ and $y^\lambda = \lambda \bar{y} + (1 - \lambda)\bar{\mu}$. Then, by the concavity of R , for all $\lambda \in (0, 1]$, $R[p^\lambda, y^\lambda] \geq \lambda R[\bar{p}, \bar{y}] + (1 - \lambda)R[p', \bar{\mu}] > R[\bar{p}, \bar{y}]$. By the continuity of u and Q , there is a $\lambda \in (0, 1]$ small enough that, $Q[u(p^\lambda), y^\lambda] \geq Q[u^*, \underline{y}]$. Now, $R[p^\lambda, y^\lambda] \geq \lambda R[\bar{p}, \bar{y}] + (1 - \lambda)R[p', \bar{\mu}] = y^\lambda$. It follows using the monotonicity property of R and definition of Y that $y^\psi := Y(p^\lambda, p^\lambda, \dots) \geq y^\lambda > \bar{y}$. But then, since Q is increasing in its second argument, $Q[u(p^\lambda), y^\psi] \geq Q[u(p^\lambda), y^\lambda] \geq Q[u^*, \underline{y}]$. But this contradicts Proposition 9 (ii). \square

Appendix C. Proof of Theorem 1

Proof. Throughout this proof, we repeatedly use the continuity, monotonicity and discounting properties of u , Q and R as described in Assumption 2 and, in particular, their implication that W is continuous. We also use Assumption 3, \mathcal{R}^* does not depend on y . Let $\hat{p} \in \mathcal{R}^* \subset \mathcal{U}^*$ and \bar{p} be as in the preceding text. Let $\bar{\mathbb{N}} = \{0, \infty\} \cup \mathbb{N}$. Define $P : \bar{\mathbb{N}} \rightarrow \mathcal{P}^\infty$ as follows. For each $T \in \{0\} \cup \mathbb{N}$, let $P(T) = (\hat{p}, \dots, \hat{p}, \bar{p}, \bar{p}, \dots)$, with T the number of periods until the sequence $(\bar{p}, \bar{p}, \dots)$ begins. Let $P(\infty) = \hat{P} = (\hat{p}, \hat{p}, \dots)$. Choose $\tau : \mathcal{P} \rightarrow \mathbb{N}$ to satisfy for each $p \in \mathcal{P} \setminus \mathcal{U}^*$, $\hat{w} > W(p, P(\tau(p)))$, which is possible given the definition of \hat{w} and \mathcal{U}^* and the continuity of W .

Set σ so that $\Phi(\sigma) = \tilde{P}$. For each $t \in \mathbb{N}$, if $\Phi(\sigma|\tilde{P}^{t-1}) \neq P(T)$, $T \in \bar{\mathbb{N}}$ and $p \neq \tilde{p}_t$, then set $\Phi(\sigma|\tilde{P}^{t-1}, p) = P(\infty)$. For all $t \in \mathbb{N}$ and P^{t-1} , (i) if $\Phi(\sigma|P^{t-1}) = P(\infty)$, then for $p \in \mathcal{P} \setminus \mathcal{U}^*$, set $\Phi(\sigma|P^{t-1}, p) = P(\tau(p))$ and for $p \in \mathcal{U}^*$, set $\Phi(\sigma|P^{t-1}, p) = P(\infty)$, (ii) if $\Phi(\sigma|P^{t-1}) = P(T)$, $T \in \mathbb{N}$, then for each $p \in \mathcal{P}$, set $\Phi(\sigma|P^{t-1}, p) = P(T - 1)$, (iii) if $\Phi(\sigma|P^{t-1}) = P(0)$, and $p \neq \bar{p}$, then set $\Phi(\sigma|P^{t-1}, p) = P(\infty)$.

It is easy to see that σ is sub-game perfect. We now show that it is revision-proof. By Lemma 2, it is sufficient to show that there is no history P^t , $t \in \{0\} \cup \mathbb{N}$, and revision path $P'' = \{p_r''\}_{r=1}^\infty$ with histories P''^r and continuations P_r'' such that for $r \in \mathbb{N}$, $W(P_r'') \geq W(\Phi(\sigma|P^t, P''^{r-1}))$ with at least one of these inequalities strict. Suppose that such a history P^t and revision path P'' exists, we seek a contradiction. For each $r \in \mathbb{N}$, let $W_r := W(P_r'')$ and $Y_r := Y(P_r'')$. It is immediate that $P'' \neq \Phi(\sigma|P^t)$. We distinguish three further cases. Case 1: $\Phi(\sigma|P^t) = P(\infty)$ and for all $r \in \mathbb{N}$, $p_r'' \in \mathcal{U}^*$. Case 2: either $\Phi(\sigma|P^t) = \tilde{P}_{t+1} \neq P(T)$, $T \in \bar{\mathbb{N}}$, or $\Phi(\sigma|P^t) = P(0)$, there is a first date $r_0 \in \mathbb{N}$ at which $p_{r_0}'' \neq \sigma_{t+r_0}(P^t, P''^{r_0-1})$ and $\Phi(\sigma|P^t, P''^{r_0}) = P(\infty)$ and from $r_0 + 1$ onwards each $p_r'' \in \mathcal{U}^*$. Case 3: There is a first date r_0 at which $\Phi(\sigma|P^t, P''^{r_0-1}) = P(T)$, $T \in \mathbb{N}$. It is easily checked that all P^t and P'' with $P'' \neq \Phi(\sigma|P^t)$ fall into one of these cases.

Suppose that P'' belongs to Case 1. Since in this case every p_r'' belongs to \mathcal{U}^* , the definition of σ implies that for all $r \in \mathbb{N}$, $\Phi(\sigma|P^t, P''^{r-1}) = P(\infty)$ and $W(\Phi(\sigma|P^t, P''^{r-1})) = \hat{w}$. On the other hand since some p_r'' may belong to $\mathcal{U}^* \setminus \mathcal{R}^*$ and, therefore, may not be maximal for R on \mathcal{U}^* , we have that $W_r = Q[u(p_r''), Y(P_{r+1}'')] \leq Q[u(p_r''), \hat{y}] = \hat{w}$. Hence, there is no player for whom $W_r > W(\Phi(\sigma|P^t, P''^{r-1}))$. But this contradicts the assumed property of P'' and so, in fact, P'' cannot belong to Case 1.

Next suppose that P'' belongs to Case 2. Similar to Case 1, for $r > r_0$, $W_r \leq \hat{w} = W(\Phi(\sigma|P^t, P''^{r-1}))$. By (i) in the proposition and Assumption 4, $\hat{w} \leq W(\Phi(\sigma|P^t, P''^{r_0-1}))$. Also, since p_{r_0}'' need not be in \mathcal{U}^* and some later p_r'' values need not be in \mathcal{R}^* , $W_{r_0} = Q[u(p_{r_0}''), Y(P_{r_0+1}'')] \leq Q[u(p_{r_0}''), \hat{y}] \leq \hat{w}$. In addition, by (ii) in the proposition and Assumption 4, either the player at $t + r_0$ is strictly worse off adhering to P'' , $W_{r_0} < W(\Phi(\sigma|P^t, P''^{r_0-1}))$, or she is no worse off, $W_{r_0} = \hat{w} = W(\Phi(\sigma|P^t, P''^{r_0-1}))$, and, if $t + r_0 > 1$, the player at $t + r_0 - 1$ receives a weakly lower continuation payoff from P'' , $Y_{r_0} \leq Y(\Phi(\sigma|P^t, P''^{r_0-1}))$. Since P'' coincides with $\Phi(\sigma|P^t)$ between 1 and $r_0 - 1$, it follows that all players between to t and $t + r_0 - 1$ are weakly worse off adhering to P'' . But then there is no player for whom $W_r > W(\Phi(\sigma|P^t, P''^{r-1}))$ and so, in fact, P'' cannot belong to Case 2.

Finally, suppose that P'' belongs to Case 3. Then there is a first date $t + r_0 + 1$ at which $\Phi(\sigma|P^t, P''^{r_0}) = P(T)$ for some $T \in \mathbb{N}$. By Lemma 4 below, for $r = 1, \dots, T$, $W(P_{r_0+r}'') = W(\Phi(\sigma|P^t, P''^{r_0+r-1}))$ and, if $t + r_0 > 0$, $Y_{r_0+1} \leq Y(\Phi(\sigma|P^t, P''^{r_0}))$. There are two sub-cases. In sub-case 3A, $r_0 > 0$, $\Phi(\sigma|P^t, P''^{r_0-1}) = P(\infty)$, $p_{r_0}'' \notin \mathcal{P} \setminus \mathcal{U}^*$ and

$\Phi(\sigma|P^t, P''^{r_0}) = P(\tau(p''_{r_0}))$. In this case, using the definition of $\tau(p''_{r_0})$, Lemma 4 and the monotonicity of Q , the player at $t + r_0$ is made strictly worse off since $W(\Phi(\sigma|P^t, P''^{r_0-1})) = \hat{w} > W(p''_{r_0}, P(\tau(p''_{r_0}))) = Q[u(p''_{r_0}), Y(P(\tau(p''_{r_0})))] \geq Q[u(p''_{r_0}), Y_{r_0+1}]$. But then P'' cannot belong to sub-case 3A. In sub-case 3B, $r_0 = 1$ and $\Phi(\sigma|P^t) = P(T)$. By Lemma 4, for $r = 1, \dots, T$, $W(P''_r) = W(\Phi(\sigma|P^t, P''^{r-1}))$. If $P''_{T+1} = P(0) = (\bar{p}, \bar{p}, \dots)$, i.e. there are no further defections from the strategy, then for all $r \in \mathbb{N}$, $W(P''_r) = W(\Phi(\sigma|P^t, P''^{r-1}))$ and there is no player for whom $W(P''_r) > W(\Phi(\sigma|P^t, P''^{r-1}))$. If there is a first $r_1 > T$ such that $p''_{r_1} \neq \bar{p}$ and for all $r > r_1$, $p''_r \in \mathcal{U}^*$, then following our analysis of Case 2, for all r , $W(P''_r) \leq W(\Phi(\sigma|P^t, P''^{r-1}))$ and again there is no player for whom $W(P''_r) > W(\Phi(\sigma|P^t, P''^{r-1}))$. Finally, if there is a first $r_1 > T$ such that $p''_{r_1} \neq \bar{p}$ and a first $r_2 > r_1$ such that $p''_{r_2} \in \mathcal{P} \setminus \mathcal{U}^*$, then following our analysis of sub-case 3A some player is made worse off. Thus, either no player is made strictly better off or some player is made strictly worse off and P'' cannot belong to sub-case 3B. This exhausts all possible cases and so there is no history P^t and revision path P'' such that latter weakly raises the payoff of all players relative to reversion to the strategy and strictly raises the payoff of some. It follows from Lemma 2 that the strategy is revision-proof. \square

Lemma 4. Let $\sigma, P : \{0, \infty\} \cup \mathbb{N} \rightarrow \mathcal{P}^\infty$ and P'' be as in the previous proof. For any $t \in \{0\} \cup \mathbb{N}$ and P^t , if $\Phi(\sigma|P^t) = P(T)$, $T \in \mathbb{N}$, then for $r = 1, \dots, T$, $W(P''_r) = W(\Phi(\sigma|P^t, P''^{r-1}))$ and $Y(P'') \leq Y(\Phi(\sigma|P^t))$.

Proof. As in the proof of Theorem 1, for all $r \in \mathbb{N}$, let $W_r = W(P''_r)$ and $Y_r = Y(P''_r)$. Consider play at $t + T$. Given that $\Phi(\sigma|P^t) = P(T)$, the construction of σ implies: $\Phi(\sigma|P^t, P''^{T-1}) = P(1)$. Thus, σ implements the best sub-game perfect path, $P(1) = (\hat{p}, \bar{p}, \bar{p}, \dots)$, for the $t + T$ -th player. Since P'' at least matches the payoffs received under the strategy, it follows that:

$$W_T \geq W(P(1)) = Q[u(\hat{p}), Y(P(0))] = Q[u^*, \bar{y}].$$

We claim that $W_T = W(P(1))$ and $Y_T \leq Y(P(1))$. Suppose the claim is false, then either (A) $W_T > W(P(1))$ or (B) $W_T = W(P(1))$ and $Y_T > Y(P(1))$. We show that in either of these cases, $Y_{T+1} > \bar{y}$ and P''_{T+1} is a sub-game perfect path. This contradiction establishes the claim.

Suppose (A) $W_T > W(P(1))$, then $Q[u(p''_T), Y_{T+1}] = W_T > W(P(1)) = Q[u^*, \bar{y}]$ and, since $u^* \geq u(p''_T)$ and Q is monotone, $Y_{T+1} > \bar{y}$. Suppose (B) $W_T = W(P(1))$ and $Y_T > Y(P(1))$, then $R[p''_T, Y_{T+1}] = Y_T > Y(P(1)) = R[\hat{p}, \bar{y}]$ and since R is weakly monotone in its second argument, either $Y_{T+1} > \bar{y}$ or $p''_T \neq \hat{p}$. If $Y_{T+1} \leq \bar{y}$ and $p''_T \neq \hat{p}$, then, by definition of \hat{p} , $p''_T \notin \mathcal{U}^*$ and $u^* = u(\hat{p}) > u(p''_T)$. However, $W_T = Q[u(p''_T), Y_{T+1}] = Q[u^*, \bar{y}]$ and so either $p''_T \in \mathcal{U}^*$ or $Y_{T+1} > \bar{y}$ must occur. Hence, in fact, $Y_{T+1} > \bar{y}$.

Suppose that for some $k \in \mathbb{N}$, $Y_{T+k} > \bar{y}$. Since

$$\bar{y} = \max_{(p, y')} \{R[p, y'], y' \in [\underline{y}, \bar{y}], Q[u(p), y'] \geq Q[u^*, \underline{y}]\} \tag{C.1}$$

and $R[p''_{T+k}, Y_{T+k+1}] = Y_{T+k} > \bar{y}$, (p''_{T+k+1}, Y_{T+k+1}) is not feasible for (C.1). Either (1) $Q[u(p''_{T+k}), Y_{T+k+1}] < Q[u^*, \underline{y}]$, (2) $Y_{T+k+1} < \underline{y}$ or (3) $Y_{T+k+1} > \bar{y}$. If (1) holds then the player at $t + T + k$ is strictly worse off under P'' than under the continuation of σ (which is sub-game perfect and, thus, gives a payoff weakly above $Q[u^*, \underline{y}]$ after all histories). But this contradicts the assumed property of P'' . So (1) cannot hold. If (2), but not (1) holds, then $R[p''_{T+k}, \underline{y}] \geq R[p''_{T+k}, Y_{T+k+1}] = Y_{T+k} > \bar{y}$. But since $(p''_{T+k}, \underline{y})$ is clearly feasible for (C.1), this contradicts the optimality of \bar{y} in (C.1). Hence, (3) holds and $Y_{T+k+1} > \bar{y}$. By induction for

all $k \in \mathbb{N}$, $Y_{T+k} = R[p''_{T+k}, Y_{T+k+1}] > \bar{y}$ with each $Q[u(p''_{T+k}), Y_{T+k+1}] \geq Q[u^*, \underline{y}]$. But then, in fact, P''_{T+1} is a sub-game perfect path with continuation payoff $Y_{T+1} > \bar{y}$. This is a contradiction. We conclude that $W_T = W(P(1)) = W(\Phi(\sigma|P''^{T-1}))$ and $Y_T \leq Y(P(1))$.

At $t + T - 1$ after (P^t, P''^{T-2}) , σ implements the path $P(2) = (\hat{p}, \hat{p}, \bar{p}, \bar{p}, \dots)$. The $t + T - 1$ -th player receives $Q[u^*, Y(P(1))]$ and since $u(p''_{T-1}) \leq u^*$ and $Y_T \leq Y(P(1))$, we have: $W_{T-1} = Q[u(p''_{T-1}), Y_T] \leq Q[u^*, Y(P(1))] = W(\Phi(\sigma|P''^{T-2}))$. However, by the assumed property of P'' , $W_{T-1} \geq W(\Phi(\sigma|P''^{T-2}))$. So, $W_{T-1} = W(\Phi(\sigma|P''^{T-2}))$, $u(p''_{T-1}) = u(\hat{p}) = u^*$ and $Y_T = Y(P(1))$. Moreover, $Y_{T-1} = R[p''_{T-1}, Y_T] \leq R[\hat{p}, Y(P(1))] = Y(P(2))$ since $Y_T = Y(P(1))$, $p''_{T-1} \in \mathcal{U}^*$ and \hat{p} is maximal for $R[p, Y(P(1))]$ on \mathcal{U}^* . Continuing with this logic back to $t + 1$, we find that P'' must give each player between $t + 1$ and $t + T$ current and continuation payoffs equal to those obtained under σ and the player at t a continuation payoff less than or equal to that obtained under σ , i.e. less than or equal to $Y(P(T))$. This completes the proof. \square

Corollary 2. *Let Assumptions 2 to 4 hold. Let \tilde{P} be any path such that (i) for all t , $W(\tilde{P}_t) \geq \hat{w}$ and (ii) if $W(\tilde{P}_t) = \hat{w}$, then $Y(\tilde{P}_t) \geq \hat{y}$. Then there is a strategy σ that implements \tilde{P} and is revision-proof and there is no alternative σ' that delivers the same payoffs to all players as σ and satisfies $Y(\Phi(\sigma')) > Y(\tilde{P})$.*

Proof. Given the strengthening of (ii) to hold at $t = 1$ as well as later dates, the strategy constructed in the proof of Theorem 1 has the desired properties. \square

Appendix D. Proof of Theorem 2

We begin with some preliminary lemmas. Define $\gamma : (\underline{y}, \bar{y}] \rightarrow (\underline{y}, \bar{y}]$ according to:

$$\gamma(y) = \begin{cases} y' \in \operatorname{argmax}_{(\underline{y}, \bar{y}]} R^{-1}[\underline{p}, \cdot](y) & y \in (\underline{y}, R[\underline{p}, \bar{y}]] \\ y' = \bar{y} & y \in (R[\underline{p}, \bar{y}], \bar{y}]. \end{cases}$$

Thus, if $y \in (\underline{y}, R[\underline{p}, \bar{y}]]$, then $y = R[\underline{p}, \gamma(y)]$ and $\gamma(y)$ is next period’s best continuation payoff when y is today’s continuation payoff and \underline{p} is played in the next period. If $\beta = 0$ (e.g. in two period-lived OLG models), then R is constant in its second argument, $\underline{y} = R[\underline{p}, \bar{y}]$ and $(\underline{y}, R[\underline{p}, \bar{y}]]$ is empty. Thus, if $y \in (\underline{y}, R[\underline{p}, \bar{y}]]$, then, necessarily, $\beta > 0$.

Lemma 5. *Let Assumption 2 hold. γ is strictly increasing on $(\underline{y}, R[\underline{p}, \bar{y}]]$. For $y \in (\underline{y}, R[\underline{p}, \bar{y}]]$, $\gamma(y) \geq y + \{(1 - \beta)/\beta\}(y - \underline{y})$.*

Proof. The strict monotonicity of γ is immediate from the definition and, by Assumption 2, the monotonicity of $R[\underline{p}, \cdot]$. For $y \in (\underline{y}, R[\underline{p}, \bar{y}]]$, we have: $R[\underline{p}, \underline{y}] \leq \underline{y} < y = R[\underline{p}, \gamma(y)]$. Thus, by the Lipschitz property of R in Assumption 2, $y - \underline{y} \leq R[\underline{p}, \gamma(y)] - R[\underline{p}, \underline{y}] \leq \beta(\gamma(y) - \underline{y})$ and so $\gamma(y) \geq y + \{(1 - \beta)/\beta\}(y - \underline{y})$. \square

Lemma 6 constructs a function $\chi : (\underline{y}, R[\underline{p}, \bar{y}]] \rightarrow (\underline{y}, \bar{y}]$ that is used to define punishment continuation payoffs in the proof of Theorem 2.

Lemma 6. *Let Assumptions 2 and 6 hold. If $R[\underline{p}, \bar{y}] > \underline{y}$, then there is a function $\chi : (\underline{y}, R[\underline{p}, \bar{y}]) \rightarrow (\underline{y}, \bar{y})$ such that (i) for each $y \in (\underline{y}, R[\underline{p}, \bar{y}])$, $\chi(y) > y$ and $Q[u(\underline{p}), \gamma(y)] > Q[u(\hat{p}), \chi(y)]$, (ii) if $y, y' \in (\underline{y}, R[\underline{p}, \bar{y}])$ and $y' > y$, then $\chi(y') - y' \geq \chi(y) - y$.*

Proof. Let $y \in (\underline{y}, R[\underline{p}, \bar{y}])$. We first show that $Q[u(\underline{p}), \gamma(y)] > Q[u(\hat{p}), y]$. Suppose not, i.e. $Q[u(\underline{p}), \gamma(y)] \leq Q[u(\hat{p}), y]$. Recall that $Q[u(\underline{p}), y'(\underline{p})] = \underline{w} = Q[u(\hat{p}), \underline{y}]$. Also, $R[\underline{p}, y'(\underline{p})] = \underline{y} < y = R[\underline{p}, \gamma(y)]$, and since $R[\underline{p}, \cdot]$ is monotone by Assumption 2, $y'(\underline{p}) < \gamma(y)$. Then by the Lipschitz properties of R and Q in Assumptions 2 and 6, $Q[u(\underline{p}), \gamma(y)] - \underline{w} \leq Q[u(\hat{p}), y] - Q[u(\hat{p}), \underline{y}] < \kappa(y - \underline{y}) = \kappa(R[\underline{p}, \gamma(y)] - R[\underline{p}, y'(\underline{p})]) \leq \kappa\beta(\gamma(y) - y'(\underline{p})) \leq Q[u(\underline{p}), \gamma(y)] - Q[u(\underline{p}), y'(\underline{p})] = Q[u(\underline{p}), \gamma(y)] - \underline{w}$. This is a contradiction. Hence, $Q[u(\underline{p}), \gamma(y)] > Q[u(\hat{p}), y]$.

Define $d : (\underline{y}, R[\underline{p}, \bar{y}]) \rightarrow (\underline{y}, \bar{y})$ as, for $y \in (\underline{y}, R[\underline{p}, \bar{y}])$, $Q[u(\hat{p}), d(y)] = Q[u(\underline{p}), \gamma(y)]$. By the continuity of Q in Assumption 2 and the previous inequality, $Q[u(\underline{p}), \gamma(y)] > Q[u(\hat{p}), y]$, d is well defined and $d(y) > y$. Let $y < y' \leq R[\underline{p}, \bar{y}]$. We show that $d(y) - y \leq d(y') - y'$. Assume to the contrary that $d(y') - d(y) < y' - y$. We have:

$$\begin{aligned} 0 &< Q[u(\underline{p}), \gamma(y')] - Q[u(\underline{p}), \gamma(y)] = Q[u(\hat{p}), d(y')] - Q[u(\hat{p}), d(y)] \\ &\leq \kappa(d(y') - d(y)) < \kappa(y' - y) \leq \kappa(R[\underline{p}, \gamma(y')] - R[\underline{p}, \gamma(y)]) \leq \beta\kappa(\gamma(y') - \gamma(y)) \\ &\leq Q[u(\underline{p}), \gamma(y')] - Q[u(\underline{p}), \gamma(y)]. \end{aligned}$$

But this is a contradiction, so that $d(y) - y \leq d(y') - y'$. Set $\chi(y) = (d(y) + y)/2$. Then, since $d(y) > y$, $d(y) > \chi(y) = (d(y) + y)/2 > y$. Also, by the monotonicity of Q , $Q[u(\underline{p}), \gamma(y)] = Q[u(\hat{p}), d(y)] > Q[u(\hat{p}), \chi(y)]$. This proves (i) in the lemma. If $y, y' \in (\underline{y}, R[\underline{p}, \bar{y}])$ and $y' > y$, then $\chi(y') - y' = (d(y') - y')/2 \geq (d(y) - y)/2 = (\chi(y) - y)$. This proves (ii) in the lemma. \square

Proof of Theorem 2. Construct a strategy σ as follows. Set $\Phi(\sigma) = \tilde{P}$. Proceed through successive dates $t \in \mathbb{N}$. At each date t , the continuation strategy following a defection from \tilde{P} , $(\sigma|\tilde{P}^{t-1}, p)$, $p \neq \tilde{p}_t$, is constructed so as to incorporate a phase in which player payoffs are “built up” until they exceed \hat{w} . Thereafter a revision-proof continuation strategy constructed as in the proof of Corollary 2 is played. At t , $W(\tilde{P}_t) > \underline{w} = Q[u(\hat{p}), \underline{y}]$. By Assumption 2 and the continuity and monotonicity of Q , there is a $y_1 \in (\underline{y}, \bar{y}]$ such that $W(\tilde{P}_t) > Q[u(\hat{p}), y_1]$. We use a function $v : \bigcup_{r \in \mathbb{N}} \mathcal{P}^r \rightarrow [\underline{y}, \bar{y}]$ to attach continuation payoffs to histories during the build-up phase. v is defined recursively as follows. For each $p \in \mathcal{P} \setminus \{\tilde{p}_t\}$, let $v(p) = y_1$. If $v(P^r) \in (\underline{y}, R[\underline{p}, \bar{y}])$ and v has not previously taken values outside of $(\underline{y}, R[\underline{p}, \bar{y}])$, i.e. for all P^s such that \tilde{P}^r is a successor to P^s , $v(P^s) \in (\underline{y}, R[\underline{p}, \bar{y}])$, then $v(P^r, p)$ is updated according to:

$$v(P^r, p) = \begin{cases} \gamma(v(P^r)) & \text{if } p = \underline{p} \\ \chi(v(P^r)) & \text{otherwise.} \end{cases}$$

Also, $\sigma_{t+r}(\tilde{P}^{t-1}, P^r)$ is set to \underline{p} . If $v(P^r)$ enters $(R[\underline{p}, \bar{y}], \bar{y})$ for the first time, then the build-up phase is concluded and for successor histories P^s to P^r , $v(P^s)$ is set arbitrarily. The continuation strategy following such a history, $(\sigma|\tilde{P}^{t-1}, P^r)$, is set as follows. First, $\Phi(\sigma|\tilde{P}^{t-1}, P^r)$ is set equal to $(p', \bar{p}, \bar{p}, \dots)$, where p' lies in the interval between \underline{p} and \bar{p} and satisfies $v(P^r) = R[p', \bar{y}]$. Such a p' exists by the convexity of \mathcal{P} , the continuity of R (Assumption 2) and the fact that $v(P^r) \in (R[\underline{p}, \bar{y}], R[\bar{p}, \bar{y}])$, since $\bar{y} = R[\bar{p}, \bar{y}]$. Now, by the concavity of u and Q in Assumption 2 and Assumptions 4 and 5, $W(\Phi(\sigma|\tilde{P}^{t-1}, P^r)) = Q[u(p'), \bar{y}] \geq$

$\lambda Q[u(\underline{p}), \bar{y}] + (1 - \lambda) Q[u(\bar{p}), \bar{y}] \geq \hat{w}$ with equality only if $\lambda = 0$. Also, by [Assumption 4](#), at each date $s \in \mathbb{N}$ and each sub-history $P^s = (p', \bar{p}, \bar{p}, \dots, \bar{p})$, $W(\Phi(\sigma|\tilde{P}^{t-1}, P^r, P^s)) = Q[u(\bar{p}), \bar{y}] \geq \hat{w}$. Thus, along the continuation path $\Phi(\sigma|\tilde{P}^{t-1}, P^r) = (p', \bar{p}, \bar{p}, \dots)$ each player receives a payoff weakly more than \hat{w} . In addition, if at any date along the path a player receives \hat{w} , then this player and all her successors are playing \bar{p} and the continuation payoff for this player's predecessors from this point onwards is at its maximal value \bar{y} . Thus, by [Corollary 2](#), there is a revision-proof strategy σ' with $\Phi(\sigma') = (p', \bar{p}, \bar{p}, \dots)$ and there is no alternative continuation strategy that weakly raises the payoff of all players after (\tilde{P}^{t-1}, P^r) and strictly raises the continuation payoff of the player at $(\tilde{P}^{t-1}, P^{r-1})$. Set $(\sigma|\tilde{P}^{t-1}, P^r) = \sigma'$.

If $\beta = 0$, then $y_1 > \underline{y} = R[\underline{p}, \bar{y}]$ and so after all defections $p \neq \tilde{p}_t$, this procedure sets $(\sigma|\tilde{P}^{t-1}, p)$ to a common revision-proof continuation strategy σ' . In this case, define $\bar{r} = 1$. If $\beta > 0$, then let $\varepsilon_1 = \frac{1-\beta}{\beta}(y_1 - \underline{y}) > 0$ and $\varepsilon_2 = \chi(y_1) - y_1 > 0$. Now, by [Lemmas 5 and 6](#), $v(P^{r+1}) - v(P^r) \geq \min(\gamma(v(P^r)) - v(P^r), \chi(v(P^r)) - v(P^r)) \geq \varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$. Thus, continuation payoffs rise by at least ε during each period of the build-up phase and this phase cannot last more than $\bar{r} = (R[\underline{p}, \bar{y}] - y_1)/\varepsilon + 1$ periods. Thus, after all histories (\tilde{P}^{t-1}, P^r) with the first element of P^r not equal to \tilde{p}_t and $r \geq \bar{r}$, the continuation strategy $(\sigma|\tilde{P}^{t-1}, P^r)$ is revision-proof.

We now verify that σ is revision-proof. Let P^t be a history and P'' a candidate revision path beginning at $t + 1$. We consider two cases. In the first case, P^t enters a sub-game in which a revision-proof continuation strategy constructed as in the proof of [Corollary 2](#) is played, i.e. P^t incorporates a first deviation from \tilde{P} at some date $t_0 < t$, passes through a phase in which player continuation payoffs are built up until at some date t_1 with $t_0 < t_1 \leq t$, $v(P^{t_1}) \in (R[\underline{p}, \bar{y}], \bar{y}]$. From P^{t_1} a revision-proof continuation strategy is prescribed by σ . Since P'' revises this continuation strategy, by the definition of revision-proofness, it cannot make all players from $t + 1$ onwards weakly better off and some strictly better off relative to the strategy.

In the second case, P^t does not enter a sub-game in which a revision-proof continuation strategy constructed as in [Corollary 2](#) is played, i.e. $P^t = \tilde{P}^t$ or P^t incorporates a first deviation from \tilde{P} at some date $t_0 \leq t$ and passes through a payoff build-up phase that is not concluded by t , $v(P^t) \in (\underline{y}, R[\underline{p}, \bar{y}]]$. There are two sub-cases.

In the first, P'' reverts to the play prescribed by the strategy before a revision-proof continuation strategy is reached, i.e. at a date $t_1 + 1 \in \mathbb{N}$ such that $v(P^t, P''^{t_1+1}) \in (\underline{y}, R[\underline{p}, \bar{y}]]$. Then, either $t_1 = 0$ and $P'' = \Phi(\sigma|P^t)$, in which case, trivially, no player is made strictly better off by the revision or $t_1 > 0$ and $P'' \neq \Phi(\sigma|P^t)$. In the latter situation, the player at $t + t_1$ is the last deviator from the play prescribed by the strategy. The choice of the y_1 and χ terms in the construction of σ ensures that this player is strictly worse off relative to reversion to the strategy. For example, if $P^t = \tilde{P}^t$ and $t_1 = 1$, then the $t + 1$ -th player is the sole defector and she obtains $W(\tilde{P}_{t+1})$ by adhering to the strategy and $Q[u(p'_1), y_1] \leq Q[u(\hat{p}), y_1] < W(\tilde{P}_{t+1})$ if she defects. If $P^t \neq \tilde{P}^t$ or $t_1 > 1$, then the last defector deviates from a prescribed play of \underline{p} at $t + t_1$. She forfeits a payoff of $Q[u(\underline{p}), \gamma(v(P''^{t_1-1}))]$ and obtains $Q[u(p''_1), v(P''^{t_1})] = Q[u(p''_1), \chi(v(P''^{t_1-1}))] \leq Q[u(\hat{p}), \chi(v(P''^{t_1-1}))] < Q[u(\underline{p}), \gamma(v(P''^{t_1-1}))]$.

In the second sub-case, P'' does not revert to the play prescribed by σ before a revision-proof continuation strategy is reached. However, by the argument above a date $t_2 \leq \bar{r}$ is reached at which $(\sigma|P^t, P''^{t_2-1})$ is revision-proof. Then either some player from $t + t_2$ onwards is made worse off by P''_{t_2} relative to reversion to the strategy or, by [Corollary 2](#), all players from $t + t_2$ onwards receive the same payoff from P''_{t_2} as from reversion to the strategy and the continuation

payoff $Y(P''_{t_2})$ is no more than that under the strategy. In the latter situation, if there are deviating players between $t + 1$ and $t + t_2 - 1$, then the last such deviator (at $t + t_1$ with $1 \leq t_1 \leq t_2 - 1$) is made strictly worse off relative to the strategy since she receives a continuation payoff $Y(P''_{t_1+1})$ weakly less than that available under the strategy, $v(P''^{t_1})$, and so

$$\begin{aligned} Q[u(p''_{t_1}), Y(P''_{t_1+1})] &\leq Q[u(p''_{t_1}), v(P''^{t_1})] = Q[u(p''_{t_1}), \chi(v(P''^{t_1-1}))] \\ &\leq Q[u(\hat{p}), \chi(v(P''^{t_1-1}))] < Q[u(\underline{p}), \gamma(v(P''^{t_1-1}))] \\ &= W(\Phi(\sigma | P^t, P''^{t_1-1})). \end{aligned}$$

Thus, either some player is made worse off or no players are made better off by the revision. These cases exhaust all possibilities. There is no history P^t and path P'' such that the latter strictly raises the payoffs of some players and does not reduce the payoff of any relative to the strategy. σ is revision-proof. \square

Appendix E. Sub-game perfection with quasi-recursive payoffs and state variables

Given ψ as in Assumption 2', let

$$\mathcal{C}_\psi = \left\{ f : \text{Graph } \Pi \rightarrow \mathbb{R} \mid f \text{ continuous and } \sup_{\text{Graph } \Pi} |f(k, P)/\psi(k)| < \infty \right\}.$$

The next result is a variant of a theorem in Becker and Boyd [4, p. 49]. It ensures the existence of a payoff function Y belonging to \mathcal{C}_ψ .

Proposition 10. *Let $\mathcal{K}, \mathcal{P}, \Pi, R$ and Λ satisfy Assumption 2'. There exists a unique function $Y \in \mathcal{C}_\psi$ satisfying for all $(k, p, P') \in \text{Graph } \Pi, Y(k, p, P') = R[k, p, Y(\Lambda(k, p), P')]$.*

Proof. Available upon request. \square

For $\mathcal{Y}' : \mathcal{K} \rightarrow 2^{\mathbb{R}} \setminus \emptyset$, the \mathcal{B} -operator is given by:

$$\begin{aligned} \mathcal{B}(\mathcal{Y}')(k) &= \left\{ y \in \mathbb{R} \mid \exists (p, y') \in \Gamma(k) \times \mathbb{R} \text{ with } y = R[k, p, y'], y' \in \mathcal{Y}'(\Lambda(k, p)), \right. \\ &\quad \left. Q[u(k, p), y'] \geq \sup_{p'' \in \Gamma(k)} \inf_{y'' \in \mathcal{Y}'(\Lambda(k, p''))} Q[u(k, p''), y''] \right\}. \end{aligned}$$

The following analogue of Proposition 9 is then be obtained by similar arguments.

Proposition 11. *(i) For all $k \in \mathcal{K}, \mathcal{Y}(k) = \mathcal{B}(\mathcal{Y})(k)$. (ii) If for all $k \in \mathcal{K}, \mathcal{B}(\mathcal{Y}')(k) \subseteq \mathcal{Y}'(k)$, then for all $k \in \mathcal{K}, \mathcal{Y}'(k) \subseteq \mathcal{Y}(k)$. (iii) \mathcal{Y} is compact-valued.*

Proof. Available upon request. \square

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