



# A model of banknote discounts

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## Abstract

Prior to 1863, state-chartered banks in the United States issued notes—dollar-denominated promises to pay specie to the bearer on demand. Although these notes circulated at par locally, they usually were quoted at a discount outside the local area. These discounts varied by both the location of the bank and the location where the discount was being quoted. Further, these discounts were asymmetric across locations, meaning that the discounts quoted in location A on the notes of banks in location B generally differed from the discounts quoted in location B on the notes of banks in location A. Also, discounts generally increased when banks suspended payments on their notes. In this paper we construct a random matching model to qualitatively match these facts about banknote discounts. To attempt to account for locational differences, the model has agents that come from two distinct locations. Each location also has bankers that can issue notes. Banknotes are accepted in exchange because banks are required to produce when a banknote is presented for redemption and their past actions are public information. Overall, the model delivers predictions consistent with the behavior of discounts.

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## 1. Introduction

Between 1783, when the United States won independence from Great Britain, until the passage of the National Currency Act in 1863, the largest component of U.S. currency in circulation was notes issued by state-chartered banks. These banknotes were dollar-denominated promises to pay

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specie to the bearer on demand and were distinguishable by issuing bank. Virtually all banks that existed during this period issued banknotes. Since the country had approximately 325 banks in 1820, close to 850 banks in 1840, and more than 1500 in 1860, large numbers of distinct currencies were in circulation in the country throughout the antebellum period.

Banknotes circulated in the location of the issuing bank, and the notes of at least some banks also circulated outside the local area. In the vernacular of the day, such banknotes were known as “foreign notes.” We know this from several sources. The balance sheets for virtually every bank during this period have an asset account for notes of other banks, and the balance sheets for banks in several states have separate account listings for out-of-state notes. An 1842 report of banks in Pennsylvania lists the banks’ holdings of notes of other banks by bank. It shows that Pennsylvania banks held the notes of banks in at least 61 cities and in 15 other states. The clearing system for New England banknotes run by Suffolk Bank in Boston cleared a large volume of notes. Such a system would not have been necessary if the circulation of banknotes had been entirely local. Finally, banknotes bearing the stamp of a business in another location are still in existence.

Due to the concurrent circulation of a large number of notes of different banks, specialized publications, generically called banknote reporters, came into existence. They were usually published by a note broker or in conjunction with one. They were published at least monthly (or more frequently in some cases), and each issue listed virtually all of the banks in existence in the country at the time and quoted the discount at which the notes of each bank would be exchanged for notes of local banks (banks in the city where the reporter was published). In other words, banknote reporters listed the exchange rates of the notes of each bank in the country in terms of local banknotes. Banknote reporters are known to have been published in many cities, including New York, Philadelphia, Boston, Pittsburgh, Cleveland, Cincinnati, Chicago, and Zanesville (OH).

If we examine the discounts quoted in banknote reporters, four facts emerge:

- (1) Local banknotes were always quoted at par.
- (2) “Foreign” banknotes usually were quoted at a discount to local banknotes, and this discount varied by the location of the bank and the reporter.
- (3) Discounts were asymmetric across locations, meaning that the discounts quoted in location A on the notes of banks in location B generally differed from the discounts quoted in location B on the notes of banks in location A.
- (4) Discounts on foreign notes were higher when those notes were not being redeemed at par.

The purpose of this paper is to build a model that can qualitatively match these facts about banknote discounts. The model we construct builds on the basic search model of money by Shi [5] and Trejos and Wright [6]. To attempt to account for the locational differences in banknote discounts, our model has agents that come from two distinct locations, whereas the Trejos–Wright framework implicitly assumes that all agents are in one location. In addition, unlike the model of Trejos–Wright, our model does not have fiat money but instead has banknotes issued by banks in the two locations. We find that an equilibrium exists in which these banknotes are valued and act as media of exchange.

We model the fact that banknotes had to be redeemed in specie on demand by adapting the [2] innovation that banks are required to produce when a banknote is presented for redemption and their past actions are public information. We present two versions of the model. In the first, banks are threatened with permanent autarky if they ever fail to produce the same quantity of goods when their notes are presented for payment as they obtain when issuing a note. We refer to this case as par redemption. In the second, banks are allowed to suspend payments on their notes by

redeeming them below their issue value. This second version of the model is intended to capture a feature of the actual experience of the period under consideration.

By requiring that banks must redeem their notes, we capture the aspect of banknotes that made them like debt instruments as modeled by Gorton [3]. However, Gorton is interested in determining whether prices of banknotes reflected banks' risk-taking behavior in the pre-Civil War period, where a risky bank is one that is likely to overissue banknotes. The mechanism that allows market participants to adequately price these risks is the redemption option: banknotes can be sent for redemption to the issuing bank, and if this is not too costly, then the issuing bank can be discouraged from overissuing by the threat of bank runs. In Gorton's model, the main determinant of the costs related to sending notes for redemption is the physical distance between the location at which trade occurs and the location of the bank that issued the banknotes used to trade. In our model, banknote prices are instead exclusively related to trading opportunities and the value of each one of these opportunities.

Our model is also related to [7]. They build a two-country, two-currency random-matching model in which money serves as a medium of exchange. Depending on the parameters in the model, they find that three types of equilibria exist: one in which both monies circulate in both countries, one in which both monies circulate only in the home country, and one in which one money circulates only in the home country and the other circulates in both. A currency is more likely to circulate in the other country the more likely it is that agents from that country meet agents from the other country. We obtain the same types of equilibria in our model. However, in our model the major determinant of whether banknotes circulate outside their own area is how easy it is for noteholders to contact a bank.

The paper proceeds as follows. In the next section, we document the four facts about banknote discounts that we want to match. In Section 3, we present the model environment. In Section 4 we define a monetary symmetric steady-state equilibrium for this economy. In Section 5, we show that for certain values of the parameters, we can explain facts 1–3 for the case in which bankers are required to redeem their notes at par. In Section 6, we examine the case in which banks have suspended payment on their notes and show that for certain values of the parameters, we can explain fact 4. The final section concludes.

## 2. Facts about banknote discounts

In this section we present some documentation for the four facts about banknote discounts that we want our model to be consistent with.<sup>1</sup> The first fact is that banknotes went at par in the local area. An examination of the banknote reporters for New York, Philadelphia, Cleveland, Cincinnati, and Pittsburgh shows that notes of local banks that had not failed were always listed at par.

The same examination reveals it was generally the case that the notes of all banks in a location were quoted at the same discount and that the notes of nonlocal banks (i.e., "foreign" banknotes) generally were listed at a discount that varied by the location of the bank and the location of the banknote reporter. For example, Thompson's banknote reporter published in New York on April 1, 1852, lists notes of New England banks at a  $\frac{1}{4}\%$  discount, those of South Carolina banks at a  $1\frac{1}{4}\%$  discount, those of Ohio banks at a  $1\frac{1}{2}\%$  discount, and those of Tennessee banks at a 3% discount. For the same date, Van Court's banknote reporter published in Philadelphia lists these

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<sup>1</sup> The facts about banknote discounts that we discuss here are based on the data set recently compiled from banknote reporters from New York, Philadelphia, Cleveland, Cincinnati, and Pittsburgh by Warren Weber.

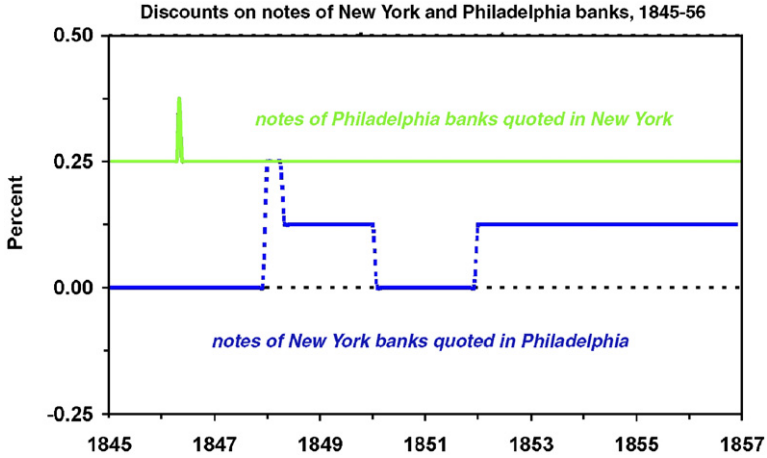


Fig. 1. Discounts on notes of New York and Philadelphia banks, 1845–1856.

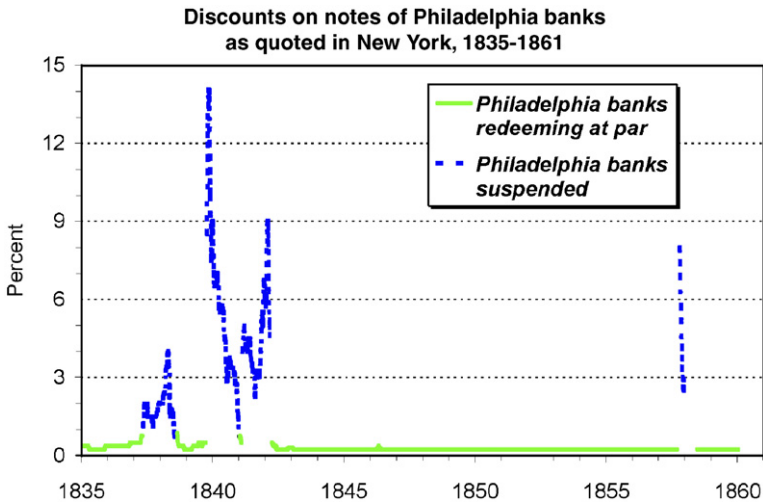


Fig. 2. Discounts on notes of Philadelphia banks as quoted in New York, 1835–1860.

discounts as  $\frac{3}{8}\%$ , 1%,  $1\frac{1}{2}\%$ , and  $2\frac{1}{2}\%$ , respectively. This dependence of discounts on the location of the bank and the banknote reporter is our second fact.

The third fact is that the discounts on banknotes were asymmetric: the discount quoted in location A on notes of banks from location B was not generally the same as the discount quoted in location B on notes of banks from location A. This is illustrated in Fig. 1, where we plot the discounts on New York banknotes as quoted in Philadelphia and the discounts on Philadelphia banknotes as quoted in New York for the period 1845–1856. The figure shows that with the exception of a short period in 1847, the notes of New York banks were quoted at a lower discount in Philadelphia than the notes of Philadelphia banks were quoted in New York. Asymmetry of discounts is also found when the discounts of the notes of Cincinnati banks in New York and

Philadelphia are compared with the discounts on the notes of banks in those cities as quoted in Cincinnati. The notes of Cincinnati banks were always quoted at a discount in New York and Philadelphia, but the notes of banks in those cities were generally quoted at par in Cincinnati.

The final fact is that the discounts on foreign banknotes were higher when banks had suspended payments than when they were redeeming their notes at par. Fig. 2 shows the behavior of the discounts on notes of banks in Philadelphia as quoted in New York for the period 1835–1860. The figure shows that banknote discounts were larger for Philadelphia banks when these banks were suspended than when they were not. The behavior of discounts on Philadelphia banknotes is consistent with that of the discounts on the notes of banks in other locations and quotes from Philadelphia instead of New York.

We did find one exception to this behavior, however. When New York banks suspended in May 1837, the discount on their notes in Philadelphia fell rather than rose. This exception is the only one that we have been able to find.

### 3. Model environment

The model environment is similar to that of [5,6]. Time is discrete and infinite. There is a single, nondurable good that is perfectly divisible. There are two types of agents, nonbankers and bankers. Both types of agents can produce and consume the nondurable good. Bankers also have access to a technology to produce banknotes—pieces of paper that bear the name of the banker. Banknotes are indivisible. We assume that nonbankers are anonymous and their past trading histories are private information. In contrast, bankers are not anonymous, and their past histories are public information. All nonbankers are identical, and all bankers are identical.

There are two locations—home and foreign. There is a measure  $B_H$  of bankers in the home location (home bankers) and a measure  $B_F$  of bankers in the foreign location (foreign bankers). In addition, there is a measure  $H$  of nonbankers from the home location (home nonbankers) and a measure  $F$  of nonbankers from the foreign location (foreign nonbankers).

At the beginning of each period, a nonbanker has a probability  $\pi$  of being a consumer but not a producer, the same probability of being a producer but not a consumer, and a probability  $1 - 2\pi$  of being neither. Also at the beginning of the period, a banker has an equal probability of being either a consumer or a producer, but cannot be both.

In every period, nonbankers from location  $j$  have a probability  $0 < \theta_j < 1$ , where the subscript denotes the location of the nonbanker  $j \in \{H, F\}$ , of meeting pairwise with a banker in their location. However, they cannot meet bankers in the other location. The restriction that nonbankers do not meet bankers in their location with certainty is intended to capture the fact that it was costly for nonbankers to go to local banks to obtain banknotes (by taking out loans or paying specie) or to redeem banknotes. The restriction that nonbankers cannot meet bankers from the other location is intended to capture the fact that it was more costly for nonbankers to go to nonlocal bankers than to local ones. We assume that  $\theta_H H \leq B_H$  and  $\theta_F F \leq B_F$ , so that a nonbanker that has the opportunity to be matched with a banker can be. Bankers cannot trade with other bankers.

Nonbankers who do not meet a banker meet randomly pairwise with another nonbanker with probability  $\delta$ . Given our random matching assumption, the probability that a nonbanker meets a home nonbanker is

$$P_H = \delta \frac{(1 - \theta_H) \frac{H}{F}}{(1 - \theta_H) \frac{H}{F} + (1 - \theta_F)}$$

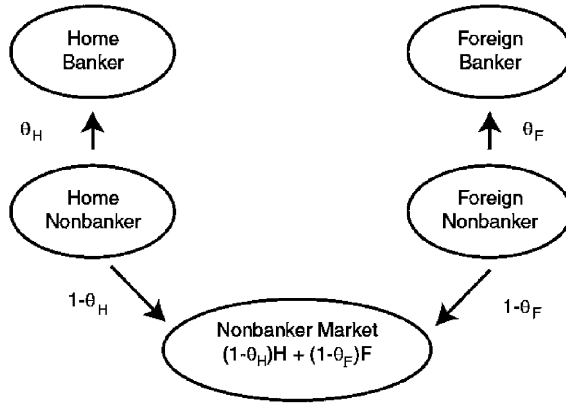


Fig. 3. Within-period meeting possibilities for private agents.

and the probability that a nonbanker meets a foreign nonbanker is

$$P_F = \delta \frac{(1 - \theta_F)}{(1 - \theta_H)\frac{H}{F} + (1 - \theta_F)}$$

The within-period meeting possibilities for nonbankers are illustrated in Fig. 3. That nonbankers cannot be producers and consumers at the same time introduces a problem of lack of double coincidence of wants into meetings between nonbankers. The lack of double coincidence of wants and the anonymity of nonbankers give rise to the need for a medium of exchange. This role is filled by banknotes. We assume that nonbankers can hold the notes of either home or foreign bankers subject to a unit upper-bound restriction on banknote holdings. We assume that nonbankers can trade banknotes for the nondurable good, but cannot trade notes of banks in one location for notes of banks in the other location. This assumption is made for tractability. In pairwise meetings between nonbankers in which one has a banknote and the other does not, we assume that the note holder makes a take-it-or-leave-it (TIOLI) offer to the other. We further assume that nonbanker producers always trade if indifferent.

We now consider the maximization problems of bankers and nonbankers.

### 3.1. Bankers

In any period, bankers can either produce or consume the nondurable good, and whether a banker is a producer or consumer is known before any trading occurs.<sup>2</sup> In addition, they have the technology to costlessly issue banknotes, which are pieces of paper distinguishable by the issuer. Banknotes are indivisible. In meetings with nonbankers, bankers either produce a note for  $q_j \in \mathbb{R}^+$  units of the nondurable good or redeem a previously issued note for  $Q_j \in \mathbb{R}^+$  units of the nondurable good, where the subscript denotes the location of the banker  $j \in \{H, F\}$ . Thus,  $q_j$  is the issue value of a banknote, and  $Q_j$  is the redemption value of a banknote. Note that bankers do not redeem notes of bankers in the other location. This assumption is consistent with the fact

<sup>2</sup> This assumption guarantees that whenever a nonbanker decides to meet a banker, he can direct his search to be in a single coincidence meeting with certainty.

that during the period we are considering, a bank would not give specie for the notes of another bank unless that bank had previously established an account for that purpose.

The momentary utility function of a banker is assumed to be

$$u(q_j, Q_j) = u(q_j) - Q_j, \quad j \in \{H, F\}.$$

The function  $u$  has the properties  $u(0) = 0$ ,  $u' > 0$ ,  $u'(0) = \infty$ ,  $u'(\infty) = 0$ , and  $u'' < 0$ . Bankers are assumed to maximize expected discounted lifetime utility, where  $\beta$  is the discount factor. The choice variable for bankers is  $q_j$ .

When a nonbanker producer meets a banker, we assume that the banker makes a TIOLI offer to the producer. Given  $Q_j$ , the banker will offer the nonbanker producer a banknote in exchange for the largest quantity of the good such that the producer is indifferent between producing and receiving a banknote and not producing. Let  $Z_j$  be the expected discounted lifetime utility of a  $j$  banker. Bankers are willing to issue and redeem banknotes as long as  $Z_j \geq 0$ .

Rather than being set exogenously, we assume that  $Q_j$  is determined by a rule set by the regulations under which banks operated. During the period we are considering, banks were required to redeem their notes at par. Failure to do so would mean that, except in extraordinary circumstances, a bank could face penalties and ultimately lose its banking privileges. We interpret the par redemption requirement to require  $Q_j = q_j$ ; that is, banks must redeem their notes at the issue value. The fact that bankers are not anonymous and that their histories are public information in our model means that bankers could potentially be punished for failing to redeem their notes at this exogenously determined price. Hence, it will not be utility maximizing for a bank to deviate.

Despite the requirements that banks always redeem their notes, there were a few times during this period when banks in large parts of the country suspended specie payments on notes. Because of the widespread nature of these suspensions and fears of the consequences if banks were required to redeem notes at par, state banking authorities usually did not take actions against banks during these suspensions. We model the case in which bankers have suspended payments on notes by setting  $Q_j = k_j q_j$ ,  $k_j \in [0, 1)$ . The reason we choose such a rule rather specifying a value for  $Q_j$  is that the issue value of a note,  $q_j$ , does not necessarily remain invariant when banks are not redeeming notes at par. Hence, if we specified an exogenous value for  $Q_j$ , it could be that in the resulting equilibrium  $Q_j > q_j$ , the redemption value of the note is actually greater than the issue value. Our formulation ensures that this cannot occur. The reason for not modeling suspensions as  $Q_j = 0$  is that banks were usually willing to provide some small amounts of specie to noteholders during these suspension periods.

### 3.2. Nonbankers

We assume that nonbankers in both locations have a momentary utility function of the form

$$u(c) - y,$$

where  $c$  denotes consumption and  $y$  denotes production. Nonbankers are assumed to maximize expected discounted lifetime utility, where  $\beta$  is the discount factor.

Let  $V_k$  and  $W_k$ ,  $k \in \{0, H, F\}$  denote the beginning-of-period expected value of having no banknotes ( $k = 0$ ), holding a home banknote ( $k = H$ ), and holding a foreign banknote ( $k = F$ ) for a home nonbanker and a foreign nonbanker, respectively. Further, let  $x_j \mathbb{R}^+$  and  $z_j \mathbb{R}^+$ ,  $j \in \{H, F\}$  denote the production of a home nonbanker and a foreign nonbanker, respectively, in a meeting with the holder of a home note ( $j = H$ ) or a foreign note ( $j = F$ ). Since we consider

only steady-state equilibria, the value functions and the production quantities are independent over time. We say that foreign banknotes go at a discount in the local market if nonbankers with foreign notes obtain fewer goods from local producers than nonbankers with local notes. The discount on foreign notes in the home market is

$$d_H = 1 - \frac{x_F}{x_H}.$$

If we apply the same concept in the foreign market, the discount on home banknotes in the foreign market is

$$d_F = 1 - \frac{z_H}{z_F}.$$

If either discount is negative, then nonlocal notes are said to be going at a premium.

Let  $0 \leq m_{ji} \leq 1$  be the fraction of nonbankers from location  $j$  holding an  $i$  banknote and  $0 \leq \lambda_{kj}^i \leq 1$  be the probability that a  $k$  (consumer) with an  $i$  banknote trades with a  $j$  (producer). Further, define  $\Omega_{ki}$  to be the expected value for a  $k$  consumer with an  $i$  banknote of going to the nonbanker market. Specifically,

$$\begin{aligned} \Omega_{Hi} = & \max_{\lambda_{HH}^i, \lambda_{HF}^i} \pi \{ P_H m_{H0} \lambda_{HH}^i \Lambda_H^H [u(x_i) - \Delta_{Hi}] \\ & + P_F m_{F0} \lambda_{HF}^i \Lambda_F^H [u(z_i) - \Delta_{Hi}] \}, \quad i \in \{H, F\}, \end{aligned} \tag{1}$$

$$\begin{aligned} \Omega_{Fi} = & \max_{\lambda_{FH}^i, \lambda_{FF}^i} \pi \{ P_H m_{H0} \lambda_{FH}^i \Lambda_H^F [u(x_i) - \Delta_{Fi}] \\ & + P_F m_{F0} \lambda_{FF}^i \Lambda_F^F [u(z_i) - \Delta_{Fi}] \}, \quad i \in \{H, F\}, \end{aligned} \tag{2}$$

where  $\Lambda_j^i$  denotes the probability that nonbankers from location  $j$  without banknotes trade with nonbankers with an  $i$  note,  $\Delta_{Hi} = \beta(V_i - V_0)$ , and  $\Delta_{Fi} = \beta(W_i - W_0)$ . The first term on the right-hand side of (1) is the expected value of meeting another home nonbanker without a banknote, and the second term is the expected value of meeting a foreign nonbanker without a banknote. The sum of these terms is multiplied by the probability that the other nonbanker in the meeting is a producer to obtain the expected value to a home consumer of going to the nonbanker market with an  $i$  banknote. The terms in (2) are similarly interpreted. Note that in our definitions of  $\lambda_{kj}^i$  and  $\Lambda_j^k$  we have implicitly assumed that although consumers can recognize whether potential sellers are from the home or foreign location, producers cannot. This means that producers have to produce the same quantity for a banknote whether the consumer is from the home or foreign location.

Let  $0 \leq \gamma_j \leq 1$  be the probability that a nonbanker from location  $j$  trades with a banker. Then the flow Bellman equations for home nonbankers with a banknote are

$$(1 - \beta)V_H = \pi \left\{ \theta_H \max_{\gamma_H \in [0,1]} \gamma_H [u(Q_H) - \Delta_{HH}] + (1 - \theta_H)\Omega_{HH} \right\}, \tag{3}$$

$$(1 - \beta)V_F = (1 - \theta_H)\pi\Omega_{HF} \tag{4}$$

and those for foreign nonbankers with a banknote are

$$(1 - \beta)W_F = \pi \left\{ \theta_F \max_{\gamma_F \in [0,1]} \gamma_F [u(Q_F) - \Delta_{FF}] + (1 - \theta_F)\Omega_{FF} \right\}, \tag{5}$$

$$(1 - \beta)W_H = (1 - \theta_F)\pi\Omega_{FH}. \tag{6}$$

The incentive compatibility conditions are

$$\gamma_H \in \begin{cases} \{1\} & \text{if } u(Q_H) - \Delta_{HH} > 0, \\ \{\phi\} & \text{if } u(Q_H) - \Delta_{HH} = 0, \\ \{0\} & \text{if } u(Q_H) - \Delta_{HH} < 0, \end{cases} \quad \gamma_F \in \begin{cases} \{1\} & \text{if } u(Q_F) - \Delta_{FF} > 0, \\ \{\phi\} & \text{if } u(Q_F) - \Delta_{FF} = 0, \\ \{0\} & \text{if } u(Q_F) - \Delta_{FF} < 0, \end{cases} \quad (7)$$

$$\lambda_{HH}^i \in \begin{cases} \{1\} & \text{if } u(x_i) > x_i, \\ \{\phi\} & \text{if } u(x_i) = x_i, \\ \{0\} & \text{if } u(x_i) < x_i, \end{cases} \quad \lambda_{FH}^i \in \begin{cases} \{1\} & \text{if } u(x_i) > z_i, \\ \{\phi\} & \text{if } u(x_i) = z_i, \\ \{0\} & \text{if } u(x_i) < z_i, \end{cases} \quad (8)$$

$$\lambda_{HF}^i \in \begin{cases} \{1\} & \text{if } u(z_i) > x_i, \\ \{\phi\} & \text{if } u(z_i) = x_i, \\ \{0\} & \text{if } u(z_i) < x_i, \end{cases} \quad \lambda_{FF}^i \in \begin{cases} \{1\} & \text{if } u(z_i) > z_i, \\ \{\phi\} & \text{if } u(z_i) = z_i, \\ \{0\} & \text{if } u(z_i) < z_i, \end{cases}$$

where  $\phi = [0, 1]$ .

The Bellman equations for nonbankers without a banknote are obtained in a similar manner. Define  $\Omega_{k0}$  to be the expected value for a  $k$  producer with no banknote of going to the nonbanker market. Specifically,

$$\begin{aligned} \Omega_{H0} = \max_{\lambda_H^0, \lambda_F^0} & \pi[(P_H m_{HH} \Lambda_{HH}^H + P_F m_{FH} \Lambda_{FH}^H) \lambda_H^0 (\Delta_{HH} - x_H) \\ & + (P_H m_{HF} \Lambda_{HF}^F + P_F m_{FF} \Lambda_{FF}^F) \lambda_F^0 (\Delta_{HF} - x_F)], \end{aligned} \quad (9)$$

$$\begin{aligned} \Omega_{F0} = \max_{\lambda_H^0, \lambda_F^0} & \pi[(P_H m_{HH} \Lambda_{HF}^H + P_F m_{FH} \Lambda_{FF}^H) \lambda_H^0 (\Delta_{FH} - z_H) \\ & + (P_H m_{HF} \Lambda_{HF}^F + P_F m_{FF} \Lambda_{FF}^F) \lambda_F^0 (\Delta_{FF} - z_F)], \end{aligned} \quad (10)$$

where  $\lambda_j^0$  and  $\Lambda_{kj}^i$  are defined analogously to  $\Lambda_j^0$  and  $\lambda_{kj}^i$  above. The first term on the right-hand side of (9) is the expected value of meeting a nonbanker with a home banknote, and the second term is the expected value of meeting a nonbanker with a foreign banknote. The sum of these terms is multiplied by the probability that the other nonbanker in the meeting is a consumer to obtain the expected value to a home producer of going to the nonbanker market with an  $i$  banknote. The terms in (10) are similarly interpreted. With this notation, the Bellman equations for nonbankers without a banknote are

$$(1 - \beta)V_0 = \pi\{\theta_H \max[\Delta_{HH} - q_H, 0] + (1 - \theta_H)\Omega_{H0}\}, \quad (11)$$

$$(1 - \beta)W_0 = \pi\{\theta_F \max[\Delta_{FF} - q_F, 0] + (1 - \theta_F)\Omega_{F0}\}. \quad (12)$$

Bankers and nonbankers with banknotes are assumed to make TIOLI offers to nonbankers who are producers without banknotes. These offers extract all of the surplus from the trade. Note that the agent making the offer takes  $V_j$  and  $W_j$  as given. These offers will be

$$q_H = \beta(V_H - V_0), \quad (13)$$

$$q_F = \beta(W_F - W_0), \quad (14)$$

$$x_j = \beta(V_j - V_0), \quad j \in \{H, F\}, \quad (15)$$

$$z_j = \beta(W_j - W_0), \quad j \in \{H, F\}. \quad (16)$$

Substituting (13)–(16) into (11) and (12) yields  $V_0 = W_0 = 0$ .

### 3.3. Steady-state banknote holdings

Because we are focusing only on steady-state equilibria, we also require that the distribution of banknote holdings remains the same from period to period. Listing the outflows from a given banknote holding on the left-hand side and the inflow on the right-hand side, the conditions on the distribution of banknote holdings that have to be satisfied for a steady-state equilibrium are as follows.

*For nonbankers without a banknote*

$$m_{i0} \left[ \theta_i + (1 - \theta_i)\pi P_j \sum_{k=H,F} m_{jk}\Lambda_{ji}^k \right] \quad (17)$$

$$= m_{ii}[\Gamma_i\theta_i + (1 - \theta_i)\pi P_j m_{j0}\Lambda_{ij}^i] + m_{ij}(1 - \theta_i)\pi P_j m_{j0}\Lambda_{ij}^i, \quad i, j \in \{H, F\}, \quad i \neq j. \quad (18)$$

*For nonbankers with a banknote of a bank in their location*

$$m_{ii}[\Gamma_i\theta_i + (1 - \theta_i)\pi P_j m_{j0}\Lambda_{ij}^i] = m_{i0}[\theta_i + (1 - \theta_i)\pi P_j m_{ji}\Lambda_{ji}^i], \quad i, j \in \{H, F\}, \quad i \neq j. \quad (19)$$

*For nonbankers with a banknote of a bank in the other location*

$$m_{ij}m_{j0}\Lambda_{ij}^j = m_{i0}m_{ji}\Lambda_{ji}^j, \quad i, j \in \{H, F\}, \quad i \neq j. \quad (20)$$

## 4. Equilibrium

We are now ready to define a symmetric steady-state equilibrium for this economy.

**Definition** (*Symmetric steady-state equilibrium, SSE*). Given  $Q_H$  and  $Q_F$ , an SSE is a set

$$\Phi = \{q_i, x_i, z_i, V_i, W_i, Z_i, m_{ji}, \gamma_i, \Gamma_i, \lambda_{jk}^i, \Lambda_{jk}^i, \forall i, j, k \in \{H, F\}\}$$

that satisfies maximization with TIOLI offers, the steady-state banknote holding equations,  $\lambda_{jk}^i = \Lambda_{jk}^i, \gamma_i = \Gamma_i, V_i \geq 0, W_i \geq 0, Z_i \geq 0, \forall i, j, k \in \{H, F\}$ .

**Definition** (*Nonmonetary SSE*). A nonmonetary SSE is the set

$$\Phi_N = \{\phi \in \Phi : \Lambda_{jk}^i = \Gamma_i = V_i = W_i = Z_i = 0, \forall i, j, k \in \{H, F\}\}.$$

That is, in a nonmonetary equilibrium there is no trade and banknotes are not valued.

**Proposition 1.** *A nonmonetary SSE exists for this economy.*

**Proof.** Substitution of  $\phi \in \Phi_N$  into (3)–(8) and (11)–(16) shows that all conditions of an SSE are satisfied and  $q_H = q_F = x_H = x_F = z_H = z_F = 0$ .  $\square$

**Definition** (*Monetary SSE*). A monetary SSE is a set

$$\Phi_M = \{\phi \in \Phi : \Gamma_i, \Lambda_{ii}^i, Z_i > 0 \text{ for } i \in \{H, F\} \text{ and } V_H, W_F > 0\}.$$

This definition of a monetary SSE is strong in that it requires that both types of banknotes are valued ( $V_H, W_F > 0$ ) and circulate ( $B_i, \Gamma_i > 0$ ). We adopt this definition because banknote discounts were quoted in terms of notes of local banknotes, which implies that more than one type of banknote was in circulation. Note, however, that our definition of a monetary SSE does not require interlocation circulation of banknotes. A question that we explore below is, for what parameter values there exist equilibria in which there is interlocation banknote circulation, that is, monetary equilibria in which either  $\Lambda_{ij}^i > 0$  or  $\Lambda_{ji}^i > 0$  for some  $i, j \in \{H, F\}, i \neq j$ .

**Lemma 1.** *In a monetary SSE with  $\Gamma_i = 1, m_{i0} = m_{ii}, i \in \{H, F\}$ .*

This result is proved in the Appendix.

**Lemma 2.** *In a monetary SSE,  $\Gamma_H(x_H - q_H) = \Gamma_F(z_F - q_F) = 0$ ; that is, banknotes circulate locally at par (at their issue value).*

This result follows from the assumption that bankers and nonbankers with notes get to make TIOLI offers to nonbankers without notes.

## 5. Par redemption equilibria

The first type of monetary SSE we examine is one in which the issue, redemption, and circulation values of banknotes are the same.

**Definition** (*Par redemption SSE*). A par redemption SSE is a set

$$\Phi_P = \{\phi \in \Phi_M : Q_i - q_i = 0, i \in \{H, F\}\}.$$

We can prove existence of a par redemption SSE for small values of  $\theta_H$  and  $\theta_F$ . In this case, notes of both home and foreign bankers trade in all single coincidence meetings.

**Theorem 1.** *Suppose  $u(\cdot)$  is homogeneous of degree  $n$ , with  $n \in (0, 1), n = \frac{m}{k}$ , where  $m$  and  $k$  are natural numbers.<sup>3</sup> Then for small values of  $\theta_H$  and  $\theta_F$ , a unique monetary SSE with  $\Lambda_{jk}^i = \Gamma_i = 1, \forall i, j, k$  exists.*

**Proof.** In the appendix.  $\square$

### 5.1. Numerical analysis

For larger values of  $\theta_H$  and  $\theta_F$ , we present the equilibria for a numerical example in which we assume that  $u(c) = c^\alpha, \alpha = \frac{1}{2}, \rho = 0.01, \pi = 0.5$ , and  $\delta = 0.9$ .<sup>4</sup> Initially, we set  $H/F = 1$ , so that there is the same measure of nonbankers in both locations. The properties of the par redemption SSE for this numerical example are shown in Fig. 4. Region 1 is the region to which the theorem applies. Both home and foreign consumers play pure strategies and exchange both home and

<sup>3</sup> To show existence of the proposed equilibrium, we need to show that the system of equations described by the Bellman equations has a solution and that such solution is incentive compatible. For the first task would be sufficient to assume concavity of the utility function (as shown in [4]). The additional assumption we require allows us to show in an easy way for what parameter values such solution is incentive compatible.

<sup>4</sup> We have also computed the SSE for various  $\alpha \in (0, 1)$ . The qualitative results are the same as those presented below.

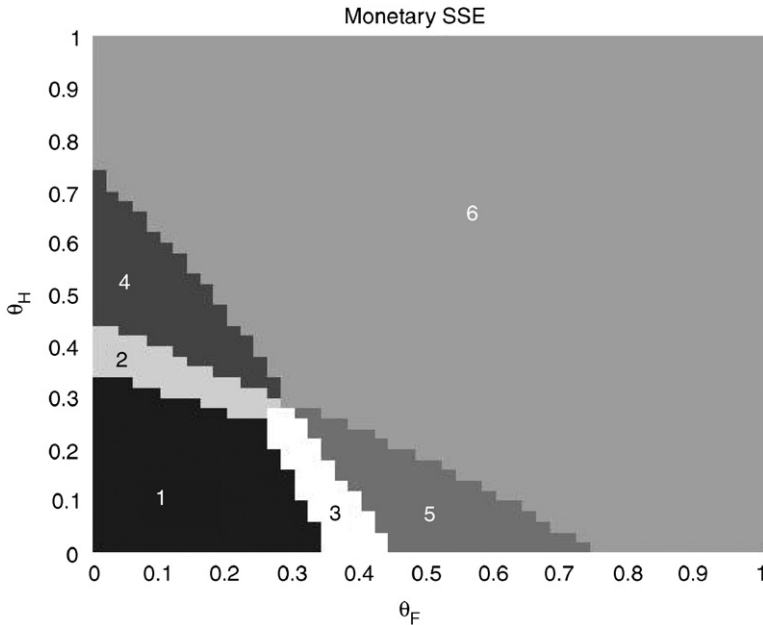


Fig. 4. Par redemption SSE.

foreign banknotes with nonlocal producers with probability 1 ( $\Lambda_{jk}^i = 1, \forall i, j, k$ ). (Recall that in a monetary equilibrium, consumers always trade local notes with local producers by definition.) In regions 2 and 3, one type of nonbanker continues to play a pure strategy while the other plays a mixed strategy in meetings with nonlocal producers when holding a local banknote. Specifically, in region 2, foreign consumers play a pure strategy and trade both types of notes with both types of producers with probability 1 ( $\Lambda_{Fj}^i = 1, \forall i, j$ ). However, home consumers holding a home note now play a mixed strategy in meetings with foreign producers ( $\Lambda_{HF}^H \in (0, 1)$ ), although they trade foreign notes with foreign producers with probability 1 ( $\Lambda_{HF}^F = 1$ ). In region 3, the behavior of the two types of consumers is reversed. In regions 4 and 5, both consumers again play pure strategies. However, now consumers from one location—home consumers in the case of region 4 and foreign consumers in the case of region 5—do not trade local notes with producers from the other location. That is,  $\Lambda_{HF}^H = 0$  in region 4 and  $\Lambda_{FH}^F = 0$  in region 5. Consumers from the other location continue to trade both types of notes with both types of producers with probability 1. Finally, in region 6, consumers play pure strategies, but trade only local notes and only with producers from their own location. That is, in region 6, the two locations are distinct; there is no interlocation trade.<sup>5</sup>

<sup>5</sup> We find that for each  $(\theta_H, \theta_F)$  a unique equilibrium exists in terms of where notes are traded (that is, only a single set of  $\Lambda_{jk}^i$  satisfies the incentive compatibility conditions). This raises the question of why we obtain unique note-trading equilibria for all parameter values whereas [7] obtain multiple equilibria in some cases. We think the reason is that Wright–Trejos permit producers to make a strategic decision about whether or not to enter into bargaining with consumers. We do not. We redid the numerical computations with the Wright–Trejos assumption and obtained multiple note-trading equilibrium for some regions of the parameter space. However, for small  $(\theta_H, \theta_F)$ , which is the case that we will consider below, there remains a unique equilibrium in which both notes were always traded with both home and foreign agents, although now this region is smaller than under our assumption.

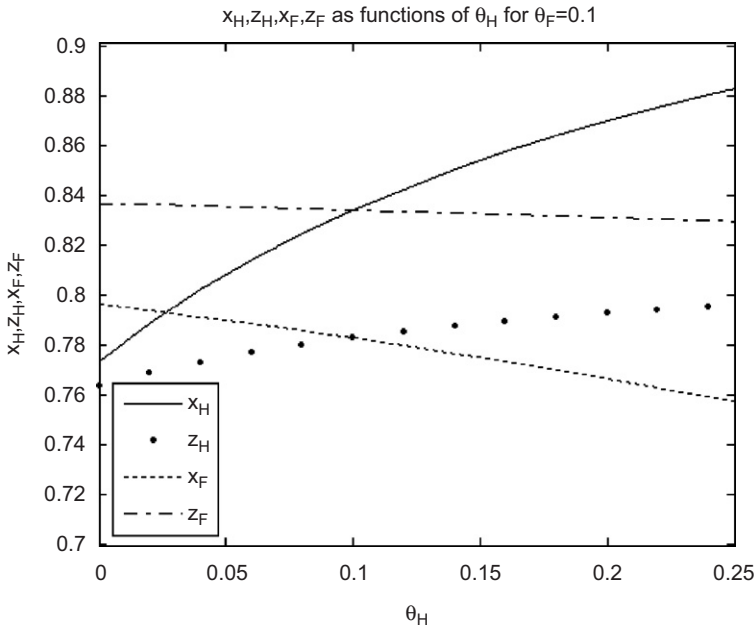


Fig. 5. Effects of changes in  $\theta_H$  on quantities of purchased goods.

Because we are interested in the discounts on banknotes, the case of interest to us is region 1 where banknotes trade in all meetings between nonbankers from both locations. Thus, from this point, we will restrict our attention to the region in which  $\theta_H$  and  $\theta_F$  are small. Because the probability of nonbanker–banker meetings is low in this case, focusing on this case is equivalent to making Cavalcanti and Wallace’s assumption that the measure of bankers in the economy is small. Because we are unable to obtain analytic comparative statics results, we continue with the numerical analysis.

The effects of changes in  $\theta_H$  on the quantities of the good that can be purchased with banknotes in the two locations are shown in Fig. 5. The quantity of goods that can be obtained with a home banknote from both home producers ( $x_H$ ) and foreign producers ( $z_H$ ) increases with  $\theta_H$ , and the quantity of goods that can be obtained with a foreign banknote from both home producers ( $x_F$ ) and foreign producers ( $z_F$ ) decreases with  $\theta_H$ . Because of the symmetry of the model, increases in  $\theta_F$  would have the opposite effects. To see the intuition behind these results, consider the case of a home nonbanker holding a home banknote. The larger  $\theta_H$ , the higher the probability of meeting a home banker and being able to redeem the note at par with certainty, as opposed to going to the nonbanker market where the nonbanker can exchange a banknote for goods only in matches with a nonbanker who happens to be a producer without a banknote. Thus, the larger  $\theta_H$ , the higher the expected value of home banknotes to a home nonbanker. Hence,  $x_H$  is larger, the larger  $\theta_H$ . Further, the higher  $x_H$  increases the expected payoffs to foreign holders of home banknotes, so that  $z_H$  increases as well.

The intuition for the response of the quantities of goods that are obtained in exchange for foreign notes to changes in  $\theta_H$  is similar. The larger  $\theta_H$ , the higher the probability that a home nonbanker holding a foreign note will not be able to trade because home nonbankers now have

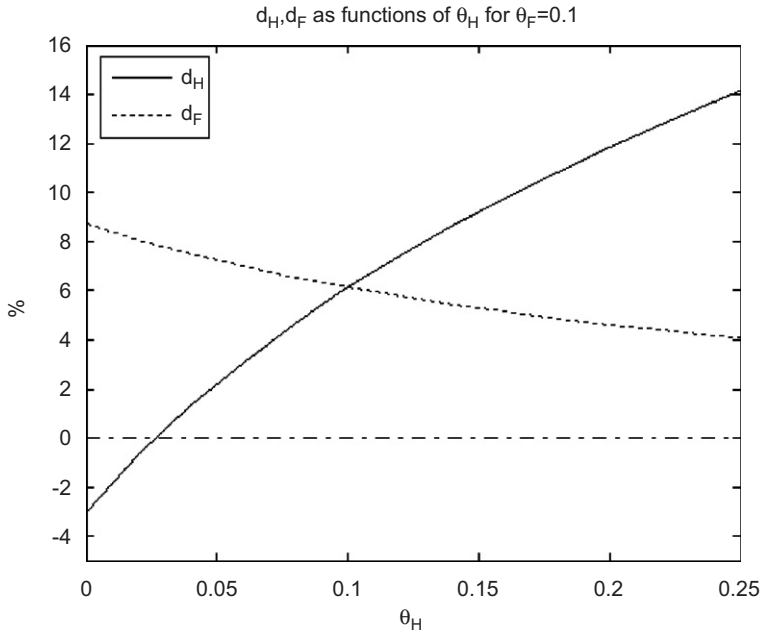


Fig. 6. Effects of changes in  $\theta_H$  on banknote discounts.

a higher probability of meeting a home banker who will refuse to redeem the note and a lower probability of meeting another nonbanker who might potentially trade. Thus, foreign banknotes become less valuable to home nonbankers, so that  $x_F$  is lower. In turn, the lower  $x_F$  reduces the payoff to foreign nonbankers of holding a foreign banknote, although the effect is partially offset by the fact that a larger  $\theta_H$  means that foreign nonbankers have a higher probability of meeting foreign producers.

The effects of changes in  $\theta_H$  on banknote discounts are shown in Fig. 6. The discount on foreign notes in the home market ( $d_H$ ) increases with  $\theta_H$ . This occurs because  $x_H$  is increasing in  $\theta_H$ , whereas  $x_F$  is decreasing as shown in Fig. 5. (Foreign notes are actually trading at a premium for low  $\theta_H$  because the figure is drawn for  $\theta_F = 0.1$ , and this difference between  $\theta_H$  and  $\theta_F$  makes foreign notes more valuable in the home market.) The discount on home notes in the foreign market ( $d_F$ ) is decreasing in  $\theta_H$ , however. This occurs because  $z_H$  is increasing in  $\theta_H$ , whereas  $z_F$  is decreasing. Because of symmetry, the opposite effects occur with increasing  $\theta_F$ . Note that because discounts depend on  $\theta_H$ , the figure shows that the predictions of the model are consistent with the fact that actual banknote discounts depended on the location of the quote. The results in Fig. 6 are roughly consistent with the data on banknote discounts. For banks in 30 cities, we obtain the discounts on their notes as quoted in Cincinnati, New York, and Philadelphia in November 1853, the number of banks in existence near that date, and the aggregate capital of these banks also near that date. We consider the number of banks and bank capital as proxies for  $\theta_H$  and find that their correlations with the discounts are between  $-0.13$  and  $-0.34$ , which is consistent with the fact that in our model  $d_F$  falls as  $\theta_H$  increases.

The figure also shows that the model's predictions are consistent with another fact about actual banknote discounts: these discounts were generally asymmetric, meaning that the discount on

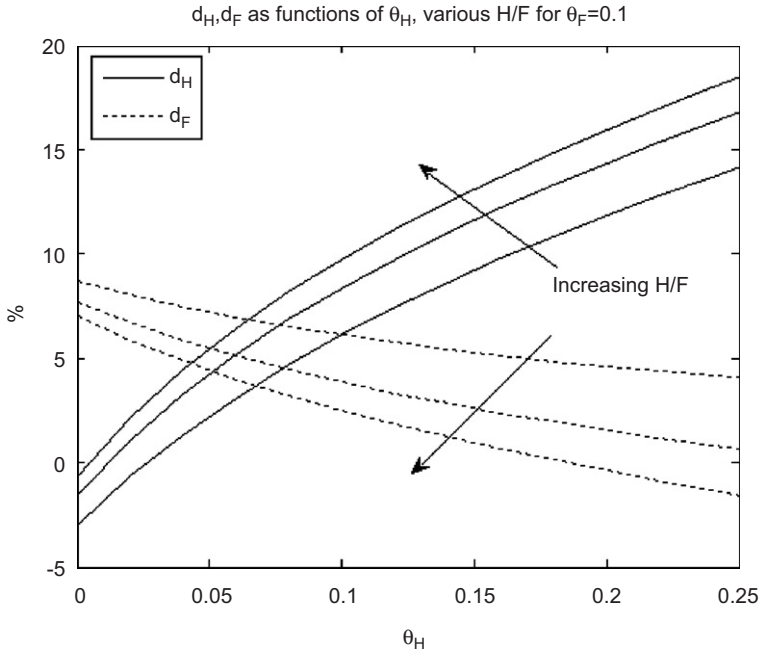


Fig. 7. Effects of increasing  $H/F$ .

foreign notes in the home market differed from the discounts on home notes in the foreign market. The figure shows that discounts on foreign notes in the home market can be either larger or smaller than discounts on home notes in the foreign market and that the difference  $d_H - d_F$  increases with  $\theta_H$ . In our example, discounts are equal only when  $\theta_H = \theta_F$ .

Next we determine the effects on  $d_H$  and  $d_F$  of changes in the relative measures of home and foreign nonbankers. Specifically, in Fig. 7, we show the effects of increasing  $H/F$ . The figure shows that as  $H/F$  increases, discounts on foreign notes in the home market increase and discounts on home notes in the foreign market increase. Increasing  $H/F$  increases the probability that the holder of a banknote will meet a producer from the home location rather than a producer from the foreign location in the nonbanker market. Because a home producer is willing to exchange more goods for a home banknote than for a foreign banknote, this increases the value of the home banknote relative to a foreign banknote. The discount on foreign banknotes in the home market increases. And because a foreign producer is willing to produce more goods for a foreign banknote than for a home banknote, the lower probability of meeting such a producer reduces the value of foreign banknotes relative to home banknotes. Because discounts in the model depend on  $H/F$ , we interpret the results in the figure as showing that the model's predictions are consistent with the fact that actual banknote discounts depended on the location of the bank. The results in Fig. 7 are also roughly consistent with the data on banknote discounts. For the same 30 cities that we used before, we use their population in 1850 as a proxy for  $H/F$ . We find the correlation between discounts on the notes of banks in these cities again as quoted in Cincinnati, New York, and Philadelphia with population to be roughly  $-0.3$ , which again is consistent with the fact that in our model  $d_F$  decreases as  $H/F$  increases.

### 6. Suspension equilibria

We now turn to the case in which bankers in at least one location have suspended note redemption in the sense that the redemption value of their notes is less than the issue value. We discuss the predictions of the model and whether these predictions are consistent with the data.

**Definition** (*Suspension SSE*). A suspension SSE is a set

$$\Phi_S = \{\phi \in \Phi_M : Q_i - q_i < 0, \text{ for some } i \in \{H, F\}\}.$$

We present results for the same numerical example as for the case of par redemption SSE. We model suspension by assuming that  $Q_i = (1 - k_i)q_i, k_i \in [0, \bar{k}_i], i \in \{H, F\}$ , where  $\bar{k}_i$  is that value such that  $u[(1 - \bar{k}_i)q_i] = q_i$ . That is,  $\bar{k}_i$  is the smallest redemption value of a banknote as a percentage of issue value that is consistent with the existence of a monetary SSE.

We first consider the case in which  $k_H = 0, k_F > 0$ , that is, the case in which home bankers redeem their notes at par, but foreign bankers have suspended (in our sense) note redemption. Fig. 8 shows that as  $k_F$  increases, discounts on foreign notes in the home market increase and discounts on home notes in the foreign market fall. Increasing  $k_F$  is similar to decreasing  $\theta_F$  because both make the option of going to a banker less valuable to the holder of a note. Thus, one can follow the reasoning for the case of changes in  $\theta_H$  to get the intuition for why changing  $k_F$  affects discounts as it does. These predictions are consistent with the data shown in Fig. 2 since, except for a short period from May to August 1837, New York banks were redeeming their notes while Philadelphia banks were suspended.

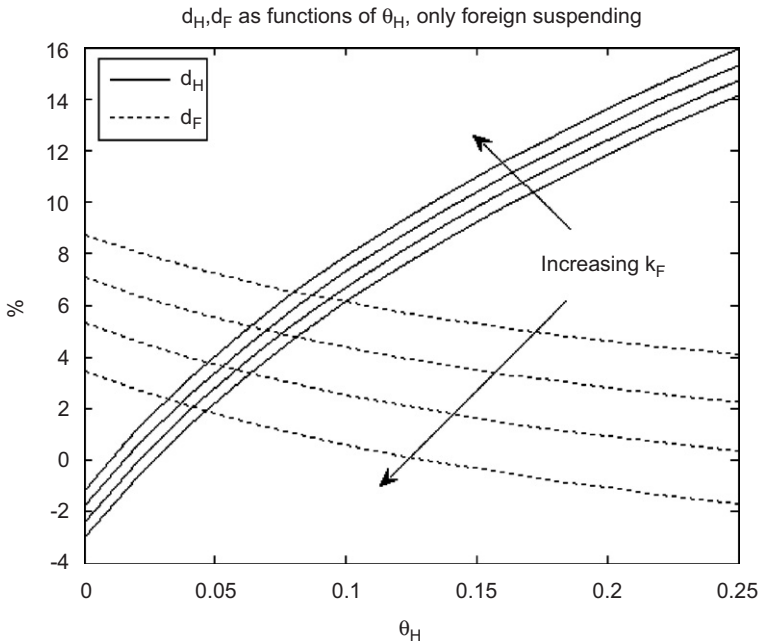


Fig. 8. Effects of increasing  $k_F$  on discounts on home and foreign notes.

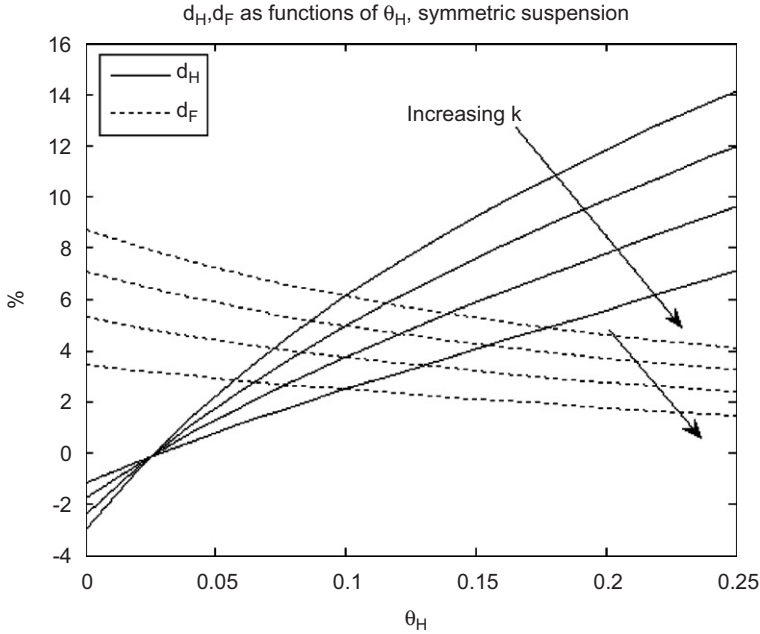


Fig. 9. Effects of increasing  $k$  on  $d_H$  and  $d_F$ .

Because virtually all banks in the United States were suspended from May to August 1837, and banks in Philadelphia and most of the southern and western part of the country were suspended from October 1839 to March 1842, we also examine the case in which all banks are suspended. Specifically, we examine the case in which  $k_H = k_F = k > 0$ ; that is, all banks have suspended by the same percentage. Fig. 9 shows that both  $d_H$  and  $d_F$  decline with  $k$ . The intuition for this result is that as  $k$  increases, the amount of goods that local nonbankers can get from a local banker decreases. This reduces the wedge between the values of local and nonlocal notes to a local nonbanker. The discount on nonlocal notes declines as a result. This prediction of the model is consistent with the data on the discounts on New York banknotes quoted in Philadelphia from May to August 1837, when banks in both New York and Philadelphia were suspended. However, it is inconsistent with the behavior of discounts quoted in both New York and Philadelphia on the notes of banks in all other locations when both that bank and New York or Philadelphia banks were also suspended. However, if we allow for asymmetric suspensions by New York and Philadelphia banks with respect to banks in other locations, then the predictions of the model would be consistent with the data if  $k_H - k_F$  were large enough.

Our model also predicts that the value of banknotes declines when bankers suspend payment. This is shown in Fig. 10 where we plot the quantity of goods that are exchanged for a home note ( $x_H$ ) when home bankers are suspended and when they are redeeming at par. Note that the difference increases with  $\theta_H$  because as  $\theta_H$  increases, the redemption option is a large fraction of the value of a banknote, and suspension reduces this option value. If we interpret the value of  $x_H$  when banks are redeeming at par as the value of specie, then this result can be interpreted as saying that banknotes go at a discount to specie when banks suspend. Such discounts are observed in the data and are the final fact we want to match.

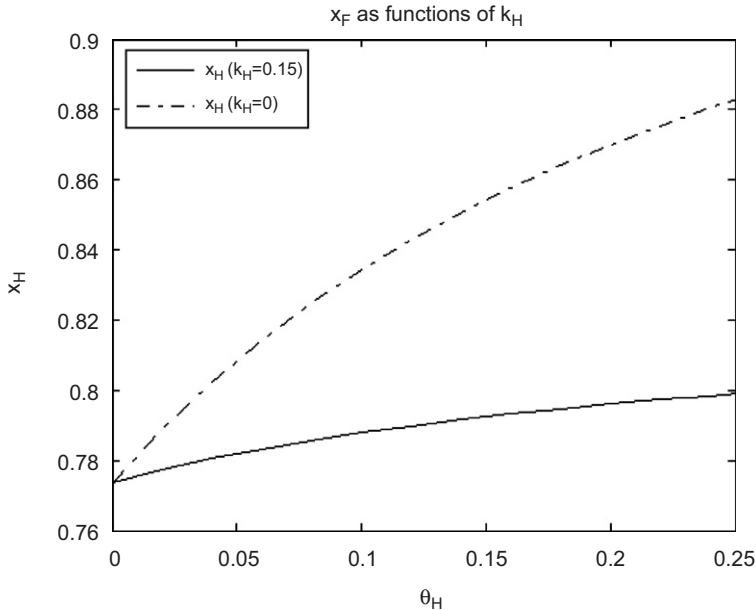


Fig. 10. Quantity of goods exchanged for home note ( $x_H$ ) when home bankers suspended or redeeming at par.

## 7. Conclusion

In this paper we have constructed a model to qualitatively match some facts about the discounts on the notes—dollar-denominated promises to pay specie to the bearer on demand—issued by state banks in the United States prior to 1863. The model we construct builds on the basic search model of money by [5] and [6]. To attempt to account for locational differences in banknote discounts, our model has nonbankers that come from two distinct locations. Each location also has bankers that can issue notes. We model the fact that banknotes had to be redeemed in specie on demand by adapting the [2] innovation that banks are required to produce when a banknote is presented for redemption and their past actions are public information. This implies that a nonbanker will be willing to accept a banknote when trading with a banker. Banknotes are also valued because they overcome the absence of double coincidence meetings in matches between two nonbankers.

We present two versions of the model. In the first, banks are threatened with permanent autarky if they ever fail to produce the same quantity of goods when their notes are presented for payment as they obtain when issuing a note. We refer to this case as par redemption. In the second, banks are allowed to suspend payments on their notes by redeeming them at an exogenously determined amount below their issue value.

The specific goal of the paper was to qualitatively match four facts about the discounts on banknotes. One fact is that notes of local banks were always quoted at par. Our model is consistent with this fact, but trivially because we allow bankers and nonbankers with notes to make TIOLI offers. A second fact is that foreign banknotes usually were quoted at a discount to local banknotes, and this discount varied by the location of the bank and by where the discount was being quoted. Our model is consistent with this fact. It is also consistent with a third fact: discounts were

asymmetric across locations, meaning that the discounts quoted in location A on the notes of banks in location B generally differed from the discounts quoted in location B on the notes of banks in location A. The final fact is that the discounts on foreign notes increased when banks had suspended payment on their notes. Here the model is consistent with the data only under the assumption that local bankers have suspended to a much lesser degree than have nonlocal bankers. Otherwise, the model predicts that discounts will fall, not increase.

Thus, we think the overall performance of the model is quite good. It delivers predictions consistent with three of the facts we wanted to explain and consistent with the fourth fact in some special cases. Further, we examined the relationship between actual banknote discounts and the number of banks, bank capital, and population, variables that are reasonable proxies for parameters in our model. We find that in all cases, the correlations between discounts and these variables was negative, as our model predicts.

## Acknowledgments

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## Appendix

**Proof of Lemma 1.** A stationary distribution of banknote holdings must satisfy the following equations:

$$\begin{aligned} m_{H0}[\theta_H + (1 - \theta_H)\pi P_F(m_{FH}\Lambda_{FH}^H + m_{FF}\Lambda_{FH}^F)] \\ = m_{HH}[\Gamma_H\theta_H + (1 - \theta_H)\pi P_F m_{F0}\Lambda_{HF}^H] + (1 - \theta_H)m_{HF}\pi P_F m_{F0}\Lambda_{HF}^F, \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} m_{HH}[\Gamma_H\theta_H + (1 - \theta_H)\pi P_F m_{F0}\Lambda_{HF}^H] \\ = m_{H0}[\theta_H + (1 - \theta_H)\pi P_F m_{FH}\Lambda_{FH}^H], \end{aligned} \quad (\text{A.2})$$

$$m_{HF}m_{F0}\Lambda_{HF}^F = m_{H0}m_{FF}\Lambda_{FH}^F, \quad (\text{A.3})$$

$$\begin{aligned} m_{F0}[\theta_F + (1 - \theta_F)\pi P_H(m_{HF}\Lambda_{HF}^F + m_{HH}\Lambda_{HF}^H)] \\ = m_{FF}[\Gamma_F\theta_F + (1 - \theta_F)\pi P_H m_{H0}\Lambda_{FH}^F] + (1 - \theta_F)m_{FH}\pi P_H m_{H0}\Lambda_{FH}^H, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} m_{FF}[\Gamma_F\theta_F + (1 - \theta_F)\pi P_H m_{H0}\Lambda_{FH}^F] \\ = m_{F0}[\theta_F + (1 - \theta_F)\pi P_H m_{HF}\Lambda_{HF}^F], \end{aligned} \quad (\text{A.5})$$

$$m_{FH}m_{H0}\Lambda_{FH}^H = m_{F0}m_{HH}\Lambda_{HF}^H, \quad (\text{A.6})$$

$$m_{H0} + m_{HF} + m_{HH} = 1, \quad (\text{A.7})$$

$$m_{F0} + m_{FF} + m_{FH} = 1. \quad (\text{A.8})$$

Eqs. (A.1) and (A.4) are redundant. Substituting (A.3) into (A.2) and (A.6) into (A.5), we obtain

$$m_{H0} = m_{HH},$$

$$m_{F0} = m_{FF}. \quad \square$$

**Proof of Theorem 1.** The strategy is to guess that  $\Lambda_{HF}^H = \Lambda_{FH}^F = \Lambda_{FH}^H = \Lambda_{HF}^F = 1$  are equilibrium strategies and then to verify under what conditions all the equilibrium conditions are satisfied. Note that this guess implies that under a par redemption equilibrium,  $\Gamma_H = \Gamma_F = 1$ .

**Lemma 3.** *If  $\Lambda_{HF}^H = \Lambda_{FH}^F = \Lambda_{FH}^H = \Lambda_{HF}^F = 1$ , then stationary note holdings are equal to  $\frac{1}{3}$ .*

**Proof.** From Lemma 1 we have

$$m_{F0} = m_{FF}, \tag{A.9}$$

$$m_{H0} = m_{HH}. \tag{A.10}$$

Combining (A.9) with (A.3) yields

$$m_{HF} = m_{H0} = m_{HH}. \tag{A.11}$$

Combining (A.10) with (A.6) yields

$$m_{FH} = m_{F0} = m_{FF}. \tag{A.12}$$

Now combining (A.11) and (A.12) with (A.7) and (A.8), it follows that  $m_{ij} = \frac{1}{3}, \forall i, j \in \{0, H, F\}$ . □

We will focus only on the home market; for the foreign market the same arguments apply.

**Lemma 4.** *The quantities  $x_H$  and  $z_H$  are part of a monetary SSE and  $\Lambda_{HF}^H = \Lambda_{FH}^F = \Lambda_{FH}^H = \Lambda_{HF}^F = 1$  if they solve the following two equations and four incentive constraints.*

$$\rho x_H = \pi \theta_H [u(x_H) - x_H] + (1 - \theta_H) \frac{\pi^2}{3} \{P_H [u(x_H) - x_H] + P_F [u(z_H) - x_H]\}, \tag{A.13}$$

$$\rho z_H = (1 - \theta_F) \frac{\pi^2}{3} \{P_H [u(x_H) - z_H] + P_F [u(z_H) - z_H]\}, \tag{A.14}$$

$$u(z_H) \geq x_H, \tag{A.15}$$

$$u(x_H) \geq z_H, \tag{A.16}$$

$$u(z_H) \geq z_H, \tag{A.17}$$

$$u(x_H) \geq x_H. \tag{A.18}$$

To show existence of equilibrium, we proceed in two steps. First we establish that a solution to (A.13) and (A.14) exists and is unique. Then we show that for small values of  $\theta_H$  and  $\theta_F$ , such a solution is incentive compatible.

**Lemma 5.** Suppose  $u(\cdot)$  is homogeneous of degree  $n$ , with  $n \in (0, 1)$ ,  $n = \frac{m}{k}$ , where  $m$  and  $k$  are natural numbers. For any value of  $\theta_H$  and  $\theta_F$  in  $[0, 1)$ , there exists a unique solution  $(x_H, z_H)$  to Eqs. (A.13) and (A.14).

**Proof.** Rewrite (A.13) as

$$\left\{ \rho + \pi\theta_H + (1 - \theta_H)\frac{\pi^2}{3}\delta \right\} x_H = \left\{ \pi\theta_H + \frac{\pi^2}{3}(1 - \theta_H)P_H \right\} u(x_H) + \frac{\pi^2}{3}(1 - \theta_H)P_F u(z_H) \tag{A.19}$$

and (A.14) as

$$\left\{ \rho + \frac{\pi^2}{3}\delta(1 - \theta_F) \right\} z_H = \frac{\pi^2}{3}(1 - \theta_F)P_H u(x_H) + \frac{\pi^2}{3}(1 - \theta_F)P_F u(z_H). \tag{A.20}$$

The above system can be written as

$$Ax = Bu(x) + Cu(z), \tag{A.21}$$

$$Dz = Eu(x) + Fu(z) \tag{A.22}$$

with all six coefficients positive. Solving for  $(x, z)$  is equivalent to solving for  $(x, \alpha)$  with  $\alpha \in \mathbb{R}$ , defined as  $z = \alpha x$ . We are looking for a pair  $(x, z)$  strictly positive, so we exclude the case  $\alpha = 0$ . Substituting, we then have

$$Ax = Bu(x) + C\alpha^n u(x), \tag{A.23}$$

$$D\alpha x = Eu(x) + F\alpha^n u(x), \tag{A.24}$$

that is,

$$x = \frac{B + C\alpha^n}{A} u(x), \tag{A.25}$$

$$x = \frac{E + F\alpha^n}{D\alpha} u(x). \tag{A.26}$$

Given  $x \neq 0$ , the system becomes

$$x = \Gamma u(x), \tag{A.27}$$

$$\frac{B + C\alpha^n}{A} = \frac{E + F\alpha^n}{D\alpha} = \Gamma. \tag{A.28}$$

We know that (A.27) has a solution for any  $\Gamma > 0$ . We look at Eq. (A.28) and rewrite it as

$$CD\alpha^{n+1} + BD\alpha - AF\alpha^n - AE = 0. \tag{A.29}$$

Since this is a continuous function of  $\alpha$ ,  $p(\alpha)$ , note that  $p(0) = -AE < 0$  (so that  $\alpha = 0$  is not a solution). Also, since  $CD > 0$ ,  $\lim_{\alpha \rightarrow \infty} p(\alpha) = \infty$ . So a positive solution must exist. Moreover, since  $n$  is a rational number, we can change variables, setting  $\hat{\alpha} = \alpha^{\frac{1}{k}}$ , and (A.29) becomes

$$CD\hat{\alpha}^{m+k} + BD\hat{\alpha}^k - AF\hat{\alpha}^m - AE = 0. \tag{A.30}$$

The above is now a polynomial of degree  $m + k$  with the same zeros as (A.29). Also (A.30) has only one sign change, so by Descartes' sign rule, there can be only one positive solution.  $\square$

Denoting the solution of (A.13) and (A.14) by  $(\hat{x}(\theta), \hat{z}(\theta))$  with  $\theta = (\theta_H, \theta_F)$ , we now need to show for what parameter values such a solution is incentive compatible.

**Lemma 6.** *If  $\theta_H = \theta_F = 0$ , a monetary SSE exists and  $\hat{x}(0) = \hat{z}(0) = \tilde{x}$ , where*

$$\left(\rho + \frac{\pi^2 \delta}{3}\right) \tilde{x} = \frac{\pi^2 \delta}{3} u(\tilde{x}). \tag{A.31}$$

**Proof.** When  $\theta_H = \theta_F = 0$ , the right-hand sides of (A.19) and (A.20) are equal. Given that  $P_H + P_F = \delta$ , then  $\hat{x} = \hat{z} = \tilde{x}$ . The existence of  $\tilde{x} > 0$  can easily be proved by noting that the left-hand side and right-hand side of (A.31) are continuous, are equal at zero, and that  $u$  satisfies the Inada conditions. This is a standard argument in Trejos–Wright models.  $\square$

**Definition.** Given the utility function  $u(\cdot)$ , define  $\tilde{u} \in \mathbb{R}$  as the unique value such that  $u(\tilde{u}) = \tilde{u}$ .

Note that by (A.31), since  $\left(\rho + \frac{\pi^2 \delta}{3}\right) / \frac{\pi^2 \delta}{3} > 1$ ,  $\tilde{x} < \tilde{u}$ . Define  $\tilde{w} = u(\tilde{x}) - \tilde{x} > 0$ . The strategy to show that  $(\hat{x}(\theta), \hat{z}(\theta))$  are incentive compatible is as follows. We show that for small values of  $\theta$ ,  $\hat{x}$  and  $\hat{z}$  are arbitrarily close to each other and arbitrarily close to  $\tilde{x}$ , so that both are less than  $\tilde{u}$ . Finally, we show that arbitrarily close  $\hat{x}$  and  $\hat{z}$ , together with being less than  $\tilde{u}$ , implies that the incentive constraints are satisfied.

The coefficients  $A, B, C, D, E, F$  are continuous in  $\theta$ . The solution for  $\alpha$  is the unique positive zero of (A.30), and since the zero of a polynomial changes continuously with its coefficients,<sup>6</sup> we conclude that  $\hat{\alpha}$  changes continuously with  $\theta$ . The solution  $\hat{x}$  to (A.27) is given by

$$\hat{x} = (u(1)\Gamma)^{\frac{1}{1-n}}, \tag{A.32}$$

which depends continuously on  $\Gamma$ , so  $\hat{x}$  depends continuously on  $\theta$ . There exists a unique solution for every  $\theta \in [0, 1) \times [0, 1)$ . We have that as  $\theta \rightarrow 0$ , (A.30) converges to

$$\hat{\alpha}^{m+k} + \hat{\alpha}^k - \hat{\alpha}^m - 1 = 0, \tag{A.33}$$

so that  $\hat{\alpha} = 1$  at  $\theta = 0$ , and for  $\theta = 0$ ,  $\hat{x} = \tilde{x}$ . We then conclude that  $(\hat{x}, \hat{\alpha}) \rightarrow (\tilde{x}, 1)$  as  $\theta \rightarrow 0$ , or in other words,  $(\hat{x}, \hat{z}) \rightarrow (\tilde{x}, \tilde{x})$ .

Define the distance in  $\mathbb{R}^2$  between  $\theta$  and 0 as  $d(\theta, 0) = \|(\theta_H, \theta_F) - (0, 0)\|$ .

**Lemma 7.** *Let  $\hat{x}$  and  $\hat{z}$  be a solution to (A.13) and (A.14). Then for small  $d(\theta, 0)$ ,  $\hat{x}$  and  $\hat{z}$  satisfy the incentive constraints (A.15) to (A.18).*

**Proof.** Given  $\tilde{x} < \tilde{u}$ , and given  $(\hat{x}, \hat{z}) \rightarrow (\tilde{x}, \tilde{x})$  for small values of  $d$ , we have that  $\hat{x} < \tilde{u}$  and  $\tilde{z} < \tilde{u}$  so that (A.17) and (A.18) hold. As before, let  $\hat{z} = \hat{\alpha}\hat{x}$  so that we rewrite (A.15) and (A.16) as

$$u(\hat{x}) \geq \frac{1}{\hat{\alpha}^n} \hat{x}, \quad u(\hat{x}) \geq \hat{\alpha}\hat{x}. \tag{A.34}$$

But given  $(\hat{x}, \hat{\alpha}) \rightarrow (\tilde{x}, 1)$  as  $d \rightarrow 0$ , the result follows for small enough  $d$ , given  $\tilde{x} < \tilde{u}$ .  $\square$

Combining the previous lemmas, we prove the theorem.

<sup>6</sup> A statement of this theorem can be found in [1].

## References

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