Non-exclusive Dynamic Contracts, Competition, and the Limits of Insurance *

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Abstract

In this paper, we study how the presence of non-exclusive contracts limits the amount of insurance provided in a decentralized economy. We consider a dynamic Mirrleesian economy in which agents are privately informed about idiosyncratic labor productivity shocks. Agents sign privately observable insurance contracts with multiple firms (i.e., they are non-exclusive). Contracts include both labor supply and savings aspects. Firms have no restriction on the contracts they can offer and interact strategically. In equilibrium, contrary to the case with exclusive contracts, a standard Euler equation holds and the marginal rate of substitution between consumption and leisure is equated to the worker's marginal productivity. Also, each agent receives zero net present value of transfers. To sustain this equilibrium, more than one firm must be active and must also offer latent contracts to deter deviations to more profitable contingent contracts. In this environment, the non-observability of contracts removes the possibility of additional insurance beyond self-insurance.

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1 Introduction

What type of contractual arrangements are available to workers in a decentralized economy when firms compete for the provision of social insurance? In this paper, we study how, in a decentralized economy, the presence of non-exclusive contracts endogenously limits the contracts offered and hence the amount of insurance. We find that competition and non-exclusivity of insurance contracts significantly reduce the amount of insurance provided: the equilibrium allocation in our environment is equivalent to a self-insurance economy, and only linear contracts are offered.

Multiple credit and labor relations are an important aspect of everyday life. Survey data shows that individuals and households receive insurance against idiosyncratic risk from a multitude of sources: publicly provided insurance (unemployment, Medicare, Medicaid, disability, food stamps, progressive income taxation), privately provided insurance (employer, between and within family transfers),[1] financial instruments in credit markets, and housing and other large durable goods. The same consideration is true for labor relationships. Paxson and Sicherman (1994) look at the number of concurrent labor relationships held by survey respondents of the Panel Study of Income Dynamics (PSID) between 1977 and 1990 and the Current Population Survey (CPS) of 1991. They find that for any given year, 20% of working males held at least a second job, and during their working life there is at least a 50% probability of holding a second job. However, monitoring all the transactions an agent might engage in with other firms is very costly for an individual firm, especially if these relationships include activities in the informal labor market, private savings, and the ability to transfer leisure into consumption through either home production or shopping time (see Aguiar and Hurst (2005)). Motivated by these considerations, the key friction addressed in this paper is the *non-exclusivity* and *non-observability* of contractual relations. In this paper, we characterize the optimal contract under the assumption that *none* of the labor

[1] The Panel Study of Income Dynamics reports a measure of income transfer received by households for the years 1969 to 1985. We find that, in a given year, 24% of the households report receiving a transfer and 67% of the households received a transfer at some stage. These transfers are significant, averaging $1,930 (1983 dollars) and represent between 70% to 90% of total food expenditures.

and credit relations an agent engages in can be observed by an individual firm in an economy where the agent's productivity is privately known by the worker.[2] We interpret this friction as reflecting both the costs that a firm might incur when monitoring the transactions agents engage in and the inability of firms to offer contracts contingent on the agents' actions with other firms in the economy.

The environment studied is a finite horizon dynamic Mirrleesian economy in which agents are privately informed about idiosyncratic labor productivity shocks that evolve over time. Agents wish to insure this risk by signing contracts with insurance providers (firms). Agents are not limited to a single insurance/labor relationship and can sign contracts with multiple firms. The contracting arrangements are private information of the contracting parties. Given this friction, in general, the communication between agent and firms cannot be limited to the exogenous private shock of agents as in the case of observable contracts. Firms might also seek information about the other relations the agent has engaged in. To accommodate for this need, we extend the results in the common agency literature (see Peters (2001), Martimort and Stole (2002), and Epstein and Peters (1999)) to our dynamic environment and characterize equilibrium using a menu game. In this game, each firm offers collections of payoff relevant alternatives – menus – and delegates to the agent the choice within these menus. The choice of the agent from a menu can reveal information about his type and the other contractual arrangements in which he might be involved. We impose no restriction on the contracts that firms can offer. A firm can, for example, offer a spot labor contract, a linear inter-temporal borrowing and saving contract, a state contingent dynamic insurance contract, and so on. Hence any contract offered –either on or off-equilibrium– is determined as a result of the strategic competition among firms. This highlights a key feature of this environment; any side-trading opportunity agents have access to arises endogenously.

Our main result is that the non-exclusivity of contracts removes the possibility of additional insurance beyond self-insurance, with only linear contracts arising in equilibrium. We show this result in two steps; we characterize the conditions an equilibrium must satisfy and

[2] The characterization under exclusive contracts is well understood. See Prescott and Townsend (1984).

prove that an equilibrium exists.

The equilibrium allocation must satisfy three optimality conditions. First, the inter-temporal marginal rate of substitution between consumption at time t and consumption at $t+1$ is equal to the marginal rate of transformation (a standard Euler equation holds).[3] Second, the marginal rate of substitution between consumption and leisure is equated to the marginal productivity for any time and any history.[4] Third, the net present value of the transfers received in equilibrium is equal to zero for every agent in the economy. These optimality conditions imply that the unique equilibrium allocation is equivalent to an economy in which agents can trade non-contingent bonds and are paid their marginal productivity and in which there is no redistribution. The intuition for this result is the following. In our environment the constrained efficient provision of insurance is provided both intra-temporally with wages and inter-temporally with non linear returns on savings. The provision of insurance, due to private information on types, implies that some types receive a wage below marginal product or a return on borrowing and saving different than the marginal rate of transformation. This introduces profitable side trades opportunities for either labor services or credit. We show that the availability of side trades are sufficient to remove any possibility for insurance. Is worth mentioning that even for a good such as labor that cannot be freely traded (a worker cannot work a negative amount with a given firm) the opportunity to work on the side is sufficient to remove any non-linearity in wages.

Finally we show equilibrium exists. This is a key step since, as noted by Myerson (1982), the existence of equilibria with multiple principals is not always guaranteed. We show that to sustain the unique equilibrium allocation two ingredients are necessary. First, more than one firm (the incumbents) must be active in equilibrium. Second, incumbent firms must offer contracts that will not be chosen in equilibrium: latent contracts. These contracts

[3] If contracts are exclusive, the Euler equation does not hold and agents are savings constrained (see Rogerson (1985) and Golosov, Kocherlakota, and Tsyvinski (2003)).

[4] This is also different with respect to the exclusive contracting environment (see, for example, Mirrlees (1971) and Golosov, Tsyvinski, and Werning (2006)), where this relation holds only for the highest skill type, while all of the remaining types face a distortion on the intratemporal margin that discourages consumption and hours provided.

have the specific role of deterring deviations of other entrants.[5] To see this, suppose the incumbent were to offer the equilibrium allocation without any additional latent contract. Since the allocation is the most profitable non-redistributive contract, any entrant must offer a contract that features some redistribution. The worker will accept the entrant's contract. However, if the incumbent offers a latent contract that allows agent to perform a side trade, the contingent contract offered by the entrant can be made unprofitable.

Related Literature

The results, linking side trading and linear contracts, are reminiscent of Allen (1985), Hammond (1987), Cole and Kocherlakota (2001).[6] We contribute to this literature by explicitly modeling the non-cooperative competition between firms, determining endogenously the market structure and showing that an equilibrium exists. Specifically, Hammond (1987) characterizes, in a static exchange economy, the constrained efficient allocation which is robust to re-trading among agents. A linear price for goods emerges in equilibrium. Some of the differences with respect to this paper is the equilibrium concept adopted and that in our environment some of the trades agents can make are one-sided; agents can sell leisure to firms but cannot buy it.

This paper is related to the literature on optimal social insurance contracts and its implementation through taxation, commonly referred to as *new dynamic public finance*.[7] In general, the environment studied in these papers assumes that insurance is provided by a unique provider –the government– who perfectly controls both consumption and labor decision of the agents. With respect to this literature, this paper has two distinct implications. Our main result suggests that the constrained efficient allocation cannot be implemented in decentralized environments unless every aspect of the contract is observable, thus mak-

[5] In our environment, restricting to direct mechanisms, while not restrictive in an environment with exclusive contracts, results in non-existence of equilibrium.

[6] For linearity in the context of price discrimination, refer to Armstrong and Vickers (2010) and references therein.

[7] For a review, refer to Kocherlakota (2010) and Albanesi (2008).

ing necessary the provision of insurance via taxes or a centralized institution that makes information public. However, our results also highlight that the presence of hidden and self-enforcing activities (for both consumption and labor) might undo any incentives the government provides through taxes. Related to this last point, our work is also related to the literature on optimal contracts in the presence of hidden trades.[8] In particular, Cole and Kocherlakota (2001) show that, in a private information endowment economy, equilibrium is equivalent to self-insurance when agents can secretly save in a storage technology. In an environment similar to ours, Golosov and Tsyvinski (2007) characterize equilibrium when agents can engage in hidden trades of Arrow-Debreu securities. They show that a standard Euler equation holds and that the decentralized equilibrium is not efficient, since firms do not internalize the effects of the contracts offered on the market rate of return. This paper can be seen as a generalization of the previous two papers, in the sense that, in those, the re-contracting possibilities are assumed exogenously (a market with linear prices or a storage technology) while in this paper the re-contracting market is a result of an equilibrium game between insurance providers.

This paper also relates to Bisin and Guaitoli (2004), who analyze a static moral hazard environment under non-exclusive contracting. Their main result shows that latent contracts are used to sustain the equilibrium. However, the nature of the moral hazard environment, differently from our environment, enables latent contracts to prevent any profitable entry by additional insurance providers, thus delivering a positive profit equilibrium to the incumbents. Also, Ales and Maziero (2009) study a static adverse selection environment with non-exclusive contracts.

The paper is organized as follows. In Section 2, we describe the environment and show that any equilibrium can be implemented by a menu game. Section 3 characterizes the equilibrium of our benchmark environment and shows that it is equivalent to self-insurance. In section 4 we show that an equilibrium exists and also show that latent contracts are

[8] For example, Cole and Kocherlakota (2001), Golosov and Tsyvinski (2007), and Abraham and Pavoni (2005).

necessary to implement the equilibrium allocation. Section 5 is the conclusion.

2 Environment

Consider an economy populated by a continuum of measure one of ex ante identical agents and I firms (insurance providers), with $I \geq 2$. The economy lasts for a finite number T of periods. Agents' period utility is defined over consumption c and labor l and is given by $u(c) - v(l)$. Agents discount future utility at rate $0 < \beta < 1$. Assume $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, increasing and strictly concave function, $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$; and $v : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, increasing and strictly convex function, $\lim_{l \to 0} v'(l) = 0$ and $\lim_{l \to \infty} v'(l) = \infty$. At every time $t = 1, 2, ..., T$, each agent draws a privately observed productivity shock $\theta_t \in \Theta$, where Θ is a finite set and its smallest element is strictly positive. We assume the law of large numbers holds. The shock is distributed according to probability distribution $\pi(\cdot)$ and is independent and identically distributed over time and across agents. Let $\theta^t = (\theta_1, ..., \theta_t)$ denote the history of uncertainty of an agent up to time t. Given a sequence of consumption and labor $\{c, l\} = \{c_t, l_t\}_{t=1}^T$, the expected discounted utility of an agent is given by

$$U(\{c, l\}) = \mathbb{E}_0 \sum_{t=1}^{T} \beta^{t-1} [u(c_t) - v(l_t)]. \tag{1}$$

For a given realization of the labor productivity shock θ, an agent providing l units of labor to a firm can produce y units of output according to $y = \theta l$. Agents can contracts with firms for employment and insurance against the productivity shocks.

A novel feature of our environment is that agents can sign contracts simultaneously with more than one firm: contracts are non-exclusive. Each firm $i \in \{1, ..., I\}$ can offer a contract that prescribes, at every time t, output requirement y_t^i and consumption transfer $y_t^i + b_t^i$, where b_t^i denotes the net transfer from the firm and can be negative. The period profit of firm i is given by the net-transfers to workers, denoted by $V^i(b^i) = -b^i$. Firms can transfer resources over time at constant rate q.

Following the Mirrleesian tradition, we assume the productivity and labor input are private information of the agent and each firm i observes only output produced y_t^i. In addition, in this paper, the terms of the contract between an agent and a firm i are only observed by the parts involved and are not observed by other firms.

We do not impose any restriction on the contracts offered by each firm. For example, a firm can offer a contract for the entire time horizon $t = 1, ..., T$; for a particular set of dates; only credit contracts ($y_t = 0,\ \forall t$); only labor contracts, or both. We also do not impose any specific structure on the contracts; in particular, we do not restrict to linear contracts.

At time 0, before any uncertainty is realized, agents sign a contract with each firm i. To take into account the voluntary participation of agents, every firm is required to offer at time 0 a null contract that determines no output requirement and no consumption transfers in every period. The contracts offered by a firm at time 0 are contingent on the future communication between that firm and the agent. We assume that contracts must be honored and neither firms nor agents can renege on them.[9]

2.1 Communication and Menu Games

The presence of private information implies that communication between firms and agents are necessary in our environment.[10] If contracts are exclusive, the environment is equivalent to a standard dynamic Mirrleesian environment as in Golosov, Kocherlakota, and Tsyvinski (2003) and Albanesi and Sleet (2006). In this case the revelation principle shows that the communication between agents and firms can without loss of generality be restricted to agents reporting their labor productivity to firms – a direct mechanism. Under non-exclusive

[9]Enforcement of contracts could be explicitly introduced in the following way. Both agents and firms have access to an enforcement mechanism ("a court") upon the payment of a cost. If this cost is paid, the terms of the contract between the two parties in consideration become public, and this court can enforce a punishment to the party that reneged on the contract. If either firms or agents falsely report a breach of the contracts, they can also be punished by court. We assume this punishment can be made large enough so that in equilibrium neither firms nor agents will renege on the contracts signed.

[10]We do not allow communication between firms. We conjecture that this is not restrictive due to the one shot and non cooperative nature of the competition between firms.

contracts restricting the firm to communicate via a direct mechanism may not be sufficient to separate agents with different labor productivities. This is because the preference ordering of agents over allocations is influenced not only by their exogenous productivities, but also by the set of contracts they have access to from other firms (for example the willingness to work an additional hour might be different depending if an agent is already working 20 hours or not with a separate firm). Since firms need to elicit from the agent what additional contracts he has access to and has accepted, limiting the communication to the agents' productivity may not be without loss of generality.

Menu Games

Expanding the message space beyond the type space provides a significant challenge for characterization. To resolve this issue we characterize the environment defining a *menu game* between firms. The key idea is that any communication in the original communication mechanism can be replaced by firms offering menus of payoff-relevant alternatives and delegating to the agents the choice within this menu.

We extend the *delegation principle* proved by Peters (2001) and Martimort and Stole (2002) to our environment.[11] This principle states that, without loss of generality, the equilibrium outcomes of any communication game can be implemented as an equilibrium of a menu game. In appendix A we formally show how to construct menus from a general message space and prove the delegation principle for our environment. This result allows us to focus on menu games, which we now describe.

At time 0, before any uncertainty is realized, each firm i offers a collection of menus $\mathcal{S}^i$.[12] Each agent chooses a menu $C^i \in \mathcal{S}^i$ from each firm i. The set C^i contains the life-time history dependent allocation of net-tranfers and output.

Formally the timing of the game is as follows:

[11] Our environment differs from the previous literature along two dimensions. First, the environment is dynamic in the sense that the exogenous uncertainty is realized in every period. Second, agents choose a communication-contingent contract from each firm i *before* any uncertainty is realized. This is important since at time 0, agents are identical. Thus it might be possible to extract more information about the contracts being offered by other firms.

[12] We do not allow for random menus, this is not restrictive given the nature of preferences.

- At time 0:

  1. Each firm i simultaneously offers menu $\mathcal{S}^i$;
  2. For all $i \in I$, agents choose $C^i \in \mathcal{S}^i$;

- At time t:

  1. Agent learns his private type θ_t;
  2. For all $i \in I$, agents choose $(b_t^i, y_t^i) \in C^i$
  3. Payoffs are realized.

Let $C = \prod_i C^i$, $\mathcal{S} = \prod_i \mathcal{S}^i$ and $(b_t, y_t) = \prod_i (b_t^i, y_t^i)$. Finally let $b^t = \{b_1, \ldots, b_t\}$.

Definition 1 (**Equilibrium of Menu Games**). *A pure strategy equilibrium of a menu game is a collection of menus $\hat{\mathcal{S}}$; agents' choices at time zero $\hat{C} \in \hat{\mathcal{S}}$; agents choices at time t: $(\hat{b}_t, \hat{y}_t)$, for all $t \in \{1, ..., T\}$, such that:*

1. *For all t, $(\hat{b}_t, \hat{y}_t)$ solves:*

$$U_t\left(\theta_t | b^{t-1}, y^{t-1}, \hat{C}\right) = \max_{(b_t, y_t) \in C_t\left(\hat{b}^{t-1}, \hat{y}^{t-1} | \hat{C}\right)} u\left(\sum_{i=1}^{I} (b_t^i + y_t^i)\right) - v\left(\frac{\sum_{i=1}^{I} y_t^i}{\theta_t}\right) + \beta \sum_{\theta_{t+1}} \pi(\theta_{t+1}) U_{t+1}\left(\theta_{t+1} | b^t, y^t, \hat{C}\right),$$

   subject to $\sum_{i=1}^{I} (b_t^i + y_t^i) \geq 0$, $\quad \sum_{i=1}^{I} y_t^i \geq 0$ and $U_{T+1}(\cdot) \equiv 0$.

2. *Agents' choice at time 0, $\hat{C}$ solves:*

$$\max_{C \in \hat{\mathcal{S}}} \sum_{\theta_1} \pi(\theta_1) U_1\left(\theta_1 | C\right).$$

3. *Firm choices at time 0, for each $i \in I$, taking as given the choices of the other firms $\hat{\mathcal{S}}^{-i}$, $\mathcal{S}^i$ solves:*

$$V^i(\hat{\mathcal{S}}^i, \hat{\mathcal{S}}^{-i}) \equiv \min \sum_{t=0}^{T} \sum_{\theta^t} \pi(\theta^t) q^t \hat{b}_t^i(\theta^t)$$

where:

a) $\hat{b}^i_t(\theta^t) \in \hat{C}^i_t(\hat{b}^{i,t-1}, \hat{y}^{i,t-1}|\hat{C}^i)$, $\hat{C}^i_t(\hat{b}^{i,t-1}, \hat{y}^{i,t-1}|\hat{C}^i) \in \hat{\mathcal{S}}^i$

b) $\hat{b}^{-i}_t(\theta^t) \in \hat{C}^{-i}_t(\hat{b}^{-i,t-1}, \hat{y}^{-i,t-1}|\hat{C}^{-i})$ *and* $\hat{C}^{-i}_t(\hat{b}^{-i,t-1}, \hat{y}^{-i,t-1}|\hat{C}^{-i}) \in \hat{\mathcal{S}}^{-i}$.

In the definition of menu games each firm i is affected by the choices of other firms via the action of the agent. This is true both at time zero and at each period t. Firms compete playing a Nash game at time zero taking this into account.

The above definition does not impose any restriction on the size of a menu. A menu can contain more alternatives than the cardinality of the type space, implying that in every period some allocations are not chosen in equilibrium. Similarly, at time 0 a firm might offer more than one set of contracts, also implying that some contracts are offered and not chosen by agents in equilibrium. We denote a contract as **latent** if it is offered in equilibrium by a firm but is not chosen in equilibrium by any agent. In this paper we show that in environments with competition under non-exclusivity, latent contracts have a fundamental role in sustaining equilibrium allocations by preventing other firms from deviating to other contracts.[13]

Throughout the next section, we will make extensive usage of strategies that take the following form:

$$\tilde{C}^j_t = \{(0,0), (b^j, y^j)\}.$$

In this case, if an agent accepts the menu $\tilde{C}^j_t$ at time 0, he has the option of receiving a net transfer b^j in exchange of an output requirement y^j at time t. The time zero cost of this option is zero since the agent at time t can also choose no interaction with firm j, i.e. allocation $(0,0)$. The menu $\tilde{C}^j_t$ is reminiscent of entry strategies used in static duopoly competition.

We denote the equilibrium allocation of a menu game by $(\hat{b}, \hat{y})$. In the next section we show that an equilibrium exists and fully characterize the unique equilibrium allocation.

[13] This feature of off equilibrium contracts was first noted by Arnott and Stiglitz (1991).

3 Equilibrium Characterization

The Delegation Principle allow us to restrict to menu games in order to characterize the equilibrium of an arbitrary communication mechanism. In the following two sections we show the main result of this paper: there is a unique equilibrium allocation of a menu game and it coincides with the equilibrium of a self-insurance economy. In this section, we prove that any equilibrium must satisfy three conditions: the marginal rate of substitution between consumption and leisure is equal to marginal productivity for all agents, the standard Euler equation holds, and the net-present value of the transfers received under any history of shocks is zero. These conditions are the sufficient first-order conditions of a self-insurance economy. In section 4 we show an equilibrium exists.

3.1 Characterization under Exclusive Contracts

Before characterizing the optimality conditions in our environment, we review two robust equilibrium conditions in an environment in which there is competition between insurance providers and *contracts are exclusive*. Prescott and Townsend (1984) shows that in a general class of private information economy, the first welfare theorem holds. The decentralized economy is equivalent to a planning problem that maximizes the ex ante lifetime utility of the agents subject to feasibility and incentive compatibility constraints (in every period for every realization, agents weakly prefer the allocation designed for them).

In an environment similar to ours, and in the presence of exclusive contracting, the equilibrium allocation has the following features:[14]

1. The marginal rate of substitution between consumption and leisure is equated to the

[14] For a review of the results of constrained efficient allocation in dynamic Mirrleesian environments, refer to Golosov, Tsyvinski, and Werning (2006).

marginal productivity only for the highest type (originally shown by Mirrlees (1971)):

$$u'(c(\bar{\theta})) = \frac{1}{\bar{\theta}} v' \left(\frac{y(\bar{\theta})}{\bar{\theta}} \right), \tag{2}$$

$$u'(c(\theta)) > \frac{1}{\theta} v' \left(\frac{y(\theta)}{\theta} \right), \quad \forall\, \theta \neq \bar{\theta}, \theta \in \Theta, \tag{3}$$

where $\bar{\theta} \equiv \max_{\theta \in \Theta} \theta$. The intuition for this result is the following: in order to separate types, it is optimal to discourage less productive agents from working. This implies that all but the most productive agents work and consume less than they would in an competitive environment.

2. If preferences are separable in consumption and leisure, the marginal rate of substitution of consumption between any two periods differs from the inter-temporal rate of transformation for all types (the standard Euler equation does not hold):

$$\frac{1}{u'(c(\theta^t))} = \frac{1}{\beta R} E \left[\frac{1}{u'(c(\theta^{t+1}))} | \theta^t \right], \quad \forall\, t, \theta^t. \tag{4}$$

This equation, derived originally by Rogerson (1985) and generalized in Golosov, Kocherlakota, and Tsyvinski (2003), implies that for all periods

$$u'(c(\theta^t)) < \beta R E \left[u'(c(\theta^{t+1})) | \theta^t \right].$$

This means that it is optimal to make any type of agent saving constrained in order to encourage the truthful revelation of productivity in future periods.

3.2 Optimality Conditions under Non-exclusivity

We now derive the equilibrium conditions in the presence of non-exclusive contracting. The first result refers to the intra-temporal consumption and leisure choice. Under exclusivity, the optimal contract provides incentives to more skilled workers by discouraging less skilled agents to work (with respect to the full information allocation). The next lemma shows that an equation as (3) cannot hold when contracts are non-exclusive, since agents can work an additional amount for other firms.

Lemma 1. *In any equilibrium for every $\theta^t \in \Theta^t$, for all t the following holds:*

$$u'(b(\theta^t) + y(\theta^t)) \leq v'\left(\frac{y(\theta^t)}{\theta_t}\right)\frac{1}{\theta_t}, \tag{5}$$

where $b(\theta^t) = \sum_i b^i(\theta^t)$ and $y(\theta^t) = \sum_i y^i(\theta^t)$ and $(b^i(\theta^t), y^i(\theta^t))$ are the contracts chosen by an agent with history θ^t from firm i at time t.

Proof. Suppose that for some history θ^t equation (5) does not hold:

$$u'(b(\theta^t) + y(\theta^t)) > v'\left(\frac{y(\theta^t)}{\theta_t}\right)\frac{1}{\theta_t}. \tag{6}$$

In this case, the agent would like to consume and work more than the equilibrium contract. An entrant[15] can make strictly positive profits offering a supplemental contract with more consumption and output. Consider an entrant that offers the null contract at time $\tau \neq t$ and at time t, the contract $C_t^E = \{(-\varepsilon, \delta^*(\varepsilon)), (0, 0)\}$ where δ^* and ε are constructed as follows. Let $\delta^*(\varepsilon|\theta_t)$ be the solution of the following problem:

$$U(\varepsilon|\theta_t) \equiv \max_{\delta \geq 0} u(b(\theta^t) + y(\theta^t) + \delta - \varepsilon) - v\left(\frac{y(\theta^t) + \delta}{\theta_t}\right). \tag{7}$$

A necessary first order condition for this problem is:

$$u'(b(\theta^t) + y(\theta^t) + \delta^*(\varepsilon|\theta_t) - \varepsilon) \leq v'\left(\frac{y(\theta^t) + \delta^*(\varepsilon|\theta_t)}{\theta_t}\right)\frac{1}{\theta_t}. \tag{8}$$

If $\varepsilon = 0$, the solution for the above problem is $\delta^*(0|\theta_t) > 0$ given that (6) holds. From the Theorem of the Maximum, the solution $\delta^*(\varepsilon)$ is continuous on ε. Fix $\epsilon_1 > 0$ such that $|\delta^*(0) - 0| > \epsilon_1$. There exists $\epsilon_2 > 0$ such that if $|\varepsilon - 0| < \epsilon_2$ then $|\delta^*(\varepsilon) - \delta^*(0)| < \epsilon_1$. Let

[15] Throughout the paper, an incumbent refers to a firm that offers a menu that contains transfers and/or output recommendations other than the null contract, and some agent chooses some of these contracts in equilibrium. An entrant refers to an insurance provider that, at all times, every agent chooses the null contract from its menus. We assume the number of firms I is large enough so that an entrant always exists.

ε be such that $0 < \varepsilon < \epsilon_2$. This contract delivers strictly positive profits, proportional to ε, and the agent is strictly better off given that his utility is higher in some history with positive probability. This contract is always profitable for the entrant even if other types $\tilde{\theta}_t$ accept the deviating contract. The only way to deter this deviation is to have some latent contract that makes no agent willing to choose it. However, if such a contract existed, it would have been chosen in the original equilibrium, contradicting the fact that it is a latent contract. □

It is worth highlighting that the result under exclusivity breaks down the moment the agent can transform in small amounts consumption into leisure. This, from an applied perspective, might be particularly relevant since workers have the option, for example, to trade consumption for leisure in small amount by hiring individuals to perform non-leisure, non-work activities.

We actually show that equation (5) holds with equality. The additional step required to show this is proved in Proposition 1 this is due to an additional difficulty. Lemma 1 only requires an additional increase in output. Ruling out the opposite inequality, thus leading to (5) holding with equality, requires a reduction in output. Since agents cannot work negative hours, this deviation cannot be performed by an entrant and can only be offered by the incumbent

The second result under non-exclusivity refers to the inter-temporal consumption choice. When contracts are exclusive, the provision of incentives implies that agents are savings constrained. The following lemma shows that this cannot happen under non-exclusivity.

Lemma 2. *In any equilibrium for every $\theta^t \in \Theta^t$, for all t, the following holds:*

$$u'(c_t(\theta^t)) = \frac{\beta}{q} \sum_{\theta_{t+1}} u'(c_{t+1}(\theta^{t+1}))\pi(\theta_{t+1}), \tag{9}$$

where $c_t(\theta^t) = \sum_{i=1}^{I} \left(b_t^i(\theta^t) + y_t^i(\theta^t)\right)$.

Proof. In appendix B. □

The intuition for the result is the following. If the equilibrium allocation does not satisfy the Euler equation, an entrant firm can offer a savings (borrowing) contract at time t with an implicit interest rate lower (higher) than the marginal rate of transformation. As long as this contract is accepted, the entrant makes strictly positive profits and said contract can be constructed in a way that provides higher utility to the agent.[16]

In the next proposition, we show that in equilibrium, for every history, the marginal rate of substitution (MRS) between consumption and leisure is equated to the marginal productivity and also that the lifetime transfer received under any history is equal to zero, so that there is no cross-subsidization between types.

Proposition 1. *In any equilibrium the following two conditions hold:*

1. *Zero net present value of transfers:*

$$\sum_{t=1}^{T}\left(\frac{1}{q}\right)^{1-t} b_t(\theta^t) = 0 \qquad \forall\ \theta^T \in \Theta^T. \tag{10}$$

2. *MRS equal to marginal productivity:*

$$u'(b(\theta^t) + y(\theta^t)) = v'\left(\frac{y\left(\theta^t\right)}{\theta_t}\right)\frac{1}{\theta_t} \qquad \forall\ \theta^t, t. \tag{11}$$

Showing that (11) holds with equality requires two steps. For any two productivity realizations, we first show that the worker with higher output must receive a higher net transfer. If not – in which case the allocation would be as described in Mirrlees (1971)– an entrant can offer an additional labor opportunity which would induce misreporting to the incumbent. Second, we show that net transfers must be the same across the two types, failure from doing this would constitute a form of "negative" insurance. Such a contract would be dominated by a more profitable contract at time zero. Whenever an incumbent

[16] The previous two results would also go through relaxing the assumption on labor productivity shocks being independent over time. This would not be the case in proposition 1 since it also includes deviations by the incumbent which are history dependent.

deviates from a proposed contract the key difficulty is to show that no latent contract might induce negative profits.

Proof. In appendix B. □

3.3 Equivalence to Self-Insurance

The previous results showed that the equilibrium allocation satisfies a standard Euler equation, the marginal rate of substitution between consumption and leisure is equated to marginal productivity in every period, and the net present value of transfers received under any history is equal to zero (there is no redistribution). These equilibrium conditions are the same optimality conditions in a decentralized economy in which agents can borrow and save at rate $R = 1/q$.

Let $\{c^*, y^*\} = \{c^*(\theta^t), y^*(\theta^t)\}_{t=1}^T$ be the solution to the following problem:

$$\max_{c,y\geq 0} \sum_{t=1}^{T} \sum_{\theta^t} \beta^{t-1} \pi\left(\theta^t\right) \left[u\left(c(\theta^t)\right) - v\left(\frac{y(\theta^t)}{\theta_t}\right)\right] \tag{12}$$
$$s.t. \qquad \sum_{t=1}^{T} \frac{c(\theta^t) - y(\theta^t)}{R^{1-t}} = 0, \quad \forall \theta^T,$$

where R is taken as given.

Proposition 2. *Let $\{\hat{b}, \hat{y}\} = \{\hat{b}(\theta^t), \hat{y}(\theta^t)\}_{t=1}^T$ be the equilibrium allocation of a menu game. Let the agents' consumption be $\hat{c}(\theta^t) = \hat{b}(\theta^t) + \hat{y}(\theta^t)$ for all θ^t and for all t. If $R = 1/q$, $c^*(\theta^t) = \hat{c}(\theta^t)$ and $y^*(\theta^t) = \hat{y}(\theta^t)$ for all θ^t and for all t.*

Proof. The first order conditions of (12) are:

$$u'(c(\theta^t)) = \beta R \sum_{\theta_{t+1}} u'(c(\theta^{t+1}))\pi(\theta_{t+1}), \tag{13}$$
$$u'(c(\theta^t)) = \frac{1}{\theta_t} v'\left(\frac{y(\theta^t)}{\theta^t}\right), \tag{14}$$
$$\sum_{t=1}^{T} \frac{c(\theta^t) - y(\theta^t)}{R^{1-t}} = 0, \quad \forall \theta^T. \tag{15}$$

A solution to (12) exists. Also, the maximization problem (12) has a strictly concave objective function and the constraint set is convex; hence, the first order conditions are necessary and sufficient for the optimum and the optimum is unique. □

The previous proposition summarizes how non-exclusivity and non-observability of contracts limit the ability to provide insurance and also limits the contracts that are offered in equilibrium. Our environment with firms interacting strategically and being allowed to offer any type of contracts is equivalent to an environment with competitive firms offering linear contracts with no redistribution.

4 Existence of Equilibrium

So far, we have characterized three necessary properties of the equilibrium allocation: (9), (10), and (11) and have shown that there is a unique allocation that satisfies these conditions, which we denote by $\{\hat{b}, \hat{y}\} = \{(b(\theta^t), y(\theta^t))_{t=1}^{T} \quad \theta^t \in \Theta^t\}$. The next proposition determines existence of equilibrium of a menu game by providing *strategies* of the firms (menus) that sustain this allocation as an equilibrium.

An important message of section 2 is that direct mechanisms might not be sufficient when characterizing the optimal contract. This means that firms might offer latent (off-equilibrium) contracts and these contracts play an important role in this environment, in particular to show that an equilibrium exists. We show that equilibrium would fail to exist if firms were restricted to offering direct mechanisms. The menus we introduce to guarantee existence are similar to the ones derived in the characterization of equilibrium to show that any contract other than self-insurance is unprofitable. Although the equilibrium of a menu game is unique in terms of allocation, there are multiple equilibrium strategies that can sustain it.

We first state a useful result to show existence. The lemma below illustrates a standard monotonicity result: more productive agents always produce more output than less productive agents. This is independent of the menu offered by an entrant.

Lemma 3. *For all time t and history θ^{t-1} if $\theta_t > \hat{\theta}_t$, then under any deviation, $y(\theta^{t-1}, \theta_t) > y(\theta^{t-1}, \hat{\theta}_t)$, where $y(\theta^{t-1}, \theta_t)$ is the total output chosen by agent with history θ^{t-1}, θ_t.*[17]

Proof. Follows from standard arguments. □

Proposition 3. *Allocation $\{\hat{b}, \hat{y}\}$ is an equilibrium allocation of a menu game.*

The proof shows that any contract offered by an entrant falls into two categories. Either it will be unprofitable for the firm and hence not offered, or it will be welfare decreasing for the agents and hence not accepted.

Proof. The proof consists of two main steps:

- **Step 1** constructs the equilibrium menus of the incumbent firm that sustain the allocation.
- **Step 2** shows that no other firm (labeled entrant) can deviate and offer a menu that is at the same time profitable and chosen by the agents.

Step 1

To show the allocation is an equilibrium we start by constructing strategies of the firms and the agents that sustain allocation $\{\hat{b}, \hat{y}\}$ as an equilibrium. Firms $i \in \{1, 2\}$ offer the menus below, while all the remaining firms offer the null contract. We define menus starting from period T:

$$\hat{C}^i_T(b^{i,T-1}, y^{i,T-1}) = \left\{(b^i_T, y^i_T) : b^i_T = 0,\ y^i_T = y(\theta) > 0 |\ u'\left(-\frac{1}{q} b^i_{T-1} + y(\theta)\right) = \frac{1}{\theta} v'\left(\frac{y(\theta)}{\theta}\right)\ \forall \theta \in \Theta\right\},$$

[17] For brevity of notation, we let $(b(\theta^{t-1}, \theta), y(\theta^{t-1}, \theta))$ denote the total transfer and output choice of the agent, which can be achieved by choosing allocation for a combination of contracts, including any deviation by a potential entrant. Similarly, $U_{t+1}(b(\theta^{t-1}, \theta), y(\theta^{t-1}, \theta))$ denotes the continuation utility associated with all the contracts the agent has chosen up to time t.

for each of the periods $t = 2, \dots, T-1$:

$$\hat{C}_t^i(b^{i,t-1}, y^{i,t-1}) = \left\{ \begin{array}{c} (b_t^i, y_t^i) : b_t^i = b(\theta),\ y_t^i = y(\theta) > 0 |\ u'\left(-\frac{1}{q}b_{t-1}^i + b(\theta) + y(\theta)\right) = \frac{1}{\theta}v'\left(\frac{y(\theta)}{\theta}\right) \\ u'\left(-\frac{1}{q}b_{t-1}^i + b(\theta) + y(\theta)\right) = \frac{\beta}{q}\sum_{\theta'}\pi(\theta')\left[u'\left(-\frac{1}{q}b(\theta) + b_{t+1}^i(\theta') + y_{t+1}^i(\theta')\right)\right]\ \forall\theta\in\Theta \end{array} \right\}.$$

For the initial period, $t = 1$, the menu is defined as above with $b_0^i = 0$. The quantity $x_t = -\frac{1}{q}b_{t-1}^i + b_t^i$ denotes the net withdrawals (or deposit if negative) from the "savings account" offered by firm i.

Together with the above menu, firm $i \in \{1, 2\}$ also offers two additional latent menus: a dynamic contract and a static contract. The *dynamic contract* is defined for all for all $t = 1, \dots, T$

$$C_t^{i,D}(b^{i,t-1}, y^{i,t-1}) = \left\{(b_t^i, 0) : b_t^i \in \mathbb{R},\ |b_t^i = \frac{1}{q}b_{t-1}^i + x_t,\ b_T^i = b_0^i = 0,\ x_t \in \mathbb{R}\right\}. \tag{16}$$

The *static contract* for all $t = 1, \dots, T$ is given by:[18]

$$C_t^{i,S} = \{(0, \delta) : \delta \geq 0\}. \tag{17}$$

Given these menus, the agents choose at time zero menu $\hat{C}^i$ from one of the two firms. We derive the agents' choices by backward induction. At time T, an agent with history (θ^{T-1}, θ_T) and past choices $(\tilde{b}(\theta^{T-1}), \tilde{y}(\theta^{T-1}))$ chooses from menu $C_T^i(\tilde{b}^i(\theta^{T-1}), \tilde{y}^i(\theta^{T-1}))$ the allocation $(\tilde{b}^i(\theta^T), \tilde{y}^i(\theta^T))$ such that $u'\left(-\frac{1}{q}\tilde{b}^i(\theta^{T-1}) + \tilde{y}^i(\theta^T)\right) = \frac{1}{\theta_T}v'\left(\frac{\tilde{y}^i(\theta^T)}{\theta_T}\right)$. For time $t \in \{1, ..., T-1\}$, an agent with history θ^t and past choices $(\tilde{b}^i(\theta^{t-1}), \tilde{y}^i(\theta^{t-1}))$ chooses from menu $C_t^i(\tilde{b}^i(\theta^{t-1}), \tilde{y}^i(\theta^{t-1}))$ allocation $(\tilde{b}^i(\theta^t), \tilde{y}^i(\theta^t))$ such that

$$u'\left(-\frac{1}{q}\tilde{b}^i(\theta^{t-1}) + \tilde{b}^i(\theta^t) + \tilde{y}^i(\theta^t)\right) = \frac{\beta}{q}\mathbb{E}_t\left[u'\left(-\frac{1}{q}\tilde{b}^i(\theta^t) + \tilde{b}^i(\theta^{t+1}) + \tilde{y}^i(\theta^{t+1})\right)\right], \tag{18}$$

$$u'\left(-\frac{1}{q}\tilde{b}^i(\theta^{t-1}) + \tilde{b}^i(\theta^t) + \tilde{y}^i(\theta^t)\right) = \frac{1}{\theta_t}v'\left(\frac{\tilde{y}^i(\theta^t))}{\theta_t}\right). \tag{19}$$

[18]Both latent contracts deliver zero profits to firms.

Given the agents' choices, firm i's profits are $\sum_{t=1}^{T} \sum_{\theta^t} q^t \pi(\theta^t) \tilde{b}^i(\theta^t) = 0$.

Step 2

To complete the equilibrium proof, we show there is no deviation by entrants or incumbents that at the same time increases agents' utility and delivers positive profits. In particular, the latent contracts $C^{i,S}$ and $C^{i,D}$ are sufficient to deter any potential deviations. These contracts play a key role: the dynamic contract allows agents to transfer resources over time (at rate $1/q$) while the static menu gives agents the opportunity to work additional hours at a wage equal to their marginal productivity. Hence, these contracts significantly limit the deviations an entrant could offer.

Consider a firm i, with $i \neq 1, 2$, offering a menu (call it the entrant firm). Let E equal the amount of resources the entrant transfers in expected terms to the agent. To be profitable, the time zero value of E must be negative. We now show that the utility attained by agents from accepting any contract offered by the entrant (which could be combined with the incumbent's menu) is less than or equal to the utility derived from an environment in which the agent can optimally borrow and save an amount E and choose hours freely. This implies that the utility under any deviation cannot be higher than the utility under the candidate equilibrium. The proof is provided for a two periods case. By backward induction, it can be extended to any finite T periods.

Throughout this step, let $\hat{x}(\theta)$ and $\hat{y}(\theta)$ denote, respectively, the transfer and the output chosen by an agent of type θ in the incumbent's menu, and let $b(\theta)$ and $y(\theta)$ denote, respectively, the transfer and the output chosen by an agent of type θ in the entrant's menu.[19] Let $\overline{W}$ be the lifetime utility agents achieve by accepting an entrant contract, and let $\hat{W}$ be the lifetime utility implied by the candidate equilibrium allocation. A key aspect of the proof is to compare the utility in period two for an agent signing the entrant contract and the utility an agent derives when he optimally choses labor. Let $\tilde{W}$ be the indirect utility attained in the second period by an agent starting that period with net resources x_θ and

[19] In the presence of a deviation by an entrant, the agent's choice in the incumbent's menus may not be the equilibrium allocation. Also in this notation, the allocations can be the result of the choices in different menus by entrants and/or incumbents.

optimally choosing labor:

$$\tilde{W}(x_\theta) = \max_y \sum_{\theta'} \pi(\theta') \left[u\left(-\frac{x_\theta}{q} + y \right) - v\left(\frac{y}{\theta'} \right) \right]. \tag{20}$$

Let $W(\hat{x}(\theta), \theta)$ be the second period utility of a type θ agent in period one derived from choosing within the entrant and the incumbent menus:

$$W(\hat{x}(\theta), \theta) =$$
$$\sum_{\theta'} \pi(\theta') \left[u\left(-\frac{\hat{x}(\theta)}{q} - \frac{E_\theta}{q} + b(\theta, \theta') + \frac{E_\theta}{q} + y(\theta, \theta') + \hat{y}(\theta, \theta') \right) - v\left(\frac{y(\theta, \theta') + \hat{y}(\theta, \theta')}{\theta'} \right) \right].$$

The variables defined next quantify (in terms of extra resources that need to be transferred) the difference in utility levels. For all θ define b^*_θ:

$$W(\hat{x}(\theta), \theta) = \tilde{W}(\hat{x}(\theta) + b^*_\theta). \tag{21}$$

From the above definition, $b^*_\theta \geq 0$ is the extra resources transferred to the agent to make him indifferent about accepting the entrant's contract or optimally choosing the output produced at time 2.

Define function $G(g)$ as

$$G(g) = \max_{y_\theta, x_\theta} \sum_\theta \pi(\theta) u\Big(-b + g \cdot (b(\theta) - E_\theta - b^*_\theta) + x_\theta + y_\theta + (1-g) \cdot E \Big) - v\left(\frac{y_\theta}{\theta} \right) + \beta \tilde{W}(x_\theta), \tag{22}$$

where E is the expected resource cost for the entrant given by

$$E = \sum_\theta \pi(\theta) \Big(b(\theta) - E_\theta \Big). \tag{23}$$

The parameter g is a "scaling " parameter that modifies the magnitude of the transfers from the entrant. The following lemma compares the value of $\hat{W}$ and $\overline{W}$ exploiting the properties of the function $G(\cdot)$.

Lemma 4.

1. *$G(1) \geq \overline{W}$ and $G(0) = \hat{W}$;*

2. *For all $g > 0$, $G'(g) \leq G'(0)$.*

3. *$G'(0) < 0$;*

Proof. In appendix C □

The previous lemma immediately implies that $\hat{W} \geq \overline{W}$. This completes the proof since it shows that no firm with $i \neq 1, 2$ will deviate from the proposed equilibrium. Also no firm $i = 1, 2$ deviates. This follows from the fact that the off-equilibrium menus $C^{i,D}$ and $C^{i,S}$ are offered by both firms. □

Summarizing, the allocation $\{\hat{b}, \hat{y}\}$ can be sustained in equilibrium by at least two incumbents simultaneously offering the menu $\hat{C}^i$ and the latent contracts $C^{i,S}$ and $C^{i,D}$. This is necessary to prevent deviations by any firm to a more profitable and ex ante welfare improving contract that features redistribution. This result highlights the importance of allowing firms to offer latent contracts. If offering such contracts were not allowed, as in direct mechanisms, equilibrium would fail to exist in this environment.

An immediate implication of the existence result and proposition 2 is that the equilibrium is unique in terms of allocation.

Corollary 1. *There is a unique equilibrium allocation of the menu game.*

Although there might exist different strategies (menus) that sustain the unique allocation as an equilibrium, in all of them latent contracts must be offered. The following example illustrates this feature. Consider a strategy in which firms offer only menus $\hat{C}^i$. This cannot be an equilibrium since either an incumbent or an entrant will deviate, offering a profitable welfare increasing menu, in the shape of a contingent contract.

As an example, consider the following profitable deviation (motivated by Abraham and Pavoni (2005)) where in the last period the constrained efficient allocation is offered. Let

$\{\tilde{b}(b^{-1}), \tilde{y}(b^{-1})\}$ be the solution to the following problem:

$$\tilde{U}(b^{-1}) = \max_{b,y} \sum_{\theta} \pi(\theta) \left[u(b(\theta) + y(\theta)) - \frac{1}{\theta} v\left(\frac{y(\theta)}{\theta}\right) \right], \tag{24}$$

$$s.t. \quad u(b(\theta) + y(\theta)) - \frac{1}{\theta} v\left(\frac{y(\theta)}{\theta}\right) \geq u(b(\hat{\theta}) + y(\hat{\theta})) - \frac{1}{\theta} v\left(\frac{y(\hat{\theta})}{\theta}\right),$$

$$\sum_{\theta} \pi(\theta) b(\theta) = b^{-1}.$$

Note that $\tilde{U}(b^{-1})$ is strictly larger than the utility of autarky with b^{-1} additional (possibly negative) resources. Firm i can deviate from the set of menus C^i defined above by substituting the time T menu with

$$\tilde{C}^T(b^{i,T-1}, y^{i,T-1}) = \left\{ \{\tilde{b}(b^i_{T-1} - \varepsilon), \tilde{y}(b^i_{T-1} - \varepsilon)\} | \text{ solves } (24) \text{ and } \varepsilon > 0 \right\}. \tag{25}$$

For ε sufficiently small, the agent prefers this contract to the original, and, in addition, this deviation provides additional $\frac{\varepsilon}{q^{1-T}}$ profits.

In the presence of the latent contracts described in (16) and (17), the deviation (25) is unprofitable. This is because at time T it implies positive net transfers from more productive to less productive agents. So, if a high productivity agent also has access to the static labor contract C^S, he would choose in (25) the allocation designed for the low productivity agent (collecting the positive transfers) and work the additional hours with the firm offering the latent contract C^S. This strictly improves the utility of the high productivity agent since he works the same amount of hours and receive higher consumption. This choice of the high productivity agent makes the profits of the deviation (25) negative since all agents choose the allocation with positive transfers.

5 Conclusion

A large fraction of the literature on optimal social insurance under private information features stark assumptions on the ability of agents to sign additional contracts. Agents are required to be either in exclusive relationships with the insurance provider or at the very least to sign contracts featuring an extreme level of cross-indexing. This cross-indexing (or interdependence of contracts) takes into account any other relationship the agent might engage, whether a labor, credit, or an insurance relationship. In many instances, the above assumptions are driven by analytic tractability rather than empirical motivation. Real-world contractual relationships rarely feature exclusivity clauses. Even more rare are instances of complete cross-indexing of contracts: for example, compensation being affected by the balance in a savings account is a rare occurrence.

This paper relaxes the exclusivity assumption taking the opposite, extreme view. In our environment, it is costless to engage in additional contractual relationships and they are available in every period. The implications of this alternative view are significant. Our main result is that competition reduces the amount of insurance provided: the equilibrium is equivalent to a self-insurance economy. In this environment, the competition between insurance providers results in linear contracts being the only contracts offered in equilibrium. This is true in labor relationships, so that the wage always reflects the marginal product. And it is true for inter-temporal contracts, so that the rate of return on debt (or savings) is equal to the inter-temporal technical rate of transformation. Finally, in equilibrium there is no redistribution. We also show that an equilibrium exists in this environment and that latent contracts play an essential role.

The results of this paper have important implications for the analysis of positive questions. It provides a micro-foundation of standard incomplete market models based entirely on equilibrium competition. The absence of exclusive relationships results in stark differences in the allocation that arises versus an environment with complete exclusivity. These differences can be tested by looking at data on consumption, income, and hours.[20] The

[20] Currently being pursued in Ales and Maziero (2010).

question is, then, which households are more likely to feature an allocation in line with an environment with non-exclusive competition?

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Appendix

A General Communication and Delegation Principle

In this appendix we define a general communication mechanism and show that the equilibrium of a communication game can be implemented as an equilibrium of a menu game in which firms offer a menu of payoff relevant alternatives, with the agents choosing from it.

This result is necessary since, under non-exclusive contracting, the preference ordering of the agents is influenced not only by their exogenous private information, but also by the set of contracts offered. This implies that restricting to a direct mechanism may not allow a firm to have a rich enough communication with the agent in order to obtain information on the other contracts.[21]

Communication

Firms and agents communicate according to a communication mechanism, which consists, for each firm $i \in \{1, ..., I\}$, of a time 0 report space $\mathcal{R}^i$ and message spaces $\mathcal{M}_t^i$ for each $t \in \{1, ..., T\}$. The set of all possible messages that an agent can send to firm i up to time t is denoted by $\mathcal{M}^{i,t} = \mathcal{M}_1^i \times ... \times \mathcal{M}_t^i$. For a given message space, each firm chooses allocation functions $g_t^i : \mathcal{M}^{i,t} \to \mathbb{R} \times \mathbb{R}_+$, which specify net-transfers of consumption and output at time t. At time 0, given the report space, a firm determines a contract $\phi^i : \mathcal{R}^i \to G_1^i(\mathcal{M}^{i,1}) \times ... \times G_T^i(\mathcal{M}^{i,T})$, where $G_t^i(\mathcal{M}^{i,t})$ is the set of all measurable mappings from message space $\mathcal{M}^{i,t}$ to the allocation space $\mathbb{R} \times \mathbb{R}_+$. The contract ϕ^i determines, conditional on the report r^i, the allocation functions an agent will have access to in all future periods. To shorten notation, let $G^i(\mathcal{M}^{i,T}) \equiv G_1^i(\mathcal{M}^{i,1}) \times ... \times G_T^i(\mathcal{M}^{i,T})$. Let $\Phi^i(\mathcal{R}^i, \mathcal{M}^{i,T})$ be the set of all measurable mappings from reporting space $\mathcal{R}^i$ to the set $G^i(\mathcal{M}^{i,T})$, and note that $\phi^i \in \Phi^i(\mathcal{R}^i, \mathcal{M}^{i,T})$. We summarize all the message spaces as follows, let $\mathcal{M} = \times_{i=1}^I \mathcal{M}^{i,T}$ and $\mathcal{R} = \times_{i=1}^I \mathcal{R}^i$. Denote the game associated with the communication mechanism $(\mathcal{M}, \mathcal{R})$ by $\Gamma_{\mathcal{M},\mathcal{R}}$.

At time 0, before any uncertainty is realized, each firm i simultaneously offers a collection of allocation functions ϕ^i, and agents communicate with firms sending a message r^i. This message determines, through ϕ^i, the functions g_t^i at every subsequent period t. The timing of the game $\Gamma_{\mathcal{M},\mathcal{R}}$ is the following:

- At time 0:
    1. Each firm i simultaneously offers contract $\phi^i : \mathcal{R}^i \to G^i(\mathcal{M}^{i,T})$;
    2. Agents send a report $r^i \in \mathcal{R}^i$ to each firm i.
- At time t:

[21] Epstein and Peters (1999) showed that in games with multiple principals, there exists a universal type space for which the revelation principle holds. This space must be rich enough to allow agents and firms to communicate the exogenous type space and the information about the contracts offered by other principals. However, this universal type space is hard to characterize in applications; hence we do not follow their approach.

1. Agent learns his private type θ_t;
2. Firm offers allocation rule $g_t^i : \mathcal{M}^{i,t} \to \mathbb{R} \times \mathbb{R}_+$ according to $\phi^i(r^i)$;
3. Agent sends a message $m_t^i \in \mathcal{M}_t^i$ to each firm i;
4. Payoffs are realized: output is produced and net-transfers are made.

Given messages $(\mathcal{M}, \mathcal{R})$, we consider a static Nash equilibrium played by firms at time 0 when choosing the contracts that are offered in future periods. Given these contracts, agents optimize choosing the report at time 0 and messages in every period $t = 1, ..., T$.

Definition 2 (**Equilibrium of Communication Game**). *A pure strategy equilibrium of* $\Gamma_{\mathcal{M},\mathcal{R}}$ *is* (r^*, m^*, ϕ^*, g^*) *such that:*

1. ***Agent's message*** m_t^**:* $G_t^1 \times ... \times G_t^I \times \Theta^t \to \mathcal{M}_t$ *solves for each* $t \in \{1, ..., T\}$*:*

$$\begin{aligned} U_t\left(\theta_t | g^*, m^{t-1}\right) &= \max_{m_t \in \mathcal{M}_t} u\left(\sum_{i=1}^{I}\left(b^i(m^{i,t}) + y(m^{i,t})\right)\right) - v\left(\frac{\sum_{i=1}^{I} y(m^{i,t})}{\theta_t}\right) + \\ &\quad + \beta \sum_{\theta_{t+1}} \pi(\theta_{t+1}) U_{t+1}\left(\theta_{t+1} | g^*, m^t\right), \end{aligned}$$

subject to $\sum_{i=1}^{I}(b^i(m^{i,t}) + y(m^{i,t})) \geq 0, \quad \sum_{i=1}^{I} y(m^{i,t}) \geq 0, \quad \forall t$
where $(b(m^{i,t}), y(m^{i,t})) = g_t^{*,i}(m^{i,t})$ *and* $U_{T+1}(\cdot) \equiv 0$.

2. ***Agent's reporting strategy at time 0****,* $r^* : G^1 \times ... \times G^I \to \mathcal{R}$ *solves:*

$$\max_{r \in \mathcal{R}} \sum_{\theta_1} \pi(\theta_1) U_1\left(\theta_1 | g\right)$$

where $g^i = \phi^{i,*}(r_i)$.

3. *Taking as given the choices of the other firms and the agents' choices, for each firm* $i \in \{1, ..., I\}$*, the* ***contract*** $\phi^{i,*}$ *solves:*

$$V^i(\phi^{i,*}, \phi^{-i,*}) \equiv \min_{\phi^i \in G^i(\mathcal{M}^i)} \sum_{t=0}^{T} \sum_{\theta^t} \pi(\theta^t) q^t b_t^{i,*}(\theta^t),$$

$b_t^*(\theta^t) = b(m^{t,*}(\theta^t))$*,* $g^i = \phi^i\left(r^{i,*}\right)$ *and* $g^{-i,*} = \phi^{-i,*}\left(r^{-i,*}\right)$.

Denote the equilibrium *allocation* of a general communication game by (b^*, y^*).

Menu Games

A communication mechanism induces allocation functions and, hence, distribution over allocations. This means that to prove the equivalence between the equilibrium allocation of a given communication mechanism and the equilibrium of a menu game, it is essential that the menus offered are rich enough to capture the strategies used to implement equilibrium in a communication mechanism. In our environment, a menu is a sequence of sets, with each set being a subset of the allocation space $\mathbb{R}\times\mathbb{R}_+$. For a message space $(\mathcal{M},\mathcal{R})$, define, for each firm i, the set $C_t^i(m^{i,t-1},\mathcal{M}_t^i|G_t^i)$ as the menu that can be implemented by a message space $\mathcal{M}_t^i$ at time t given a history of messages $m^{i,t-1}$ and a set of allocation functions G_t^i. Formally, a menu at time t is the following set:

$$C_t^i(m^{i,t-1},\mathcal{M}_t^i|G_t^i)\equiv\{X\subseteq\mathbb{R}\times\mathbb{R}_+|\ \exists\ g_t^i\in G_t^i\subseteq G_t^i(\mathcal{M}^{i,t}):X=\mathbf{Im}(g_t^i|m^{i,t-1})\}\ \ \forall t,\ \forall i \quad (26)$$

where

$$\mathbf{Im}(g_t^i|m^{i,t-1})=\left\{x\in\mathbb{R}\times\mathbb{R}_+|\ \exists\ m_t^i\in\mathcal{M}_t^i\ :x=g_t^i\left(m^{i,t-1},m_t^i\right)\right\}\quad\forall t,\ \forall i. \qquad (27)$$

Each set defined in (26) contains all subsets of $\mathbb{R}\times\mathbb{R}_+$ with cardinality at most $\mathcal{M}_t^i$.

For any subset $G_t^i\subseteq G_t^i(\mathcal{M}^{i,t})$, let $G^i\equiv G_1^i\times...\times G_t^i$ and define a sequence of menus offered by firms at time 0 as:

$$C\left(G^i\right)=\left\{X_t\subseteq C_t^i(m^{i,t-1},\mathcal{M}_t^i|G_t^i),\ t=1,...,T,\ \forall m^{i,t-1}\in\mathcal{M}^{i,t-1},m_t^i\in\mathcal{M}_t^i\right\}. \qquad (28)$$

At time 0, each agent chooses a sequence of menus in the collection offered by firm i. Define $\mathcal{S}^i$ as the collection of menus that are consistent with a communication system $(\mathcal{M},\mathcal{R})$.

$$\mathcal{S}^i(\mathcal{R}^i,\mathcal{M}^i)\equiv\{C^i\subseteq C\left(G^i\right)|\exists\ \phi^i\in\Phi^i(\mathcal{R}^i,\mathcal{M}^i):G^i=\mathbf{Im}(\phi^i)\}. \qquad (29)$$

This set contains all the collections of sets C^i with cardinality less than or equal to the cardinality of $\mathcal{R}^i$. Without explicitly writing the dependence on the message spaces, let $\mathcal{S}^i=\mathcal{C}^i(\mathcal{R}^i,\mathcal{M}^i)$ be the menus offered by firm i and let C^i be an element of $\mathcal{S}^i$. Let $\Gamma_{C,\mathcal{S}}$ be the game associated with menus $(C,\mathcal{S})$.

The following proposition shows that an equilibrium in a general communication system can be implemented as an equilibrium of a menu game. In this menu game, the collection of menus offered by each firm must be compatible with the general communication mechanism as defined above.

Proposition 4 (Delegation Principle)**.** *Let (b^*,y^*) be an equilibrium allocation of a general communication game $\Gamma_{\mathcal{M},\mathcal{R}}$. Then there exists $(\hat{b},\hat{y})$ that is an equilibrium allocation of a menu game $\Gamma_{C,\mathcal{S}}$ and $(b^*,y^*)=(\hat{b},\hat{y})$.*

Proof. The proof is by construction. Starting from the equilibrium strategies of a general communication game, we construct strategies for a menu game and show that these strategies constitute an equilibrium.

Define as in (26) and (29) respectively the menus and the collection of menus that are compatible with message spaces $(\mathcal{M},\mathcal{R})$. Define the strategy of firm i in this menu game as:

$$\hat{\mathcal{S}}^i=\{C^i\subseteq C^i\left(G^i\right)|G^i=\mathbf{Im}(\phi^{i,*})\}. \qquad (30)$$

The collection of menus $\hat{\mathcal{S}}^i$ contains all the subsets of the allocation space that are consistent with the collection of allocation functions in the original equilibrium. Agents' strategies are defined as follows.

$$\hat{C}^i = \{\hat{C}^i_t \in \hat{\mathcal{S}}^i : \hat{C}^i_t = \mathbf{Im}(g^{i,*}_t | m^{i,t-1,*}) \text{ and } g^{i,*}_t = \phi^{i,*}(r^{i,*})\}$$

$$(\hat{b}^i(\theta^t), \hat{y}^i(\theta^t)) = g^{i,*}_t(m^{i,t,*}(\theta^t)).$$

Note that by construction $\hat{C}^i_t \in \hat{\mathcal{S}}^i$ and $(\hat{b}^i(\theta^t), \hat{y}^i(\theta^t)) \in \hat{C}^i_t$, $\forall \theta^t$, $\forall t$. The menu $\hat{C}^i_t$ is the subset of allocation space, $\mathbb{R}^2$, that corresponds to the allocation function chosen by the agent in the original equilibrium. Also $(\hat{b}^i, \hat{y}^i)$ corresponds to allocation determined by the allocation function given the equilibrium message sent by each type θ^t. If agents and firms follow these strategies, the equilibrium allocation in the menu game is the same as in the original equilibrium.

First, let's show that the agents' strategies are an equilibrium. Suppose that at some time t, for some firm i $\exists$ $(b^i_t, y^i_t) \in \hat{C}^i_t$ such that:

$$u\left(\sum_{i=1}^{I}(b^i_t + y^i_t)\right) - v\left(\frac{\sum_{i=1}^{I} y^i_t}{\theta_t}\right) + \beta \sum_{\theta_{t+1}} \pi(\theta_{t+1}) U_{t+1}\left(\hat{b}^{t-1}, b_t, \hat{y}^{t-1}, y_t, \theta_{t+1} | \hat{C}\right) >$$
$$u\left(\sum_{i=1}^{I}(\hat{b}^i_t + \hat{y}^i_t)\right) - v\left(\frac{\sum_{i=1}^{I} \hat{y}^i_t}{\theta_t}\right) + \beta \sum_{\theta_{t+1}} \pi(\theta_{t+1}) U_{t+1}\left(\hat{b}^{t}, \hat{y}^{t}, \theta_{t+1} | \hat{C}\right).$$

Since $(b^i_t, y^i_t) \in \hat{C}^i_t$, there exists $m^i_t \in \mathcal{M}^i_t$ such that $(b^i_t, y^i_t) = g^{i,*}_t(m^{i,t})$. Replacing in the agents' payoff:

$$u\left(\sum_{i=1}^{I}\left(b^i(m^{i,t}) + y(m^{i,t})\right)\right) - v\left(\frac{\sum_{i=1}^{I} y(m^{i,t})}{\theta_t}\right) + \beta \sum_{\theta_{t+1}} \pi(\theta_{t+1}) U_{t+1}\left(m^{t}, \theta_{t+1} | g^*\right) >$$
$$u\left(\sum_{i=1}^{I}\left(b^i(m^{i,t,*}) + y(m^{i,t,*})\right)\right) - v\left(\frac{\sum_{i=1}^{I} y(m^{i,t,*})}{\theta_t}\right) + \beta \sum_{\theta_{t+1}} \pi(\theta_{t+1}) U_{t+1}\left(m^{t,*}, \theta_{t+1} | g^*\right).$$

But this contradicts $m^{i,*}$ being an equilibrium in the original game. Now suppose $\hat{C}^i$ is not an equilibrium for some i. There exists some $C^i \in \hat{\mathcal{S}}^i$ such that:

$$U(C^i, \hat{C}_{-i}) > U(\hat{C}).$$

Since $C^i \in \hat{\mathcal{S}}^i$, $\exists$ $r^i \in \mathcal{R}^i$ such that $C^i = \mathbf{Im}(g^i)$ and $g^i = \phi^{i,*}(r^i)$. Replacing in the agents' payoff:

$$U\left(g^i, g^*_{-i},\right) > U\left(g^{i,*}, g^*_{-i}\right).$$

But this contradicts $r^{i,*}$ being an equilibrium in the original game.

Finally, we check that firms' strategies constitute an equilibrium. Suppose $\exists$ $\mathcal{S}^i \in \mathcal{S}^i(\mathcal{R}^i, \mathcal{M}^i)$ such that $V^i(\mathcal{S}^i, \hat{\mathcal{S}}^{-i}) > V^i(\hat{\mathcal{S}}^i, \hat{\mathcal{S}}^{-i})$.

Since $\mathcal{S}^i \in \mathcal{S}^i(\mathcal{R}^i, \mathcal{M}^i)$, there exists ϕ^i such that $g^i = \phi^i(r^{i,*})$. Replacing in the firm's payoff in the original game $V^i(\phi^i, \phi^*_{-i}) > V^i(\phi^{i,*}, \phi^*_{-i})$. But this contradicts $\phi^{i,*}$ being an equilibrium in the original game. □

Proposition 4 states that for given message spaces $(\mathcal{M}, \mathcal{R})$, there exists a menu game that implements the same equilibrium allocation. It is important to note that message spaces restrict the menus that can be offered in a menu game. Hence, if firms are allowed to use unrestricted message spaces, the same equilibrium allocation can be implemented if firms can offer unrestricted menus. The same result is shown in Martimort and Stole (2002) for a similar environment (see footnote 11 for details).

B Proofs of Section 3

B.1 Proof of Lemma 2

Proof. Suppose that for some history $\hat{\theta}^t$ equation (9) does not hold.
Case 1:

$$u'(c_t(\hat{\theta}^t)) > \frac{\beta}{q} \sum_{\theta_{t+1}} u'(c_{t+1}(\hat{\theta}^t, \theta_{t+1}))\pi(\theta_{t+1}). \tag{31}$$

In this case, the agent is borrowing constrained. An entrant can make strictly positive profits offering a borrowing contract at a rate higher than $1/q$, contradicting the original allocation being an equilibrium. The first step is to construct the contract to be offered by a firm. Let $\delta^*(\varepsilon)$ be the solution of the following problem:

$$U(\varepsilon) \equiv \max_{\delta \geq 0} u(c_t(\hat{\theta}^t) + \delta) + \beta E_t u\left(c_{t+1}(\hat{\theta}^t, \theta_{t+1}) - \delta\left(\frac{1}{q} + \varepsilon\right)\right). \tag{32}$$

A necessary first order condition for this problem is:

$$u'(c_t(\hat{\theta}^t) + \delta) \leq \beta\left(\frac{1}{q} + \varepsilon\right) E_t u'\left(c_{t+1}(\hat{\theta}^t, \theta_{t+1}) - \delta\left(\frac{1}{q} + \varepsilon\right)\right). \tag{33}$$

If $\varepsilon = 0$, the solution for the above problem is $\delta^*(0) > 0$ given that (31) holds. From the Theorem of the Maximum, the solution $\delta^*(\varepsilon)$ is continuous on ε. Fix $\epsilon_1 > 0$ such that $|\delta^*(0) - 0| > \epsilon_1$. There exists $\epsilon_2 > 0$ such that if $|\varepsilon - 0| < \epsilon_2$ then $|\delta^*(\varepsilon) - \delta^*(0)| < \epsilon_1$. Let ε be such that $0 < \varepsilon < \epsilon_2$.[22] Consider an entrant that offers the contract $C_t = \{(\delta^*(\varepsilon), 0), (0, 0)\}$ and $C_{t+1} = \{(-\delta^*(\varepsilon)(\frac{1}{q} + \varepsilon), 0), (0, 0)\})$ and the contract $(0, 0)$ for all other periods. This firm is making strictly positive profits, proportional to $\delta^*(\varepsilon)\varepsilon$, and the agent is strictly better off keeping the original equilibrium together with this contract since it increases his utility in a history with positive probability and keeps the same utility in all other histories.

[22] Note that $u'(c_t(\hat{\theta}^t) + \delta^*(\varepsilon))$ is a finite, strictly positive number and hence is also the right hand side of equation (33). This implies that $c_{t+1}(\hat{\theta}^t, \theta_{t+1}) - \delta^*(\varepsilon)\left(\frac{1}{q} + \varepsilon\right) > 0$ for all $(\hat{\theta}^t, \theta_{t+1})$.

Hence, under the original equilibrium, a firm can offer a contract that makes strictly positive profits. This contradicts the allocation being an equilibrium.

The other case can be proved using a similar argument. □

B.2 Proof of Proposition 1

Proof. For a history θ^{t-1}, define the net present value of transfers received from time t onwards by:

$$A_t(\theta^{t-1}, \theta_{t-1}^T) = \sum_{n=t}^{T} \left(\frac{1}{q}\right)^{t-n} b_n(\theta^{t-1}, \theta_{t-1}^n), \tag{34}$$

where $\theta_{t-1}^n = (\theta_t, \theta_{t+1}, \ldots, \theta_n)$ is the sequence of shocks following history θ^{t-1} from time t to n and $b_n(\theta^{t-1}, \theta_{t-1}^n)$ is the equilibrium transfer chosen at time n by an agent with history θ^n. We show, using a backward induction argument, that for all t, $A_s(\theta^{s-1}, \theta_{s-1}^T)$ is independent of θ_{s-1}^T for all $s \geq t$. This implies that $A_1(\theta^T)$ is the same for all $\theta^T \in \Theta^T$. If $A_1(\theta^T) > 0$, firms make strictly negative profits in equilibrium, and would be better off offering a null contract. If $A_1(\theta^T) < 0$, an entrant can offer the same sequence of transfers, giving an additional transfer $\varepsilon > 0$ in the terminal period. Since the sequence of transfers is not contingent and is profitable for all types, there is no latent contract that makes it unprofitable.

1. **Equations (10) and (11) hold for $t = T$.**

We first show that at time T, transfers are independent of realization of time T shock, and then show that for time T equation (11) holds.

Equation (10) holds at $t = T$:

Suppose that (10) does not hold and let $b(\theta^T) = \min_{b \in C(b^{T-1}, y^{T-1})} b$ and $b(\theta^{T-1}, \hat{\theta}_T)$ the second smallest b. Denote by $\hat{\theta}^T = (\theta^{T-1}, \hat{\theta}_T)$. The contradiction argument relies on the incumbent firm deviating to an allocation that delivers higher profits. First note that it must be true that $y(\hat{\theta}^T) + b(\hat{\theta}^T) > y(\theta^T) + b(\theta^T)$. If not, given that $b(\hat{\theta}^T) > b(\theta^T)$ then $y(\hat{\theta}^T) < y(\theta^T)$, an entrant firm can offer the following contract $\tilde{C}_T = \{(-\varepsilon, y(\theta^T) - y(\hat{\theta}^T)); (0,0)\}$, for some ε small enough. An agent with type θ^T is better off by choosing allocation $(b(\hat{\theta}^T), y(\hat{\theta}^T))$ in menu C_T together with $(-\varepsilon, y(\theta^T) - y(\hat{\theta}^T))$ in menu $\tilde{C}_T$. With these choices, his utility is:

$$u\left(b(\hat{\theta}^T) - \varepsilon + y(\theta^T)\right) - v\left(\frac{y(\theta^T)}{\theta_T}\right) > u\left(b(\theta^T) + y(\theta^T)\right) - v\left(\frac{y(\theta^T)}{\theta_T}\right)$$

where the inequality holds as long as $b(\hat{\theta}^T) - \varepsilon > b(\theta^T)$. No latent contracts can prevent this deviation, since it is profitable for the entrant as long as some agent accepts it.[23]

[23] Note that this case arises in the solution of the constrained efficient allocation: high skilled agents work more and make positive transfers to less skilled agents. The deviation $\tilde{C}_T$ makes this allocation unprofitable in our environment, since it induces skilled agents to choose the allocation designed for low skilled agents and to work an additional amount with an entrant.

The equilibrium allocation, being optimal for the agent, must satisfy the following:

$$u(b(\hat{\theta}^T)+y(\hat{\theta}^T))-v\left(\frac{y(\hat{\theta}^T)}{\hat{\theta}_T}\right)\geq u(b(\theta^T)+y(\theta^T))-v\left(\frac{y(\theta^T)}{\hat{\theta}_T}\right), \tag{35}$$

$$u(b(\theta^T)+y(\theta^T))-v\left(\frac{y(\theta^T)}{\theta_T}\right)\geq u(b(\hat{\theta}^T)+y(\hat{\theta}^T))-v\left(\frac{y(\hat{\theta}^T)}{\theta_T}\right). \tag{36}$$

Case 1 If (35) holds with equality, an agent of type $\hat{\theta}_T$ is indifferent between his equilibrium choice and the choice of agent θ_T. However, the insurance providers receive strictly higher profits from the allocation θ_T, since by assumption $b(\hat{\theta}^T) > b(\theta^T)$. This incumbent can deviate to an alternative menu that differs from the original by offering at time T only the allocation chosen by agent θ_T. No latent contract can induce lower profits to deter this deviation, since now the deviating incumbent offers a subset of the allocations that were available in the original equilibrium. The argument also holds if the equilibrium allocation is divided between multiple insurance providers.
Case 2 Suppose that (35) holds with strict inequality. Following the argument in the previous case, for any type $\bar{\theta}_T$ such that $b\left(\bar{\theta}_T\right) > b_T(\theta^T)$, it must be true that:

$$u(b(\bar{\theta}^T)+y(\bar{\theta}^T))-v\left(\frac{y(\bar{\theta}^T)}{\bar{\theta}_T}\right)> u(b(\theta^T)+y(\theta^T))-v\left(\frac{y(\theta^T)}{\bar{\theta}_T}\right). \tag{37}$$

Otherwise, the incumbent firm will offer only the contract containing $b_T(\theta^T)$.

Consider the following deviation by an incumbent firm $\tilde{b}(\hat{\theta}^T) = b(\hat{\theta}^T) - \varepsilon$ and $\tilde{b}(\theta^T) = b(\theta^T) + \varepsilon - \delta$ for $\varepsilon, \delta > 0$ and $\varepsilon > \delta$ (to be defined explicitly below) and keeping unchanged all the other allocations.[24] This deviation reduces the spread of transfers and increases the incumbent's profit by a factor proportional to δ.

To show that such deviation is profitable, thus reaching a contradiction, we show that there is no latent contract $\alpha \equiv (\alpha_b, \alpha_y)$ that can induce a reduction in the profits of this firm. Suppose such a contract exists. One possibility is to induce θ_T agents, when faced with the deviating allocation $\tilde{b}$, to choose $\tilde{b}(\hat{\theta}^T)$. This would imply a reduction of profits, since $\tilde{b}(\hat{\theta}^T) > \tilde{b}(\theta^T)$. Such latent contract has to satisfy:

$$u(\tilde{b}(\hat{\theta}^T)+y(\hat{\theta}^T)+\alpha_b+\alpha_y)-v\left(\frac{y(\hat{\theta}^T)+\alpha_y}{\theta_T}\right)> u(\tilde{b}(\theta^T)+y(\theta^T))-v\left(\frac{y(\theta^T)}{\theta_T}\right). \tag{38}$$

Since α is not chosen in the original equilibrium, it must also be true that

$$u(b(\theta^T)+y(\theta^T))-v\left(\frac{y(\theta^T)}{\theta_T}\right)\geq u(b(\hat{\theta}^T)+y(\hat{\theta}^T)+\alpha_b+\alpha_y)-v\left(\frac{y(\hat{\theta}^T)+\alpha_y}{\theta_T}\right). \tag{39}$$

[24] If there are multiple θ with values equal to $b(\hat{\theta}^T)$ or $b(\theta^T)$, the same deviation applies to all such transfers.

However, $u(\tilde{b}(\theta^T)+y(\theta^T)) > u(b(\theta^T)+y(\theta^T))$ and $u(b(\hat{\theta}^T)+y(\hat{\theta}^T)+\alpha_b+\alpha_y) > u(\tilde{b}(\hat{\theta}^T)+y(\hat{\theta}^T)+\alpha_t+\alpha_y)$, which combined with (39) implies

$$u(\tilde{b}(\theta^T)+y(\theta^T)) - v\left(\frac{y(\theta^T)}{\theta_T}\right) > u(\tilde{b}(\hat{\theta}^T)+y(\hat{\theta}^T)+\alpha_b+\alpha_y) - v\left(\frac{y(\hat{\theta}^T)+\alpha_y}{\theta_T}\right), \quad (40)$$

contradicting (38). As before, consider any other type $\bar{\theta}_T \neq \theta_T$ with $b\left(\bar{\theta}_T\right) > b\left(\theta_T\right)$:

$$u(\tilde{b}(\theta^T)+y(\theta^T)+\alpha_b+\alpha_y) - v\left(\frac{y(\theta^T)+\alpha_y}{\bar{\theta}_T}\right) > u(b(\bar{\theta}^T)+y(\bar{\theta}^T)) - v\left(\frac{y(\bar{\theta}^T)}{\bar{\theta}_T}\right). \quad (41)$$

Since a latent contract is not chosen in the original equilibrium, it must also be true that

$$u(b(\bar{\theta}^T)+y(\bar{\theta}^T)) - v\left(\frac{y(\bar{\theta}^T)}{\bar{\theta}_T}\right) \geq u(b(\theta^T)+y(\theta^T)+\alpha_t+\alpha_y) - v\left(\frac{y(\theta^T)+\alpha_y}{\bar{\theta}_T}\right). \quad (42)$$

The previous equation must hold with equality, otherwise in the original equilibrium the deviating firm would not offer contract $b\left(\bar{\theta}_T\right)$. Let

$$\begin{aligned}\Delta(\bar{\theta}) \quad = \quad & \min_{\alpha\in C_T^{-i}} \left\{ u(b(\bar{\theta}^T)+y(\bar{\theta}^T)) - v\left(\frac{y(\bar{\theta}^T)}{\bar{\theta}_T}\right) + \right. \\ & \left. -u(b(\theta^T)+y(\theta^T)+\alpha_b+\alpha_y) + v\left(\frac{y(\theta^T)+\alpha_y}{\bar{\theta}_T}\right)\right\}. \end{aligned} \quad (43)$$

This gives the minimum utility gain agent $\bar{\theta}^T$ receives from choosing allocation $(b(\bar{\theta}^T), y(\bar{\theta}^T))$ instead of $(b(\theta^T), y(\theta^T))$ combined with any other latent contract α. Since (42) holds with strict inequality, $\Delta\left(\bar{\theta}\right)$ is strictly positive for each $\bar{\theta}$. Let $\bar{\alpha} \equiv \arg\min \Delta(\bar{\theta})$. There exists $\varepsilon\left(\bar{\theta}\right) > 0$ such that

$$\begin{aligned} & u(b(\bar{\theta}^T)+y(\bar{\theta}^T)) - v\left(\frac{y(\bar{\theta}^T)}{\bar{\theta}^T}\right) \geq \\ & u(b(\theta^T)+y(\theta^T)+\bar{\alpha}_t+\bar{\alpha}_y+\varepsilon\left(\bar{\theta}\right)) - v\left(\frac{y(\theta^T)+\bar{\alpha}_y}{\bar{\theta}^T}\right) > \\ & u(b(\theta^T)+y(\theta^T)+\bar{\alpha}_b+\bar{\alpha}_y+\varepsilon\left(\bar{\theta}\right)-\delta) - v\left(\frac{y(\theta^T)+\bar{\alpha}_y}{\bar{\theta}^T}\right). \end{aligned} \quad (44)$$

Let $\varepsilon = \min_{\bar{\theta}\neq\theta} \varepsilon\left(\bar{\theta}\right)$. Under this choice of ε, the above equation contradicts (42). Equation (44) also implies that for all $\bar{\theta} \neq \theta$, the choice following the deviation is the same as in the original equilibrium.

The last step in the proof requires checking that the time $T-1$ incentive constraints hold. This is necessary in order to leave the decision of the agents unchanged at time $T-1$. Note that for a given $\varepsilon > 0$, there exists $\delta^* > 0$ that makes the utility, calculated in time

$T-1$, of the modified contract the same as in the original contract. To see this, note that if $\delta = \varepsilon$ the change in utility of the agent is negative following the proposed deviation, while if $\delta = 0$ the utility change is positive, since the agent now faces a reduction in the spread of consumption at time T because $y(\hat{\theta}^T) + b(\hat{\theta}^T) > y(\theta^T) + b(\theta^T)$. This implies that there exists an intermediate value of δ^* such that $\varepsilon > \delta^* > 0$ so that the change is zero. Hence, the time $T-1$ decision will be unchanged if $\delta = \delta^*$.

Equation (11) holds at time $t = T$:

Lemma 1 implies that there is only one case left to consider. Suppose that for some $\theta^T = (\theta^{T-1}, \theta_T)$

$$u'(b(\theta^{T-1}) + y(\theta^T)) < v'\left(\frac{y\left(\theta^T\right)}{\theta_T}\right)\frac{1}{\theta_T}. \tag{45}$$

In this case, the agent would like to consume and work less than the equilibrium contract. A deviation that reduces the total output and consumption by agent θ^T cannot be provided by an entrant, since a worker cannot deliver negative hours. However, an incumbent firm will find it optimal to deviate from the equilibrium contract, offering an allocation with lower consumption and lower output requirement and making strictly positive profits. Formally, it offers the original contract at all time $t < T$ and at time T, a menu that contains a null contract, the modified allocation chosen by θ^T, and the original allocation chosen by the remaining types:

$$C_T\left(b(\theta^{T-1}), y(\theta^{T-1})\right) = \\ \left\{\left(b(\theta^T) + y(\theta^T) + \delta^*(\varepsilon|\theta_T) - \varepsilon, y(\theta^T) + \delta^*(\varepsilon|\theta_T)\right); (0,0); \left(y(\hat{\theta}^T) + b(\hat{\theta}^T), y(\hat{\theta}^T)\right)\ \hat{\theta^T} \neq \theta^T\right\}$$

where δ^* and ε are constructed in a similar fashion to the proof of Lemma 1, with the constraint $\delta \leq 0$.

With this deviation, the incumbent makes strictly positive profits, proportional to ε, and there exists ε so that agents' utility is unchanged following this deviation. This guarantees that no deviation at time $T-1$ takes place. This contract is always profitable for the incumbent even if another type $\tilde{\theta}_T$ accepts it. If an agent with type $\tilde{\theta}^T$ is able to choose the pair $(b(\theta^T) + y(\theta^T) + \delta^*(\varepsilon|\theta_T) - \varepsilon, y(\theta^T) + \delta^*(\varepsilon|\theta_T))$ at time T, it implies that he must also have chosen the allocation sequence $\{(b(\theta^n) + y(\theta^n), y(\theta^n))\}_{n=1}^{T-1}$ in previous periods. From the previous step in the proposition, transfers from any history are independent of time T; i.e., this agent will receive transfers with the same net present value as in the original choice. Hence, the deviation is profitable.

2. **Equations (10) and (11) hold for $t < T$.**

As an inductive assumption, suppose (10) holds for $t+1$. We now show it holds for period t. Rewrite the net present value of transfers as:

$$A_t(\theta^{t-1},\theta_{t-1}^T) = \sum_{n=t}^{T}\left(\frac{1}{q}\right)^{t-n} b_n(\theta^{t-1},\theta_{t-1}^n) =$$
$$b_t(\theta^{t-1},\theta_t) + q\sum_{n=t+1}^{T}\left(\frac{1}{q}\right)^{t+1-n} b_n(\theta^{t-1},\theta_{t-1}^n) = b_t(\theta^{t-1},\theta_t) + qA_{t+1}(\theta^t,\theta_t^T).$$

By way of contradiction, there exist θ_t and $\hat{\theta}_t$ following history θ^{t-1} such that

$$b_t(\theta^{t-1},\theta_t) + qA_{t+1}(\theta^{t-1},\theta_t) < b_t(\theta^{t-1},\hat{\theta}_t) + qA_{t+1}(\theta^{t-1},\hat{\theta}_t). \tag{46}$$

By the inductive assumption $b_t(\theta^{t-1},\theta_t) < b_t(\theta^{t-1},\hat{\theta}_t)$. As in the proof for time T, the contradiction argument relies on deviations by entrants to guarantee that (11) holds and on deviations by entrant and incumbent firms to imply that the net present value of transfers is zero.

Under the inductive assumption, the agent faces no distortion on both his intratemporal margin and inter-temporal margin (recall Lemma 2) from time $t+1$ onward. This implies that the equilibrium allocation from time $t+1$ onwards is equivalent to a self-insurance economy (this will be formally proved in Proposition 2). Let $S_{t+1}(x)$ be the utility the agent receives from entering time $t+1$ with a level x of net present value of assets. The value function S is monotonically increasing in the level of assets. Given this, the agents' equilibrium choices at time t satisfy the following:

$$u(b(\hat{\theta}^t)+y(\hat{\theta}^t)) - v\left(\frac{y(\hat{\theta}^t)}{\hat{\theta}_t}\right) + \beta S_{t+1}\left(qA_{t+1}(\hat{\theta}^t) - b(\hat{\theta}^t)\right) \geq$$
$$u(b(\theta^t)+y(\theta^t)) - v\left(\frac{y(\theta^t)}{\hat{\theta}_t}\right) + \beta S_{t+1}\left(qA_{t+1}(\theta^t) - b(\theta^t)\right), \tag{47}$$

and

$$u(b(\theta^t)+y(\theta^t)) - v\left(\frac{y(\theta^t)}{\theta_t}\right) + \beta S_{t+1}\left(qA_{t+1}(\theta^t) - b(\theta^t)\right) \geq$$
$$u(b(\hat{\theta}^t)+y(\hat{\theta}^t)) - v\left(\frac{y(\hat{\theta}^t)}{\theta_t}\right) + \beta S_{t+1}\left(qA_{t+1}(\hat{\theta}^t) - b(\hat{\theta}^t)\right). \tag{48}$$

If $y(\theta^t) \geq y(\hat{\theta}^t)$, an entrant can offer the following menu that enables the agent to work additional hours and move resources between time t and time $t+1$:

$$\begin{aligned} \tilde{C}_t &= \left\{ \left(b(\theta^t) - b(\hat{\theta}^t), y(\theta^t) - y(\hat{\theta}^t) \right); (0,0) \right\}, \\ \tilde{C}_{t+1} &= \left\{ \left(-\frac{1}{q}[b(\theta^t) - b(\hat{\theta}^t)] - \varepsilon, 0 \right) \right\}. \end{aligned}$$

This menu generates strictly positive profits to the entrant, proportional to ε. If this menu is offered, agent θ^t will deviate, accepting the allocation for $\hat{\theta}^t$ together with the allocation specified in the entrant's menu. This is due to the fact that the agent can now replicate his original time t level of output and have access to a strictly higher net present value of transfers at a cost equal to ε.

Suppose now that $y(\theta^t) < y(\hat{\theta}^t)$. The first case we consider is when consumption at time t is higher for the agent with a higher net present value of transfer, $y(\theta^t) + b(\theta^t) < y(\hat{\theta}^t) + b(\hat{\theta}^t)$. As in the argument for period T, inequality (47) cannot hold with equality. This enables us to reduce the time t spread of consumption between histories θ^t and $\hat{\theta}^t$. Following the same steps of time T, a contradiction can be reached.

The final case is $y(\theta^t) < y(\hat{\theta}^t)$ and $y(\theta^t) + b(\theta^t) \geq y(\hat{\theta}^t) + b(\hat{\theta}^t)$. This case violates the inter-temporal Euler equation for at least one of the two types, thus contradicting Lemma 2. To see this, suppose that the Euler equation (9) holds for agent θ^t. We have

$$\begin{aligned} & u'(y(\theta^t) + b(\theta^t)) = \frac{\beta}{q} \sum_{\theta_{t+1}} \pi(\theta_{t+1}) u'(c(\theta^{t+1})) \\ \Rightarrow \;& u'(y(\hat{\theta}^t) + b(\hat{\theta}^t)) \geq \frac{\beta}{q} \sum_{\theta_{t+1}} \pi(\theta_{t+1}) u'(c(\theta^{t+1})) \\ \Rightarrow \;& u'(y(\hat{\theta}^t) + b(\hat{\theta}^t)) > \frac{\beta}{q} \sum_{\theta_{t+1}} \pi(\theta_{t+1}) u'(c(\hat{\theta}^t, \theta_{t+1})), \end{aligned}$$

where the last implication follows from the fact that an agent with higher transfer will have higher consumption at time $t+1$, thus a lower expected marginal utility of consumption.

To conclude, given that it was shown that the net present value of transfers is independent of the time t choice, we can follow the same steps as in time T to show that equation (45) holds for time t. □

C Proof of Lemma 4

Proof.

1. If $g = 1$, we have:[25]

$$G(1) = \max_{y_\theta, x_\theta} \sum_\theta \pi(\theta) u\Big(- b + b(\theta) - E_\theta - b^*_\theta + x_\theta + y_\theta \Big) - v\left(\frac{y_\theta}{\theta}\right) + \beta \tilde{W}(x_\theta).$$

A feasible choice at at time 1 is: $y_\theta = y(\theta) + \hat{y}(\theta)$ and $x_\theta = \hat{x}(\theta) + E_\theta + b^*_\theta$. So:

$$G(1) \geq \sum_\theta \pi(\theta) u\Big(-b + b(\theta) + \hat{x}(\theta) + y(\theta) + \hat{y}(\theta) \Big) - v\left(\frac{y(\theta) + \hat{y}(\theta)}{\theta}\right) + \beta \tilde{W}(\hat{x}(\theta) + E_\theta + b^*_\theta).$$

By definition of b^*_θ:

$$G(1) \geq \sum_\theta \pi(\theta) u\Big(-b + b(\theta) + \hat{x}(\theta) + y(\theta) + \hat{y}(\theta) \Big) - v\left(\frac{y(\theta) + \hat{y}(\theta)}{\theta}\right) + \beta W(\hat{x}(\theta) + E_\theta, \theta) \geq \overline{W}.$$

If $g = 0$, we have:

$$G(0) = \max_{y_\theta, x_\theta} \sum_\theta \pi(\theta) u\Big(- b + x_\theta + y_\theta + E \Big) - v\left(\frac{y_\theta}{\theta}\right) + \beta \tilde{W}(x_\theta) = \hat{W}.$$

2. Consider the total derivative of G relative to the scale parameter g. Using the optimality condition for x_θ, y_θ, we have:

$$G'(g) \equiv \frac{dG(g)}{dg} = \sum_{\theta \in \Theta} \pi(\theta)[b(\theta) - E_\theta - b^*_\theta - E] u'\Big(-b + g \cdot (b(\theta) - E_\theta - b^*_\theta - E) + x_\theta + y_\theta + E \Big). \tag{49}$$

Define for each $\theta \in \Theta$, $A(\theta) = b(\theta) - E_\theta - b^*_\theta - E$.

Standard arguments on consumption for incomplete markets (consumption increasing in assets) imply

$$u'\Big(g \cdot A(\theta) - b + x^*_\theta + y^*_\theta + E\Big) \leq u'\Big(- b + x_\theta + y_\theta + E\Big), \text{ if } A(\theta) \geq 0,$$
$$u'\Big(g \cdot A(\theta) - b + x^*_\theta + y^*_\theta + E\Big) > u'\Big(- b + x_\theta + y_\theta + E\Big), \text{ if } A(\theta) < 0.$$

Where $x^*_\theta + y^*_\theta$ denotes the optimal choice given additional resources equal to $g \cdot A(\theta)$, the above equations imply that $G'(g) \leq G'(0)$ for all $g > 0$.

The above does not rule out the case of $G(1) = G(0)$. This could arise if the entrant offers a contract such that $b^*_\theta = 0$ for all θ together with $b(\theta) - E_\theta = E$ for all θ.

[25] This proof applies for any finite number of periods. For $T \geq 3$ the variable b is the relevant state variable (and will have a non-zero value). For $T = 2$ it is equal to zero in the definition of the function G.

3.
$$G'(0) = \sum_{\theta} \pi(\theta)[b(\theta) - E_{\theta} - b^*_{\theta} - E]u'(-b + x_{\theta} + y_{\theta} + E). \tag{50}$$

From standard arguments, $-b+x_{\theta}+y_{\theta}+E$ is increasing in θ so that $u'(-b+x_{\theta}+y_{\theta}+E)$ is decreasing in θ.

We now show for all $\theta > \tilde{\theta}$, $\quad b(\theta)-E_{\theta}-b^*_{\theta} \geq b(\tilde{\theta})-E_{\tilde{\theta}}-b^*_{\tilde{\theta}}$. Suppose not: $b(\theta)-E_{\theta}-b^*_{\theta} < b(\tilde{\theta})-E_{\tilde{\theta}}-b^*_{\tilde{\theta}}$. Since the allocation $(b(\theta), y(\theta))$ was chosen in equilibrium, it must satisfy:

$$\begin{aligned}
&u\left(-b + b(\theta) + \hat{x}(\theta) + y(\theta) + \hat{y}(\theta)\right) - v\left(\frac{y(\theta) + \hat{y}(\theta)}{\theta}\right) + \beta W(\hat{x}(\theta) + E_{\theta}, \theta) \geq \\
&u\left(-b + b(\tilde{\theta}) + \hat{x}(\tilde{\theta}) + y(\tilde{\theta}) + \hat{y}(\tilde{\theta}) + \delta_x + \delta_y\right) - v\left(\frac{y(\tilde{\theta}) + \hat{y}(\tilde{\theta}) + \delta_y}{\theta}\right) + \\
&+\beta W(\hat{x}(\tilde{\theta}) + E_{\tilde{\theta}} + \delta_x, \tilde{\theta})
\end{aligned}$$

for any feasible δ_x and δ_y. In particular, let $\delta_y = y(\theta) + \hat{y}(\theta) - y(\tilde{\theta}) - \hat{y}(\tilde{\theta})$ and $\delta_x = b(\theta) + \hat{x}(\theta) - b(\tilde{\theta}) - \hat{x}(\tilde{\theta})$. Lemma 3 implies that $\delta_y \geq 0$, and hence is feasible. Using this definition, the above becomes:

$$\begin{aligned}
&u\left(-b + b(\theta) + \hat{x}(\theta) + y(\theta) + \hat{y}(\theta)\right) - v\left(\frac{y(\theta) + \hat{y}(\theta)}{\theta}\right) + \beta W(\hat{x}(\theta) + E_{\theta}, \theta) \geq \\
&u\left(-b + b(\theta) + \hat{x}(\theta) + y(\theta) + \hat{y}(\theta)\right) - v\left(\frac{y(\theta) + \hat{y}(\theta)}{\theta}\right) + \beta W(E_{\tilde{\theta}} + b(\theta) + \hat{x}(\theta) - b(\tilde{\theta}), \tilde{\theta}) \Rightarrow \\
&W(\hat{x}(\theta) + E_{\theta}, \theta) \geq W(E_{\tilde{\theta}} + b(\theta) + \hat{x}(\theta) - b(\tilde{\theta}), \tilde{\theta}).
\end{aligned}$$

Note that
$$\tilde{W}(\hat{x}(\theta) + E_{\theta} + b^*_{\theta}) \geq \tilde{W}(E_{\tilde{\theta}} + b(\theta) + \hat{x}(\theta) - b(\tilde{\theta}) + b^*_{\tilde{\theta}}).$$

Since $\tilde{W}$ is strictly decreasing, then

$$\hat{x}(\theta) + E_{\theta} + b^*_{\theta} \leq E_{\tilde{\theta}} + b(\theta) + \hat{x}(\theta) - b(\tilde{\theta}) + b^*_{\tilde{\theta}},$$

which is a contradiction. Finally since $b(\theta) - E_{\theta} - b^*_{\theta} - E$ is increasing in θ and also

$$\sum_{\theta} \pi(\theta)[b(\theta) - E_{\theta} - b^*_{\theta} - E] < 0,$$

(since $b^*_{\theta} \geq 0$ for all θ) we find that $G'(0) < 0$.[26]

□

[26] An heuristic argument to show this is to rewrite (50) as $\sum_{\theta} \pi(\theta)A(\theta)B(\theta) = E_{\theta}[A(\theta)]E_{\theta}[B(\theta)] + cov[A(\theta)][B(\theta)]$.