

## Preference Signaling in Matching Markets

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ONLINE APPENDIX

### B1. Proofs of supplementary results.

This subsection proves Theorem A1, Propositions A1, A2, and Lemma A1.

**Proof of Proposition A1** (Binary nature of firm optimal offer). Consider firm  $f$  from some block  $\mathcal{F}_b$ ,  $b \in \{1, \dots, B\}$  with realized preference profile  $\theta^* \in \Theta_f$  and that receives signals from the set of workers  $\mathcal{W}^S \subset \mathcal{W} \cup \mathcal{N}$ . Denote worker  $S_f$  as  $w$  and select arbitrary another worker  $w' \in \mathcal{W}^S$ . We first prove that the expected payoff to  $f$  from making an offer to  $w$  is strictly greater than the expected payoff from making an offer to  $w'$ . We denote the strategies of firm  $f$  that correspond to these actions as  $\sigma_f(\theta^*, \mathcal{W}^S) = w$  and  $\sigma'_f(\theta^*, \mathcal{W}^S) = w'$ .

Workers use symmetric best-in-block strategies and firms have best-in-block beliefs. Specifically, firm  $f$  believes that it is the top firm within block  $\mathcal{F}_b$  in the preference lists of workers  $w$  and  $w'$ . Denote the set of all possible agents' profiles consistent with firm  $f$  beliefs as<sup>28</sup>

$$\bar{\Theta} \equiv \{\theta \in \Theta \mid \theta_f = \theta^* \text{ and } f = \max_{\theta_w}(f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^S\}$$

As in the proof of Proposition 1, we denote a permutation that changes the ranks of  $w$  and  $w'$  in a firm preference list as  $\rho : (\dots, w, \dots, w', \dots) \rightarrow (\dots, w', \dots, w, \dots)$  and construct preference profile  $\theta' \in \Theta$  from  $\theta^*$  as follows:

- firm  $f$ 's preferences are the same as in  $\theta^*$ :  $\theta'_f = \theta^*$ ,
- workers  $w$  and  $w'$  are exchanged in the preference lists of firms  $-f : \forall f' \in -f$ , we have  $\theta'_{f'} = \rho(\theta_{f'})$ ,
- workers  $w$  and  $w'$  preference profiles are exchanged  $\theta'_w = \theta_{w'}$ ,  $\theta'_{w'} = \theta_w$ ,
- for any other  $w^0 \in \mathcal{W} \setminus \{w, w'\}$ ,  $\theta_{w^0} = \theta'_{w^0}$ .

Since firm  $f$ 's preference list is unchanged and since  $w, w' \in \mathcal{W}^S$ , profile  $\theta'$  belongs to  $\bar{\Theta}$ . Since strategies of firms  $-f$  are anonymous, then for any  $f' \in -f$  and for any  $\mathcal{W}_{f'}^S \subset \mathcal{W} \cup \mathcal{N}$  we have

$$\sigma_{f'}(\rho(\theta_{f'}), \rho(\mathcal{W}_{f'}^S)) = \rho(\sigma_{f'}(\theta_{f'}, \mathcal{W}_{f'}^S))$$

<sup>28</sup>For the case of one block of firms, firm  $f$  beliefs also exclude preference profiles where firm  $f$  is a top firm for those workers that did not send signal to firm  $f$ . That is,  $\bar{\Theta} \equiv \{\theta \in \Theta \mid \theta_f = \theta^*, f = \max_{\theta_w}(f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^S, \text{ and } f \neq \max_{\theta_w}(f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W} \setminus \mathcal{W}^S\}$ . For simplicity, we assume that there are at least two blocks. All the derivations are also valid without change for the case of one block.

Worker  $w$  and  $w'$  send their signals to firm  $f$  under both profile  $\theta$  and  $\theta'$ . Therefore, they do not send their signals to firms  $-f$ , i.e.  $\rho(\mathcal{W}_{f'}^S) = \mathcal{W}_{f'}^S$ . Since  $\theta'_f = \rho(\theta_f)$  we have

$$\sigma_{f'}(\theta'_{f'}, \mathcal{W}_{f'}^S) = \rho(\sigma_{f'}(\theta_{f'}, \mathcal{W}_{f'}^S))$$

This means that the probability of firm  $f'$  making an offer to worker  $w$  for profile  $\theta$  equals the probability of making an offer to worker  $w'$  for profile  $\theta'$ . Moreover, since we exchange worker  $w$  and  $w'$  preference lists for profile  $\theta'$ , whenever it is optimal for worker  $w$  to accept firm  $f'$ 's offer under profile  $\theta$ , it is optimal for worker  $w'$  to accept an offer from firm  $f'$  under profile  $\theta'$ . Since firm types are independent, the probability of firm  $f$  being matched when it uses strategy  $\sigma_f$  for profile  $\theta$  equals the probability of firm  $f$  being matched when it uses strategy  $\sigma'_f$  for profile  $\theta'$ , i.e.  $m_f(\sigma_f, \sigma_{-f}, \theta) = m_f(\sigma'_f, \sigma_{-f}, \theta')$ .

Therefore, for each  $\theta \in \bar{\Theta}$  there exists  $\theta' \in \bar{\Theta}$  such that the probability that firm  $f$  gets an offer from worker  $w$  equals the probability that firm  $f$  gets an offer from worker  $w'$ . Moreover, profile  $\theta'$  is different for different  $\theta$  by our construction. Since  $\theta$  and  $\theta'$  are equally likely,

$$\mathbb{E}_{\theta} m_f(\sigma_f, \sigma_{-f}, \theta \mid \theta \in \bar{\Theta}) = \mathbb{E}_{\theta} m_f(\sigma'_f, \sigma_{-f}, \theta \mid \theta \in \bar{\Theta}).$$

Therefore, the expected probability that firm  $f$  gets a match if it makes an offer to some worker in  $\mathcal{W}^S$  is the same across all workers in  $\mathcal{W}^S$ . But within this set, a match with  $S_f$  offers the greatest utility, so the expected payoff to  $f$  from making an offer to  $S_f$  is strictly greater than the payoff from making an offer to any other worker in  $\mathcal{W}^S$ .

A similar construction is valid for the workers in set  $\mathcal{W} \setminus \mathcal{W}^S$ . That is, the probability that firm  $f$ 's offer is accepted is the same across all workers in  $\mathcal{W} \setminus \mathcal{W}^S$ . Hence, firm  $f$  prefers making an offer to its most valuable worker,  $T_f$ , than to any other worker in  $\mathcal{W} \setminus \mathcal{W}^S$ .<sup>29</sup>

**Proof of Proposition A2** (Optimality of Cutoff Strategies). If workers use best-in-block strategies and firms have best-in-block beliefs, the optimal choice of firm  $f$  for each set of received signals is either  $S_f$  or  $T_f$  (or some lottery between them) (see Proposition A1). In light of this, we break the proof into two parts. First we show that the identities of workers that have sent a signal to firm  $f$  influence neither the expected payoff of making an offer to  $S_f$  nor the expected payoff of making an offer to  $T_f$ , conditional on the total number of signals received by  $f$  remaining constant. Second we prove that if it is optimal for firm  $f$  to choose  $S_f$  when it receives signals from some set of workers, then it still optimal for firm

<sup>29</sup>If  $T_f = S_f$  the statement of the proposition is still valid. Firm  $f$  believes that it is  $T_f$ 's top firm within block  $\mathcal{F}_b$  and firm  $f$  prefers making an offer to  $T_f = S_f$  rather than to any other worker in  $\mathcal{W}$ .

$f$  to choose  $S_f$  if the number of received signals does not change and  $S_f$  has a smaller rank ( $S_f$  is more valuable to  $f$ ).

Let us consider some firm  $f$  from block  $\mathcal{F}_b$ ,  $b \in \{1, \dots, B\}$  and some realization  $\theta^*$  of its preference list. Assume that it is optimal for  $f$  to make an offer to  $S_f$  if it receives a set of signals  $\mathcal{W}^S \subset \mathcal{W} \cup \mathcal{N}$ . We want to show that if  $f$  receives the set of signals  $\mathcal{W}^{S'}$  such that  $S_f(\theta^*, \mathcal{W}^S) = S_f(\theta^*, \mathcal{W}^{S'})$  and  $|\mathcal{W}^{S'}| = |\mathcal{W}^S|$ , it is still optimal for  $f$  to make an offer to  $S_f$ . For simplicity, we only consider the case when  $\mathcal{W}^S$  and  $\mathcal{W}^{S'}$  differ only in one signal. (The general case then follows straightforwardly.) That is, there exist workers  $w$  and  $w'$  such that  $w$  belongs to set  $\mathcal{W}^S$ , but not to set  $\mathcal{W}^{S'}$ ; while  $w'$  belongs to  $\mathcal{W}^{S'}$ , but not to  $\mathcal{W}^S$ . We consider two firm  $f$ 's strategies for realization of signals  $\mathcal{W}^S$  and  $\mathcal{W}^{S'}$ .

$$\sigma_f(\theta^*, \cdot) = S_f(\theta^*, \cdot), \quad \sigma'_f(\theta^*, \cdot) = T_f(\theta^*, \cdot).$$

We denote the set of possible agents' profiles that are consistent with firm  $f$  having received signals from  $\mathcal{W}^S$  and  $\mathcal{W}^{S'}$  as<sup>30</sup>

$$\begin{aligned} \bar{\Theta}^S &\equiv \{\theta \in \Theta \mid \theta_f = \theta^* \text{ and } f = \max_{\theta_w} (f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^S\} \\ \bar{\Theta}^{S'} &\equiv \{\theta \in \Theta \mid \theta_f = \theta^* \text{ and } f = \max_{\theta_w} (f' \in \mathcal{F}_{b'}) \text{ for each } w \in \mathcal{W}^{S'}\} \end{aligned}$$

respectively. We now construct a bijection between  $\bar{\Theta}^S$  and  $\bar{\Theta}^{S'}$ . Denote a permutation that changes the ranks of  $w$  and  $w'$  in a firm preference profile as  $\rho : (\dots, w, \dots, w', \dots) \rightarrow (\dots, w', \dots, w, \dots)$ . For any profile  $\theta \in \bar{\Theta}^S$  we construct profile  $\theta' \in \Theta$  as follows:

- firm  $f$  preferences are the same as in  $\theta$ :  $\theta'_f = \theta^*$ ,
- the ranks of workers  $w$  and  $w'$  are exchanged in the preference lists of firms  $-f$ :  $\forall f' \in -f, \theta'_{f'} = \rho(\theta_{f'})$ ,
- the preference lists of workers  $w$  and  $w'$  are exchanged:  $\theta'_w = \theta_{w'}, \theta'_{w'} = \theta_w$ ,
- for any other  $w^0 \in \mathcal{W} \setminus \{w, w'\}$ ,  $\theta_{w^0} = \theta'_{w^0}$ .

Since this construction leaves the preference list of firm  $f$  unchanged, and since workers  $w$  and  $w'$  swap preference lists, we have that if  $\theta \in \bar{\Theta}^S$ , then  $\theta' \in \bar{\Theta}^{S'}$ . By construction, profile  $\theta'$  is different for different  $\theta$ . Finally, since the cardinality of sets  $\bar{\Theta}^S$  and  $\bar{\Theta}^{S'}$  are the same, the above correspondence is a bijection.

Since firm  $-f$  strategies are anonymous, for any  $f' \in -f$  and  $\mathcal{W}_{f'}^S \subset \mathcal{W} \cup \mathcal{N}$

$$\sigma_{f'}(\rho(\theta_{f'}), \rho(\mathcal{W}_{f'}^S)) = \rho(\sigma_{f'}(\theta_{f'}, \mathcal{W}_{f'}^S)).$$

<sup>30</sup>See footnote 28 for the definition of firm beliefs for the case of one block.

This means that the probability of firm  $f'$  making an offer to worker  $w$  for any profile  $\theta$  equals the probability of firm  $f'$  making an offer to worker  $w'$  for corresponding profile  $\theta'$ . Moreover, since we exchange worker  $w$  and  $w'$  preference lists for profile  $\theta'$ , whenever it is optimal for worker  $w$  to accept firm  $f$  offer for profile  $\theta$ , it is optimal for worker  $w'$  to accept firm  $f'$ 's offer for profile  $\theta'$ . Since firms types are independent, the probability of firm  $f$  being matched when it uses strategy  $\sigma_f(\theta^*, \cdot)$  for profile  $\theta$  equals the probability of firm  $f$  being matched when it uses strategy  $\sigma_f(\theta^*, \cdot)$  for profile  $\theta'$ :

$$m_f(\sigma_f, \sigma_{-f}, \theta) = m_f(\sigma_f, \sigma_{-f}, \theta').$$

Similarly, for strategy  $\sigma'_f(\theta^*, \cdot)$  we have  $m_f(\sigma'_f, \sigma_{-f}, \theta) = m_f(\sigma'_f, \sigma_{-f}, \theta')$ . Since our construction is a bijection between  $\bar{\Theta}^S$  and  $\bar{\Theta}^{S'}$ , and since  $\theta$  and  $\theta'$  are equally likely, we have

$$\begin{aligned} \mathbb{E}_\theta m_f(\sigma_f, \sigma_{-f}, \theta \mid \theta \in \bar{\Theta}^S) &= \mathbb{E}_\theta m_f(\sigma_f, \sigma_{-f}, \theta' \mid \theta' \in \bar{\Theta}^{S'}) \\ \mathbb{E}_\theta m_f(\sigma'_f, \sigma_{-f}, \theta \mid \theta \in \bar{\Theta}^S) &= \mathbb{E}_\theta m_f(\sigma'_f, \sigma_{-f}, \theta' \mid \theta' \in \bar{\Theta}^{S'}). \end{aligned}$$

Therefore, if firm  $f$  optimally makes an offer to  $S_f(T_f)$  when it has received set of signals  $\mathcal{W}^S$ , it also should optimally make an offer to  $S_f(T_f)$ , which is the same worker, for the set of signals  $\mathcal{W}^{S'}$ .

We now prove that if firm  $f$  optimally chooses  $S_f(\theta^*, \mathcal{W}^S)$  when it receives signals from  $\mathcal{W}^S$ , then it should still optimally choose  $S_f(\theta^*, \mathcal{W}^{S'})$  for set of signals  $\mathcal{W}^{S'}$ , if the number of received signals is the same  $|\mathcal{W}^{S'}| = |\mathcal{W}^S|$  and  $S_f(\theta^*, \mathcal{W}^{S'})$  has a smaller rank, that is, when the signaling worker is more valuable to  $f$ . We consider set  $\mathcal{W}^{S'}$  that differs from  $\mathcal{W}^S$  only in the best (for firm  $f$ ) worker and the difference between the ranks of top signaled workers equals one. (The general case follows straightforwardly.) That is,

$$\begin{aligned} w \in \mathcal{W}^S / S_f(\theta^*, \mathcal{W}^S) &\Leftrightarrow w \in \mathcal{W}^{S'} / S_f(\theta^*, \mathcal{W}^{S'}) \quad \text{and} \\ \text{rank}_f(S_f(\theta^*, \mathcal{W}^{S'})) &= \text{rank}_f(S_f(\theta^*, \mathcal{W}^S)) - 1. \end{aligned}$$

The construction in the first part of the proof works again in this case. Using sets of profiles and a correspondence similar to the one above, we can show that the probabilities of firm  $f$  being matched with  $S_f(T_f)$  are the same for  $\mathcal{W}^S$  and  $\mathcal{W}^{S'}$ . Observe that if firm  $f$ 's offer to  $T_f$  is accepted, naturally firm  $f$  gets the same payoff for sets  $\mathcal{W}^S$  and  $\mathcal{W}^{S'}$ . If firm  $f$ 's offer to  $S_f$  is accepted, firm  $f$  gets strictly greater payoff for set  $\mathcal{W}^{S'}$  compared to set  $\mathcal{W}^S$ , because by definition  $S_f(\theta^*, \mathcal{W}^{S'})$  has smaller rank than  $S_f(\theta^*, \mathcal{W}^S)$ . Hence, if it is optimal for firm  $f$  to make an offer to  $S_f(\theta^*, \mathcal{W}^S)$  when it receives set of signals  $\mathcal{W}^S$ , it is optimal for firm  $f$  to make an offer to  $S_f(\theta^*, \mathcal{W}^{S'})$  when firm  $f$  receives set of signals  $\mathcal{W}^{S'}$ .

Combined, the two statements we have just proved allow us to conclude that if firms  $-f$  use anonymous strategies, firm  $f$ 's optimal strategy can be represented as some cutoff strategy.<sup>31</sup>

**Proof of Theorem A1.** As discussed in Section III, in any symmetric nonbabbling equilibrium each worker sends its signal to its most preferred firm. Consequently, all information sets for firms are realized with positive probability, so firm beliefs are determined by Bayes' Law: if a firm receives a signal from a worker, it believes that worker ranks the firm first in its preference list. We now take these worker strategies and firm beliefs as fixed, and analyze the second stage of the game when firms choose offers. We will show that this reduced game is a super-modular game, and then use the results of Milgrom and Roberts (1990) to prove our theorem.

We analyze the game where we restrict firm strategies to be cutoff strategies. Denote the set of cutoff strategy profiles as  $\Sigma_{cut}$ , with typical element  $\sigma = (\sigma_{f_1}, \dots, \sigma_{f_F})$ . Recall that a cutoff strategy for firm  $f$  is a vector  $\sigma_f = (j_f^1, \dots, j_f^W)$  where  $j_f^k$  corresponds to the cutoff when firm  $f$  receives  $k$  signals. We will consider only strategies where each cutoff is a natural number, i.e.  $j_f^k \in \{1, \dots, W\}$ . As defined on p.16, vector comparison yields a natural partial order on  $\Sigma_{cut}$ :  $\sigma \geq_{\Sigma_{cut}} \sigma' \Leftrightarrow \sigma_f \geq \sigma'_f \Leftrightarrow j_f^k \geq (j'_f)^k$  for any  $f \in \mathcal{F}$  and  $k \in \{1, \dots, W\}$ . This partial order is reflexive, antisymmetric, and transitive.

To show the second stage is a game with strategic complementarities, we need to verify that  $\mathbb{E}_{\theta}(\pi_f(\sigma_f, \sigma_{-f}, \theta))$  is super-modular in  $\sigma_f$  and  $\mathbb{E}_{\theta}(\pi_f(\sigma_f, \sigma_{-f}, \theta))$  has increasing differences in  $\sigma_f$  and  $\sigma_{-f}$ . The former is trivially true because when  $f$  shifts one of its cutoff vector components, this does not influence the change in payoff from a shift of another cutoff vector component. Namely, if we consider  $\sigma_f^1 = (\dots, j_l, \dots, j_k, \dots)$ ,  $\sigma_f^2 = (\dots, j'_l, \dots, j_k, \dots)$ ,  $\sigma_f^3 = (\dots, j_l, \dots, j'_k, \dots)$ , and  $\sigma_f^4 = (\dots, j'_l, \dots, j'_k, \dots)$  for some  $l, k \in \{1, \dots, W\}$ , then

$$\mathbb{E}_{\theta}(\pi_f(\sigma_f^1, \sigma_{-f}, \theta)) - \mathbb{E}_{\theta}(\pi_f(\sigma_f^2, \sigma_{-f}, \theta)) = \mathbb{E}_{\theta}(\pi_f(\sigma_f^3, \sigma_{-f}, \theta)) - \mathbb{E}_{\theta}(\pi_f(\sigma_f^4, \sigma_{-f}, \theta))$$

That  $\mathbb{E}_{\theta}(\pi_f(\sigma_f, \sigma_{-f}, \theta))$  has increasing differences in  $\sigma_f$  and  $\sigma_{-f}$  follows from Proposition 3. Namely, for any  $\sigma_f, \sigma_{-f}, \sigma'_f$ , and  $\sigma'_{-f}$  such that  $\sigma'_f \geq \sigma_f$  and  $\sigma'_{-f} \geq \sigma_{-f}$  we have

$$\mathbb{E}_{\theta}(\pi_f(\sigma'_f, \sigma'_{-f}, \theta)) - \mathbb{E}_{\theta}(\pi_f(\sigma_f, \sigma'_{-f}, \theta)) \geq \mathbb{E}_{\theta}(\pi_f(\sigma'_f, \sigma_{-f}, \theta)) - \mathbb{E}_{\theta}(\pi_f(\sigma_f, \sigma_{-f}, \theta))$$

Hence the second stage of the game, when firms choose their strategies, is a game

<sup>31</sup>Note that there can be other optimal strategies. If firm  $f$  is indifferent between making an offer to  $S_f$  and making an offer to  $T_f$  for some set of signals, firm  $f$  could optimally make its offer to  $S_f$  or to  $T_f$  for any set of signals conditional on maintaining the same rank of the most preferred signaling worker and cardinality of signals received.

with strategic complementarities. Since in our model firms are ex-ante symmetric, Theorem 5 of Milgrom and Roberts (1990) establishes the existence of largest and smallest symmetric pure strategy equilibria.

**Proof of Lemma A1.** We first prove the statement regarding the expected number of matches. Consider firm  $f$  cutoff strategies  $\sigma_f$  and  $\sigma'_f$ , such that  $\sigma'_f$  has weakly greater cutoffs, and define two sets of preference profiles as

$$\begin{aligned}\bar{\Theta}_+ &\equiv \{\boldsymbol{\theta} \in \Theta \mid m(\sigma_f, \sigma_{-f}, \boldsymbol{\theta}) < m(\sigma'_f, \sigma_{-f}, \boldsymbol{\theta})\} \\ \bar{\Theta}_- &\equiv \{\boldsymbol{\theta} \in \Theta \mid m(\sigma_f, \sigma_{-f}, \boldsymbol{\theta}) > m(\sigma'_f, \sigma_{-f}, \boldsymbol{\theta})\}.\end{aligned}$$

For each profile  $\boldsymbol{\theta}$  from set  $\Theta^+$ , it must be the case that without firm  $f$ 's offer,  $T_f$  has an offer from another firm and worker  $S_f$  does not:

$$(B1) \quad m(\sigma'_f, \sigma_{-f}, \boldsymbol{\theta}) - m(\sigma_f, \sigma_{-f}, \boldsymbol{\theta}) = 1.$$

Similarly, if profile  $\boldsymbol{\theta}$  is from set  $\Theta^-$ , it must be the case that without firm  $f$  offer,  $S_f$  has an offer from another firm, and  $T_f$  does not

$$(B2) \quad m(\sigma'_f, \sigma_{-f}, \boldsymbol{\theta}) - m(\sigma_f, \sigma_{-f}, \boldsymbol{\theta}) = -1.$$

We now show that  $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$ . Equations (B1) and (B2), along with the fact that each  $\boldsymbol{\theta} \in \Theta^+ \cup \Theta^-$  occurs equally likely, is then enough to prove the result.

Let us denote  $T_f = w'$  and  $S_f = w$ . We construct function  $\psi : \Theta \rightarrow \Theta$  as follows. Let  $\psi(\boldsymbol{\theta})$  be the profile in which workers have preferences as in  $\boldsymbol{\theta}$ , but firms  $-f$  all swap the positions of workers  $w'$  and  $w$  in their preference lists. If profile  $\boldsymbol{\theta}$  belongs to  $\bar{\Theta}_-$ , without firm  $f$ 's offer, worker  $w$  has an offer from another firm, and worker  $w'$  does not. Therefore, when preferences are  $\psi(\boldsymbol{\theta})$ , without firm  $f$ 's offer the following two statements must be true: *i*) worker  $w'$  *must* have another offer and *ii*) worker  $w$  *cannot* have another offer.

To see *i*), note that under  $\boldsymbol{\theta}$ , worker  $w$  sends a signal to firm  $f$ , so his outside offer must come from some firm  $f'$  who has ranked him first. Under profile  $\psi(\boldsymbol{\theta})$ , firm  $f'$  ranks worker  $w'$  first. If worker  $w'$  has not sent a signal to firm  $f'$ , then by anonymity,  $w'$  gets the offer of firm  $f'$ . If worker  $w'$  has signaled to firm  $f'$ , worker  $w'$  again gets firm  $f'$ 's offer.

To see *ii*), suppose to the contrary that under  $\psi(\boldsymbol{\theta})$ , worker  $w$  does in fact receive an offer from some firm  $f' \neq f$ . Since worker  $w$  sends a signal to firm  $f$ , worker  $w$  must be  $T_{f'}$  under  $\psi(\boldsymbol{\theta})$ , so that worker  $w'$  is  $T_{f'}$  under  $\boldsymbol{\theta}$ . But then by anonymity  $w'$  receives the offer of firm  $f'$  under  $\boldsymbol{\theta}$ , a contradiction. From *i*) and *ii*), we have  $\boldsymbol{\theta} \in \bar{\Theta}_- \Rightarrow \psi(\boldsymbol{\theta}) \in \bar{\Theta}_+$ . Since function  $\psi$  is injective, we have  $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$ .

In order to prove the second statement note that the expected number of matches of each worker increases when firm  $f$  responds more to signals. Using

the construction presented above, one can show that whenever worker  $w$  “loses” a match with firm  $f$  for profile  $\theta$  (worker  $w$  is  $T_f$ ) it is possible to construct profile  $\theta'$  when worker  $w$  obtains a match (worker  $w$  is  $S_f$ ). The function that matches these profiles is again injective. Moreover, worker  $w$  values more greatly the match with firm  $f$  when she has signaled it ( $S_f$ ) rather when she is simply highest ranked ( $T_f$ ). Therefore, the ex-ante utility of worker  $w$  increases when firm  $f$  responds more to signals.

*B2. Market Structure and the Value of a Signaling Mechanism — Proofs and Extensions*

This set of results pertains to Section V: Market Structure and the Value of a Signaling Mechanism. In this section, we denote as  $u(j)$  the utility of a firm from matching with its  $j$ th ranked worker. The first proposition states that when preferences over workers are sufficiently flat, then in any nonbabbling equilibrium firms always respond to signals.

PROPOSITION B1: *Under the assumption that*

$$(B3) \quad u(W) > \frac{W}{F} \left(1 - \left(1 - \frac{1}{W}\right)^F\right) u(1)$$

*there is a unique nonbabbling equilibrium in the offer game with signals. Each worker sends her signal to her top firm. Each firm  $f$  makes an offer to  $S_f$  if it receives at least one signal; otherwise, firm  $f$  makes an offer to  $T_f$ .*

**Proof of Proposition B1.** We will show that under condition (B3) even if  $S_f$  is the worst ranked worker in firm  $f$  preferences, firm  $f$  still optimally makes her an offer.

Proposition 3 shows that if firms  $-f$  respond more to signals, i.e. increase their cutoffs, it is also optimal for firm  $f$  to respond more to signals. Therefore, if firm  $f$  optimally responds to signals when no other firm does, it will certainly optimally respond to signals when other firms respond. Hence, it will be enough to consider the incentives of firm  $f$  when firms  $-f$  do not respond to signals and always make an offer their top ranked workers.

Let us consider some realized profile of preferences of firm  $f$  and denote  $T_f$  as  $w$ . If firms  $-f$  do not respond to signals, then some firm among  $-f$  makes an offer to worker  $w$  with probability  $q = 1/W$ . Therefore, the probability that the offer of firm  $f$  to worker  $w$  is accepted equals

$$(1 - q)^{F-1} + \dots + C_{F-1}^j q^j (1 - q)^{F-1-j} \frac{1}{j+1} + \dots + q^{F-1} \frac{1}{F}$$

where  $C_x^y = x!/(y!(x-y)!)$ . Intuitively,  $j$  firms among the other  $F-1$  firms simultaneously make an offer to worker  $w$  with probability  $C_{F-1}^j q^j (1 - q)^{F-1-j}$ . Therefore, firm  $f$  is matched with worker  $w$  only with probability  $1/(j+1)$  because

worker  $w$ 's preferences are uniformly distributed. The sum over all possible  $j$  from 0 to  $F - 1$  gives us the overall probability of firm  $f$ 's offer being accepted. This expression simplifies to

$$(B4) \quad \frac{W}{F} \left( 1 - \left( 1 - \frac{1}{W} \right)^F \right)$$

Alternatively, firm  $f$ 's offer to its top signaled worker is accepted with probability one. Therefore, firm  $f$  optimally makes an offer to the signaled worker only if (B3) holds. We conclude that under Assumption (B3) there is no other nonbabbling symmetric equilibrium in the offer game with signals.

**PROPOSITION B2:** *Consider the following assumptions on agent utility functions and the discount factor:*

$$\begin{aligned} u(W) &> \frac{W}{F} \left( 1 - \left( 1 - \frac{1}{W} \right)^F \right) u(1) \\ u(W) &> \delta u(1), \quad v(W) > \delta v(1) \end{aligned}$$

*Then the following holds:*

- 1) *There is a unique symmetric sequential equilibrium in the offer game with no signals and  $L$  periods of interaction: each firm makes an offer to its most preferred worker and each worker accepts its best offer in each period.*
- 2) *There is a unique symmetric, sequential, nonbabbling (in each period) equilibrium in the offer game with signals and  $L$  periods of interaction: in period 0, each worker sends her signal to her most preferred firm; in periods  $l = 1, \dots, L$ , each firm makes an offer to its top signaling worker among workers remaining in the market; otherwise the firm makes an offer to its top ranked worker among those in the market. Each worker accepts the best available offer in each period.*

**Proof of Proposition B2.** Consider the offer game with no signals and  $L$  periods of interaction. We will apply backward induction, examining first the final stage of the game. The final stage of the game is identical to a one period offer game with no signals. Hence, each firm makes an offer to its top ranked worker and each worker accepts best available offer in the unique symmetric equilibrium of this stage.

Assumptions  $u(W) > \delta u(1)$  and  $v(W) > \delta v(1)$  guarantee that there is no incentive to hold offers or make dynamically strategic offers. Since firms  $-f$  use symmetric anonymous strategies at stage  $L - 1$  and stage  $L$ , the only optimal strategy of firm  $f$  at stage  $L - 1$  is to make an offer to  $T_f$ . Each worker who receives at least one offer in stage  $L - 1$  optimally accepts the best available offer immediately. Similar logic applies to the other stages.

Now consider the offer game with signals and  $L$  periods of interaction. The symmetry of the strategies of workers  $-w$  and the anonymity of firm strategies guarantee that the equilibrium probability that a firm makes an offer to worker  $w$  (across any of the  $L$  periods) conditional on receiving a signal from  $w$  (and also conditional on not receiving her signal) is the same for all firms. Therefore, workers optimally send their signals to their most preferred firm in period 0.

Observe that signals play a meaningful role for firms only in the first period. Since  $u(W) > \delta u(1)$  and  $u(W) > W/F \left(1 - (1 - 1/W)^F\right) u(1)$ , each firm  $f$  makes an offer at period 1 to  $S_f$  if it receives at least one signal. Since  $v(W) > \delta v(1)$  workers accept the best available offers immediately. In period 2, each remaining firm either receives no signals or else see its offer being rejected in period 1. Thereafter firm offers to their most preferred remaining workers prevail, as the logic of backward induction in the offer game with no signals and many periods applies to periods 2 through  $L$ .

**Proof of Proposition 4** (Balanced Markets). We first calculate an explicit formula for the increase in the expected number of matches from the introduction of the signaling mechanism.

LEMMA B1: *Consider a market with  $W$  workers and  $F > 2$  firms. The expected number of matches in the offer game with no signals equals*

$$(B5) \quad m^{NS}(F, W) = W \left(1 - \left(1 - \frac{1}{W}\right)^F\right)$$

*The expected number of matches in the offer game with signals equals*

$$(B6) \quad m^S(F, W) = F \left(1 - \left(\frac{F-1}{F}\right)^W\right) + F \left( \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}} \left(1 - \frac{F-1}{W} \left(1 - \left(\frac{F-2}{F-1}\right)^W\right)\right) \cdot \left(1 - \left(1 - \frac{1}{W} \left(\frac{F-2}{F-1}\right)^{W-1}\right)^{F-1}\right) \right)$$

**Proof of Lemma B1.** Let us first calculate the expected number of matches in the pure coordination game with no signals. Proposition 1 establishes that the unique symmetric nonbabbling equilibrium when agents use anonymous strategies is as follows. Each firm makes an offer to its top worker and each worker accepts the best offer among those available. We have already calculated the probability of firm  $f$  being matched to its top worker in Proposition B1. The probability of this event is  $W/F \left(1 - (1 - 1/W)^F\right)$ . Therefore, the expected total number of

matches in the game with no signals equals

$$m^{NS}(F, W) = W \left(1 - \left(1 - \frac{1}{W}\right)^F\right)$$

Let us now calculate the expected number of matches in the offer game with signals. Proposition B1 derives agent strategies in the unique symmetric nonbabbling equilibrium in the pure coordination game with signals. Each worker sends her signal to her top firm and each firm makes its offer to its top signaling worker if it receives at least one signal, otherwise it makes an offer to its top ranked worker. We first calculate the ex-ante probability of some firm  $f$  being matched. We denote the set of workers who send a signal to  $f$  as  $\mathcal{W}_f^S \subset \mathcal{W} \cup \mathcal{N}$ . If firm  $f$  receives at least one signal,  $|\mathcal{W}_f^S| > 0$ , it is guaranteed a match because each worker sends her signal to her top firm. If firm  $f$  receives no signals, it makes an offer to its top ranked worker  $T_f$ . This worker accepts firm  $f$ 's offer only if the offer is the best one among those she receives. Let us denote the probability that  $T_f$  accepts firm  $f$ 's offer (under the condition that firm  $f$  receives no signals) as

$$P_{T_f, |\mathcal{W}_f^S|=0} \equiv P(T_f \text{ accepts firm } f' \text{ s offer} \mid |\mathcal{W}_f^S| = 0)$$

The ex-ante probability that firm  $f$  is matched then equals

$$Prob\_match_f(F, W) = P(|\mathcal{W}_f^S| > 0) \cdot 1 + P(|\mathcal{W}_f^S| = 0) \cdot P_{T_f, |\mathcal{W}_f^S|=0}$$

If firm  $f$  receives no signals,  $|\mathcal{W}_f^S| = 0$ , it makes an offer to  $T_f$ , which we will call worker  $w$ . Worker  $w$  receives an offer from its top ranked firm, say firm  $f_0$ , conditional on firm  $f$  receiving no signals,  $|\mathcal{W}_f^S| = 0$ , with probability equal to

$$\begin{aligned} G &= P(|\mathcal{W}_{f_0}^S| = 1 \mid |\mathcal{W}_f^S| = 0) \cdot 1 + \dots + P(|\mathcal{W}_{f_0}^S| = W \mid |\mathcal{W}_f^S| = 0) \cdot \frac{1}{W} \\ &= \sum_{j=0}^{W-1} C_{W-1}^j \left(\frac{1}{F-1}\right)^j \left(1 - \frac{1}{F-1}\right)^{W-j-1} \frac{1}{j+1}. \end{aligned}$$

Intuitively, firm  $f_0$  receives a signal from a particular worker with probability  $1/(F-1)$  (note that firm  $f$  receives no signals). Then, if firm  $f_0$  receives signals from  $j$  other workers, worker  $w$  receives an offer from firm  $f_0$  with probability  $1/(j+1)$ . Similarly to equation (B4) the expression for  $G$  simplifies to

$$G = \frac{F-1}{W} \left(1 - \left(1 - \frac{1}{F-1}\right)^W\right).$$

Firm  $f$  can be matched with worker  $w$  only if worker  $w$  does not receive an offer from its top firm, which happens with probability  $1 - G$ . If worker  $w$  does not receive an offer from her top firm – firm  $f_0$  – firm  $f$  competes with other firms

that have received no signals from workers. The probability that some firm  $f'$  among firms  $\mathcal{F} \setminus \{f, f_0\}$  receives no signals conditional on the fact that worker  $w$  sends her signal to firm  $f_0$  and firm  $f$  receives no signals ( $|\mathcal{W}_f^S| = 0$ ) equals  $r = (1 - 1/(F - 1))^{W-1}$ . Note that the probability that firm  $f'$  does not receive a signal from a worker equals  $1 - 1/(F - 1)$ , because firm  $f$  receives no signals. There are also only  $W - 1$  workers that can send a signal to firm  $f'$ , because worker  $w$  sends her signal to firm  $f_0$ .

Therefore, the probability that some firm  $f'$  among firms  $\mathcal{F} \setminus \{f, f_0\}$  receives no signals and makes an offer to worker  $w$ , conditional on the fact that worker  $w$  sends her signal to firm  $f_0$ , equals  $r/W$ . Therefore, the probability that worker  $w$  prefers the offer of firm  $f$  to other offers (conditional on the fact that firm  $f$  receives no signals and worker  $w$  sends her signal to firm  $f_0$ ) equals<sup>32</sup>

$$\sum_{j=0}^{F-2} C_{F-2}^j \left(\frac{r}{W}\right)^j (1 - \frac{r}{W})^{F-2-j} \frac{1}{j+1} = \frac{W}{(F-1)r} \left(1 - (1 - \frac{r}{W})^{F-1}\right).$$

The probability that worker  $w$  accepts firm  $f'$ 's offer then equals

$$P_{T_f, |\mathcal{W}_f^S|=0} = (1 - G) \left( \frac{W}{(F-1)r} \left(1 - (1 - \frac{r}{W})^{F-1}\right) \right).$$

Taking into account that firm  $f$  receives no signals with probability  $P(|\mathcal{W}_f^S| = 0) = (1 - 1/F)^W$ , the probability of firm  $f$  being matched in the offer game with signals is then

$$\begin{aligned} Prob\_match_f(F, W) &= 1 - (1 - \frac{1}{F})^W + (1 - \frac{1}{F})^W \frac{W}{(F-1)r} \\ &\quad \cdot \left(1 - \frac{F-1}{W} \left(1 - (1 - \frac{1}{F-1})^W\right)\right) \left(1 - (1 - \frac{r}{W})^{F-1}\right) \end{aligned}$$

where  $r = (1 - 1/(F - 1))^{W-1}$ . The expected total number of matches in the offer game with signals equals  $m^S(F, W) = F \cdot Prob\_match_f(F, W)$ . This completes the proof of Lemma B1.

Lemma B1 establishes the expected total number of matches in the offer game with and without signals. Let us first fix  $W$  and calculate where the increase in the expected number of matches from the introduction of the signaling mechanism,  $V(F, W) = m^S(F, W) - m^{NS}(F, W)$ , attains its maximum. In order to derive the result of the proposition we consider markets with a large number of firms and workers and we use Taylor's expansion formula:

$$(B7) \quad (1 - a)^b = \exp(-ab + O(a^2b)),$$

<sup>32</sup>Note that the maximum number of offers worker  $w$  could get equals to  $M - 1$  as it does not receive an offer from its top firm  $f_0$ .

where  $O(a^2b)$  is a function that is smaller than a constant for large values of  $a^2b$ . Using the result of Lemma B1 and formular (B7) one could immediately calculate the approximation of the expected number of matches from the introduction of the signaling mechanism in large markets

$$W \left( x - xe^{-1/x} + \left( 1 - x \left( 1 - e^{-1/x} \right) \right) \left( 1 - e^{-xe^{-1/x}} \right) - 1 + e^{-x} \right) + O(1)$$

where we denote  $x = F/W$ . Therefore  $V(F, W) = W\alpha(x) + O(1)$ , where  $\alpha(x)$  is a positive quasi-concave function that attains maximum at  $x_0 \simeq 1.012113$ . Therefore, for fixed  $W$ ,  $V(F, W)$  attains its maximum value at  $F = x_0W + O(1)$ . Similar to the previous derivation, we can fix  $F$ . Then we obtain

$$F \left( 1 - e^{-1/x} + \left( 1 - x \left( 1 - e^{-1/x} \right) \right) \frac{1}{x} \left( 1 - e^{-xe^{-1/x}} \right) - \frac{1}{x} \left( 1 - e^{-x} \right) \right) + O(1)$$

Therefore  $V(F, W) = F\beta(x) + O(1)$ , where  $\beta(x)$  is a positive quasi-concave function that attains maximum at  $x_{00} \simeq 0.53074$ . Therefore, for fixed  $F$ ,  $V(F, W)$  attains its maximum value at  $W = y_0F + O(1)$ , where  $y_0 = 1/x_{00} = 1.8842$ .

**Proof of Proposition 5** (Multiple Periods). We will prove the argument under assumptions on agents' utility and discount factor of Propositions B2 that guarantee the uniqueness of equilibrium in the games with and without signals.

For clarity of the argument, we compare markets with one and two periods of interaction. Consider a market with two periods. Since workers can send only one signal and firms respond to all signals, all firms that receive at least one signal leave the market in period 1 (signals indicate that offers will be accepted for sure). Therefore, no firms remaining in period 2 have received signals, so the second period of the offer game with signals is identical to a single period offer game with no signals.

Since the introduction of the signaling mechanism increases the expected number of matches, the expected number of remaining market participants in period 2 is greater in the offer game with no signals than it is in the offer game with signals. As Proposition 4 shows, the number of matches in a market with one period is proportional to the size of the market. Therefore, the expected number of matches in the second period in the offer game with no signals is greater than in the offer game with signals. In other words, the second period plays a more significant role in the offer game with no signals. Hence, the difference between the expected number of matches in the offer game with signals and the offer game with no signals decreases upon adding the second period of interaction. This logic extends to  $L$  periods of interaction.