Reduced Form Implementation for Environments with Value Interdependencies

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Abstract

We provide a unified and simple treatment of reduced-form implementation for general social choice problems and extend it to environments with value interdependencies. We employ the geometric approach developed by Goeree and Kushnir (2016) to characterize the set of feasible interim agent values (agent utilities excluding transfers) by deriving the analytical expression of its support function. As an application, we use the reduced-form implementation to analyze second-best mechanisms in environments with value interdependencies.

Keywords: *mechanism design, convex analysis, reduced-form implementation, social choice, interdependent values, convex set, support function*

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1. Introduction

A typical problem of a mechanism designer can be expressed as a maximization of some objective over the set of feasible and incentive compatible allocation rules and transfers. In standard settings incentive compatibility identifies transfers from interim allocation probabilities up to a constant (Milgrom and Segal, 2002). This reduces the problem of the mechanism designer to an optimization over interim allocation probabilities only, i.e. *reduced form* problem.

To study the reduced form problem one should be able to identify the set of feasible interim allocation probabilities. Matthews (1984) first conjectured the set of inequalities characterizing this set in symmetric single-object auctions. These inequalities, as a necessary condition, were first used by Maskin and Riley (1984) and Matthews (1983) to analyze single-object auctions with risk-averse bidders. The sufficiency of these inequalities were subsequently proven by Border (1991) who then extended the result to asymmetric auction environments (see Border, 2007).[1] This characterization was used in numerous papers in the past and attracted attention of many recent papers.[2]

In this paper we provide a unified and simple treatment of reduced form implementation for general social choice problems and extend it to environments with value interdependencies. In the presence of value interdependencies, interim allocation probabilities alone do not determine interim agent utilities. To account for them, we characterize feasible interim agent values, i.e. interim agent utilities excluding transfers.

To characterize the set of feasible interim agent values we use the geometric approach developed by Goeree and Kushnir (2016) who exploit the one-to-one relation between a convex closed set and its support function. To obtain the support function for interim agent values we exploit the fact that they are a linear transformation of ex post allocations. For each possible type profile, feasible ex post allocations form a simplex for which the support function is well-known. Utilizing the rule how the support function transforms under a linear transformation we then determine the support function for feasible interim values. Finally, we use the duality from convex analysis to recover the inequalities characterizing the set of feasible interim values.

To illustrate our main result we apply reduced form implementation to study auction environments with multiple agents and linear value interdependencies. Maskin (1992) and Dasgupta

[1] For recent developments in reduced form auctions see Vohra (2011), Mierendorff (2011), Alaei et al. (2012), Cai et al. (2012), Che et al. (2013), Hart and Reny (2015), and Goeree and Kushnir (2016).

[2] See Armstrong (2000), Brusco and Lopomo (2002), Morand and Thomas (2006), Manelli and Vincent (2010), Asker and Cantillon (2010), Belloni et. al (2010), Hörner and Samuelson (2011), Miralles (2012), Pai and Vohra (2013, 2014ab), Pai (2014), Mierendorff (2014).

and Maskin (2000) show that for large value interdependencies the first-best social surplus cannot be implemented with incentive compatible mechanisms. Using the reduced-form implementation results we analyze the second-best mechanism for these environments. We show that the optimal Bayesian and dominant strategy incentive compatible mechanisms lead to the same level of social surplus if the object has to be allocated to agents. If the object does not have to be allocated to agents we provide a condition on the level of interdependencies when both implementation concepts achieve the same level of social surplus.

The paper proceeds as follows. Section 2 presents a social choice model. We derive the support function for the feasible set of interim agent values in Section 3. Section 4 analyzes second-best mechanisms in the environments with value interdependencies. Section 5 concludes.

2. Social Choice Model

We consider an environment with a finite set $\mathcal{I} = \{1, 2, ..., I\}$ of agents and a finite set $\mathcal{K} = \{1, 2, ..., K\}$ of social alternatives. Agent $i \in \mathcal{I}$ has a one-dimensional type x_i with finite support $X_i = \{x_i^1, \dots, x_i^{N_i}\} \subset \mathbb{R}_+$.[3] We also denote the profile of all agent types as $\mathbf{x} = (x_1 \dots, x_I)$ with support $X = \prod_i X_i$. We allow for correlation in types and denote their joint probability distribution as $f(\mathbf{x})$. Agent values are interdependent: when alternative k is selected and the profile of agent types is $\mathbf{x}$ agent i's value equals $v_i^k(\mathbf{x})$.

A direct mechanism can be characterized by $K + I$ functions, $\{q^k(\mathbf{x})\}_{k\in\mathcal{K}}$ and $\{t_i(\mathbf{x})\}_{i\in\mathcal{I}}$, where $q^k(\mathbf{x})$ is the probability that alternative k is selected and $t_i(\mathbf{x}) \in \mathbb{R}$ is agent i's transfer. We denote then agent i's ex post values as $v_i(\mathbf{x}) = \sum_{k\in\mathcal{K}} v_i^k(\mathbf{x})q^k(\mathbf{x})$ and interim expected values as $V_i(x_i) = \sum_{\mathbf{x}_{-i}} f_{-i}(\mathbf{x}_{-i}|x_i)v_i(\mathbf{x})$, where $f_{-i}(\mathbf{x}_{-i}|x_i)$ is the distribution of other agent types $\mathbf{x}_{-i}$ conditional on x_i. When agents report their types truthfully, agent i's ex post utility equals $u_i(\mathbf{x}) = v_i(\mathbf{x}) + t_i(\mathbf{x})$, and his interim expected utility equals $U_i(x_i) = V_i(x_i) + T_i(x_i)$, where $T_i(x_i) = \sum_{\mathbf{x}_{-i}} f_{-i}(\mathbf{x}_{-i}|x_i)t_i(\mathbf{x})$.

We heavily exploit the notion of the *support function* $\mathcal{S}^C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of a closed convex set $C \subset \mathbb{R}^n$, which is defined as

$$\mathcal{S}^C(\mathbf{w}) = \sup\{\mathbf{v} \cdot \mathbf{w} \,|\, \mathbf{v} \in C\}, \tag{1}$$

with the inner product $\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^n v_j w_j$. Support functions have three important properties

[3]Our main result, Theorem 1, also holds without any changes in environments with multi-dimensional types.

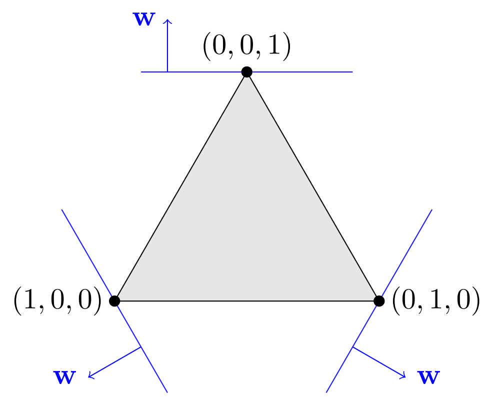

Figure 1. The figure shows how the support function for 2-simplex $\{q^k \geq 0, k = 1, 2, 3; q^1+q^2+q^3 = 1\}$ is determined. The inner product $\mathbf{q} \cdot \mathbf{w} = \sum_{k=1}^{3} q^k w^k$ achieves the maximum at one of the extreme points $(0, 0, 1)$, $(0, 1, 0)$, or $(1, 0, 0)$ resulting in $\mathcal{S}^{2-simplex}(\mathbf{w}) = \max\left(w^1, w^2, w^3\right)$.

that we outline below (for more details, see Rockafellar, 1997). First, there is the one-to-one relation between the support function and the corresponding convex set: given the support function S^C one can always recover the corresponding set as $C = \left\{\mathbf{v} \in \mathbb{R}^n \,|\, \mathbf{v} \cdot \mathbf{w} \ \leq \ \mathcal{S}^C(\mathbf{w}),\ \forall\, \mathbf{w} \in \mathbb{R}^n\right\}$. Second, the support function for a Cartesian product of sets $C_1, C_2 \subset \mathbb{R}^n$ equals the sum of the support functions: $S^{C_1 \times C_2}(\mathbf{w}') = S^{C_1}(\mathbf{w}_1) + S^{C_2}(\mathbf{w}_2)$, where $\mathbf{w}' = (\mathbf{w}_1, \mathbf{w}_2) \in \mathbb{R}^{2n}$, which directly follows from definition (1). Finally, the support function straightforwardly changes under a linear transformation. In particular, consider a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ and a closed convex set $C \subset \mathbb{R}^n$. Recall a basic property of the inner product $Aq \cdot w = q \cdot A^T w$, where $q \in C$ and A^T is the transpose of A. Therefore, the support function corresponding to image AC of linear transformation A equals $\mathcal{S}^{AC}(\mathbf{w}) = \mathcal{S}^C(A^T \mathbf{w})$, where $\mathbf{w} \in \mathbb{R}^n$.

3. Reduced Form Implementation

This section presents our main result that presents an analytical expression of the support function associated with the set of feasible interim values $V_i(x_i)$. To accomplish this goal we use the novel geometric approach developed recently by Goeree and Kushnir (2016).

For a given type profile $\mathbf{x}$, let us consider allocation $\{q^k(\mathbf{x})\}_{k \in \mathcal{K}}$ defining a $(K-1)$-dimensional simplex, i.e. $q^k(\mathbf{x}) \geq 0$ and $\sum_{k \in \mathcal{K}} q^k(\mathbf{x}) = 1$. If we consider 2-simplex $\{q^k \geq 0,\, k = 1, 2, 3\,;\, q^1 + q^2 + q^3 = 1\}$, depicted on Figure 1, it is straightforward to verify that the inner product $\mathbf{q} \cdot \mathbf{w}$ achieves the maximum at one of the extreme points $(0, 0, 1)$, $(0, 1, 0)$, or $(1, 0, 0)$ of the simplex resulting in support function $\mathcal{S}^{2-simplex}(\mathbf{w}) = \max\left(w^1, w^2, w^3\right)$. This expression straightforwardly generalizes to higher dimensions: the support function $\mathcal{S}^{simplex} : \mathbb{R}^K \to \mathbb{R}$ corresponding to $(K-1)$-simplex $\{q^k(\mathbf{x})\}_{k\in\mathcal{K}}$ for given $\mathbf{x} \in X$ equals

$$\mathcal{S}^{simplex}(\mathbf{w}(\mathbf{x})) \;=\; \max_{k\,\in\,\mathcal{K}}\, w^k(\mathbf{x}), \tag{2}$$

where $\mathbf{w}(\mathbf{x}) = (w^1(\mathbf{x}), ..., w^K(\mathbf{x}))$. Taking into account that the support function for a Cartesian product of sets equals the sum of support functions (see Section 2), we obtain the support function for the set of feasible allocations for all possible type profiles $\mathbf{x} \in X$

$$\mathcal{S}(\mathbf{w}) \;=\; \sum_{\mathbf{x}\,\in\,X}\, \max_{k\,\in\,\mathcal{K}}\, w^k(\mathbf{x}), \tag{3}$$

where $\mathbf{w} = \{\mathbf{w}^k(\mathbf{x})\}_{k\in\mathcal{K},\mathbf{x}\in X}$.

To characterize the set of ex post values we exploit how the support function changes under a linear transformation (see Section 2). The support function for ex post values equals then $\mathcal{S}_{ex\ post}(\mathbf{w}) = \mathcal{S}(A^T\mathbf{w})$, where A corresponds to linear transformation $v_i(\mathbf{x}) = \sum_{k\in\mathcal{K}} v_i^k(\mathbf{x})q^k(\mathbf{x})$. Equivalently, $\mathcal{S}_{ex\ post}(\mathbf{w})$ can be obtained from (3) by replacing $w^k(\mathbf{x})$ with $\sum_{i\,\in\,\mathcal{I}} v_i^k(\mathbf{x})w_i(\mathbf{x})$:

$$\mathcal{S}_{ex\ post}(\mathbf{w}) \;=\; \sum_{\mathbf{x}\,\in\,X}\, \max_{k\,\in\,\mathcal{K}}\, \sum_{i\,\in\,\mathcal{I}} v_i^k(\mathbf{x})w_i(\mathbf{x}), \tag{4}$$

where weight $w_i(\mathbf{x})$ corresponds to $v_i(\mathbf{x})$ for each $i \in \mathcal{I}$ and $\mathbf{x} \in X$, and we denote with slight abuse of notation $\mathbf{w} = \{w_i(\mathbf{x})\}_{i\in\mathcal{I},\mathbf{x}\in X}$.

Finally, we derive the support function for interim values $V_i(x_i) = \sum_{\mathbf{x}_{-i}} f_{-i}(\mathbf{x}_{-i}|x_i)v_i(\mathbf{x})$. Since interim values are obtained from ex post values with a linear transformation, we can again use the property how the support function changes under a linear transformation. To arrive at an expression that is symmetric in the probabilities we define interim support functions using the probability-weighted inner product

$$\mathbf{V}\cdot\mathbf{W} \;=\; \sum_{i\,\in\,\mathcal{I}}\,\sum_{x_i\,\in\,X_i} f_i(x_i)V_i(x_i)W_i(x_i).$$

In other words, we multiply the interim weight $W_i(x_i)$ associated with $V_i(x_i)$ by $f_i(x_i)$ so that all terms are weighted by $f(\mathbf{x}) = \prod_i f_i(x_i)$. Using the definition of the support function and how it allows to recover the set back, we obtain our main result.[4]

Theorem 1. *The support function for the set of feasible interim values equals*

$$\mathcal{S}_{interim}(\mathbf{W}) \;=\; E_{\mathbf{x}}\left(\max_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} v_i^k(\mathbf{x})W_i(x_i)\right),$$

and feasible interim values $\mathbf{V}$ *satisfy* $\mathbf{V}\cdot\mathbf{W} \leq \mathcal{S}_{interim}(\mathbf{W})$ *for all* $\mathbf{W} \in I\!R^{\sum_i |X_i|}$.

Overall, Theorem 1 completely characterizes the set of feasible interim values and shows how its points can be recovered using the knowledge of its support function.

4. Application: Second-Best Mechanisms

In this section we illustrate how Theorem 1 can be applied to analyze the properties of second-best mechanisms in the environments with interdependent values.[5]

We consider a single-object auction with I agents. Agent i's type x_i is independently distributed according to probability distribution $f_i(x_i)$. There are $I+1$ possible alternatives: alternative i corresponds to agent i winning the object and alternative 0 to the seller keeping the object. Agent i's utility equals $x_i + \alpha\sum_{j\neq i} x_j$ if he wins the object and 0 otherwise, where $\alpha \geq 0$ measures the degree of interdependency. Overall, agent i's ex post value equals

$$v_i(x_i, \mathbf{x}_{-i}) = (x_i + \alpha\sum\nolimits_{j\neq i} x_j)q^i(x_i, \mathbf{x}_{-i}).$$

Let $Q_i(x_i) = E_{\mathbf{x}_{-i}}(q^i(x_i, \mathbf{x}_{-i}))$ be agent i's interim probability of winning the object. To analyze the maximum level of social surplus that can be achieved with Bayesian incentive compatible (BIC) or ex post incentive compatible (EPIC) mechanism we use the following characterization.

Proposition 1. *An allocation rule* $\mathbf{q}$ *can be implemented with some BIC (EPIC) mechanism if and only if* $Q_i(x_i)$ $(q^i(x_i, \mathbf{x}_{-i}))$ *is non-decreasing in* x_i *for each* i *(and each* $\mathbf{x}_{-i} \in X_{-i}$*).*

To introduce the above monotonicity constraint into the support function we employ the

[4]To clarify, if we used the standard non-probability weighted inner-product the expression for the support function in Theorem 1 would be $\mathcal{S}_{interim}(\mathbf{W}) \;=\; \sum_{\mathbf{x}\in X}\max_{k\in\mathcal{K}}\left(\sum_{i\in\mathcal{I}} f_{-i}(\mathbf{x}_{-i}|x_i)v_i^k(\mathbf{x})W_i(x_i)\right)$.

[5]For related ideas see Hernando-Veciana and Michelucci (2009, 2014).

geometric approach developed by Goeree and Kushnir (2016). Let us first consider EPIC constraints and the support function corresponding to ex post values (4). The support function of the intersection of the ex post feasible set and half spaces $\mathbf{b}_m \cdot \mathbf{v} \geq 0$ for $m = 1, \ldots, M$ can be calculated as (see Rockafellar, 1997)

$$\inf_{\lambda_m \geq 0} \mathcal{S}_{ex\ post}(\mathbf{w} + \sum_{m=1}^{M} \lambda_m \mathbf{b}_m). \tag{5}$$

The EPIC constraints can be represented by half-spaces (see Proposition 1):

$$\frac{v_i(x_i^{n+1}, \mathbf{x}_{-i})}{x_i^{n+1} + \alpha \sum_{j \neq i} x_j} - \frac{v_i(x_i^n, \mathbf{x}_{-i})}{x_i^n + \alpha \sum_{j \neq i} x_j} \geq 0, \qquad \frac{v_i(x_i^n, \mathbf{x}_{-i})}{x_i^n + \alpha \sum_{j \neq i} x_j} - \frac{v_i(x_i^{n-1}, \mathbf{x}_{-i})}{x_i^{n-1} + \alpha \sum_{j \neq i} x_j} \geq 0.$$

for $n = 2, ..., N_i - 1$ and $i = 1, \ldots, I$. Let $\lambda_i(x_i^n, \mathbf{x}_{-i})$ be associated with the first constraint and $\lambda_i(x_i^{n-1}, \mathbf{x}_{-i})$ with the second. For $n = 1, ..., N_i$ we also define the differences $\Delta\lambda_i(x_i^n, \mathbf{x}_{-i}) = \lambda_i(x_i^n, \mathbf{x}_{-i}) - \lambda_i(x_i^{n-1}, \mathbf{x}_{-i})$ with $\lambda_i(x_i^0, \mathbf{x}_{-i}) = \lambda_i(x_i^{N_i}, \mathbf{x}_{-i}) = 0$. Using formula (5) we then obtain

$$\mathcal{S}^{EPIC}_{ex\ post}(\mathbf{w}) \;=\; \inf_{\lambda_i \geq 0} \sum_{\mathbf{x} \in X} \max_{i \in \mathcal{I}} \big(0, (x_i + \alpha \sum\nolimits_{j \neq i} x_j) w_i(\mathbf{x}) - \frac{\Delta\lambda_i(\mathbf{x})}{f_i(x_i)}\big).$$

Using the property how the ex post support function changes under a linear transformation (see Section 2) we obtain the support function for interim values satisfying EPIC constraints:

$$\mathcal{S}^{EPIC}(\mathbf{W}) \;=\; \inf_{\lambda_i \geq 0} E_{\mathbf{x}} \max_{i \in \mathcal{I}} \big(0, (x_i + \alpha \sum\nolimits_{j \neq i} x_i) W_i(x_i) - \frac{\Delta\lambda_i(\mathbf{x})}{f_i(x_i)}\big). \tag{6}$$

A similar procedure applies to BIC constraints

$$E_{\mathbf{x}_{-i}}\Big(\frac{v_i(x_i^{n+1}, \mathbf{x}_{-i})}{x_i^{n+1} + \alpha \sum_{j \neq i} x_j} - \frac{v_i(x_i^n, \mathbf{x}_{-i})}{x_i^n + \alpha \sum_{j \neq i} x_j}\Big) \geq 0, \quad E_{\mathbf{x}_{-i}}\Big(\frac{v_i(x_i^n, \mathbf{x}_{-i})}{x_i^n + \alpha \sum_{j \neq i} x_j} - \frac{v_i(x_i^{n-1}, \mathbf{x}_{-i})}{x_i^{n-1} + \alpha \sum_{j \neq i} x_j}\Big) \geq 0.$$

for $n = 2, ..., N_i - 1$ and $i = 1, \ldots, I$. Let $\Lambda_i(x_i^n)$ be associated with the first constraint and $\Lambda_i(x_i^{n-1})$ with the second. We also denote for $n = 1, \ldots, N_i$ the differences $\Delta\Lambda_i(x_i^n) = \Lambda_i(x_i^n) - \Lambda_i(x_i^{n-1})$ with $\Lambda_i(x_i^0) = \Lambda_i(x_i^{N_i}) = 0$. The support function for the set of interim values satisfying BIC constraints equals then

$$\mathcal{S}^{BIC}(\mathbf{W}) \;=\; \inf_{\Lambda_i \geq 0} E_{\mathbf{x}} \max_{i\in\mathcal{I}}\big(0, (x_i + \alpha \sum\nolimits_{j\neq i} x_j) W_i(x_i) - \frac{\Delta\Lambda_i(x_i)}{f_i(x_i)}\big). \tag{7}$$

For large degrees of value interdependencies, and in particular for $\alpha > 1$, Maksin (1992) and Dasgupta and Maskin (2000) show that the first-best social surplus cannot be implemented with BIC (and, hence, EPIC) mechanisms. We now analyze when the second-best level of social surplus, which equals the value of support function at unit weights $\mathcal{S}^{BIC}(\mathbf{1})$, can be implemented with some EPIC mechanism. We first approach this question when the object has to be always allocated to agents, and, hence, there is no zero term in the support functions (6) and (7).

Proposition 2. *If the object has to be always allocated to agents and $\alpha > 1$ the second-best level of social surplus can be implemented with some EPIC mechanism and equals*

$$\mathcal{S}^{BIC}(\mathbf{1}) \;=\; \alpha \sum\nolimits_{i\in\mathcal{I}} E(x_i) + (1-\alpha) \min_{i\in\mathcal{I}}(E(x_i)).$$

where $E(x_i)$ denotes the expected value of agent i's type.

We now identify a condition on α when one of the agents always gets the object at the second-best allocation, even though the auctioneer can keep the object. In this case the second-best level of social surplus can be again implemented with some EPIC mechanism.

Proposition 3. *Consider the case when the auctioneer can keep the object. The second-best level of social surplus can be implemented with some EPIC mechanism if*

$$1 < \alpha \leq \max_{i\in\mathcal{I}} \left(\frac{E(x_i)}{E(x_i) - \sum_{j\in\mathcal{I}} x_j^1} \right). \tag{8}$$

We finally illustrate our results with a simple auction example with two symmetric bidders and two equally likely and independently distributed types, $\underline{x} = 1$ and $\overline{x} = 10$. We compare the sets of feasible outcomes that satisfy BIC and EPIC constraints respectively. Since the bidders are ex ante symmetric, the allocation rule has no agent specific subscript and can be represented by a matrix

$$\mathbf{q} \;=\; \begin{pmatrix} q(\underline{x},\underline{x}) & q(\overline{x},\underline{x}) \\ q(\underline{x},\overline{x}) & q(\overline{x},\overline{x}) \end{pmatrix},$$

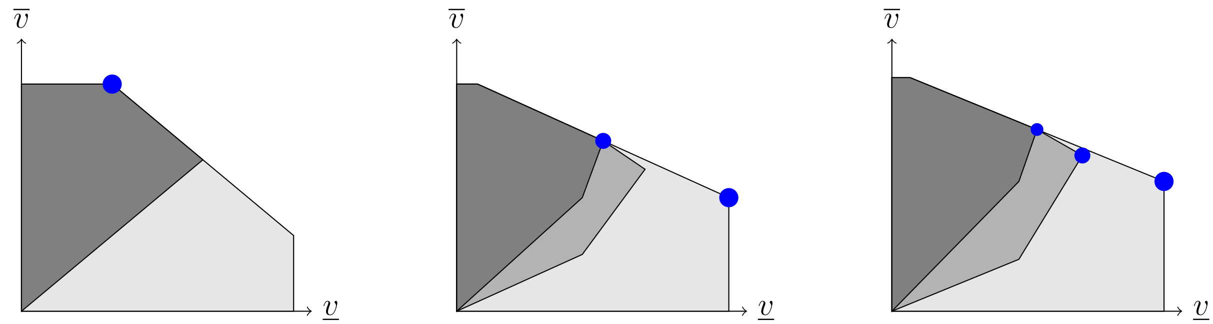

Figure 2. Feasible outcomes and the maximum level of social surplus. Shown are the feasible outcomes with no incentive constraints imposed (light), Bayesian incentive compatible outcomes (medium dark), and ex post incentive compatible outcomes (dark) for $\alpha = 0$ (left panel), $\alpha = 1.2$ (middle panel), and $\alpha = 2$ (right panel). The largest blue dot indicates the first-best outcome, the medium-sized blue dot the second-best outcome under BIC, and the smallest blue dot indicates the allocation that delivers the maximum level of social surplus under EPIC.

where the rows correspond to (say) agent 1's type and the columns to agent 2's type, and the entries correspond to the probabilities that the object is assigned to agent 1. The probability that object is assigned to agent 2 can be obtained by transposing the matrix.

Figure 2 shows the sets of interim values that result when $\alpha = 0$ (left panel), $\alpha = 1.2$ (middle panel), and $\alpha = 2$ (right panel). In each of the panels, the light area corresponds to the set of feasible values without any incentive constraints imposed, the medium dark area to the BIC values, and the dark area to the EPIC values. In case of pure private values, i.e. $\alpha = 0$, as shown by the left panel the latter two sets coincide, which corresponds to the BIC-DIC equivalence result of Gershkov et al. (2013).[6] However, the equivalence between Bayesian and ex post implementation generally fails when $\alpha > 0$ as shown by the middle and right panels.

The easiest way to describe the different sets is by their vertices.[7] For instance, the set of EPIC outcomes can be described by five vertices, which (clockwise starting at the origin) correspond to the following allocation rules

$$\mathbf{q}^{EPIC} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

[6] See also Manelli and Vincent (2010), Kushnir (2015), and Kushnir and Liu (2015).

[7] The vertices follow from the gradient of the support function at points of differentiability. The five EPIC vertices $(0,0), (0, \frac{15}{2} + 3\alpha), (\frac{1}{4} + \frac{1}{4}\alpha, \frac{15}{2} + 3\alpha), (\frac{1}{2} + \frac{11}{4}\alpha, 5 + \frac{11}{4}\alpha), (\frac{1}{4} + \frac{5}{2}\alpha, \frac{5}{2} + \frac{5}{2}\alpha)$. The first four plus $(\frac{3}{8} + \frac{15}{4}\alpha, \frac{15}{4} + \frac{21}{8}\alpha)$, $(\frac{1}{4} + \frac{5}{2}\alpha, \frac{5}{2} + \frac{1}{4}\alpha)$ constitute the six BIC vertices.

Likewise, for the BIC outcomes the six vertices correspond to the allocation rules

$$\mathbf{q}^{BIC} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Bayesian incentive compatibility requires that the sum of entries in the top row does not exceed the sum of entries in the bottom row. In contrast, ex post incentive compatibility requires that the entries in the top row do not exceed the entries in the bottom row for both columns (see Proposition 1). Notice that the final two BIC matrices violate this more stringent condition.

The blue dots in Figure 2 indicate the optimal outcomes: the largest blue dot indicates the first-best outcome, the medium-sized blue dot indicates the second-best outcome under BIC, and the smallest blue dot indicates the allocation that delivers the maximum level of social surplus under EPIC. For $\alpha \leq 1$, the first-best outcome can be implemented with EPIC mechanism and correspond to the third EPIC matrix. When $1 < \alpha \leq \frac{E(x)}{E(x)-2\underline{x}} = \frac{11}{7}$ the first-best cannot be achieved, but the second-best outcome can be implemented with EPIC mechanisms and correspond to the penultimate EPIC matrix. If $\alpha > \frac{11}{7}$ the second-best outcome can be implemented only with BIC mechanism, and correspond to the penultimate BIC matrix. In this case BIC implementation leads to more social surplus than EPIC.

5. Conclusion

Reduced-from implementation serves as a cornerstone in the developing theory of auctions with risk-averse bidders (see Maskin and Riley, 1984) and has attracted a lot of attention of many recent papers (e.g. Pai and Vohra, 2014ab). Moreover, some new approaches has been recently developed to reinterpret the Masking-Riley-Matthews conditions (see Hart and Reny, 2015) to extend them to multi-unit auctions with capacity constraints (see Che, Kim, Mierendorff, 2013) and social choice environments (Goeree and Kushnir, 2016).

The contribution of this short paper is to provide a unified and simple treatment of reduced-form implementation and to extend it to environments with interdependent values. As an application, we show how reduced-form implementation can be used to analyze the level of social surplus in the second-best mechanisms. We leave the exciting prospect of applying the reduced-form implementation to other mechanism design problems with interdependent values as a topic for future research.

A. Appendix

Proof of Proposition 1. The statement for BIC mechanisms directly follows from Theorem 3.1 in Jehiel and Moldovanu (2001). The extension to EPIC mechanisms is immediate. ∎

Proof of Proposition 2. The second-best level of social surplus equals the value of support function at unit weights $\mathcal{S}^{BIC}(\mathbf{1})$. If the auctioneer has to always allocate the object (no zeros in (7)) the support function reduces to

$$\mathcal{S}^{BIC}(\mathbf{1}) \;=\; \alpha \sum\nolimits_{i\in\mathcal{I}} E(x_i) + \inf_{\Lambda_i\geq 0} E_{\mathbf{x}} \max_{i\in\mathcal{I}}\big((1-\alpha)x_i - \frac{\Delta\Lambda_i(x_i)}{f_i(x_i)}\big). \tag{A.1}$$

Lemma A2 of Goeree and Kushnir (2016) then establishes that the minimum of the above expression is achieved when $(1-\alpha)x_i - \frac{\Delta\Lambda_i(x_i)}{f_i(x_i)} = ((1-\alpha)x_i)_+$, where $+$ denotes a sequence of f-majorized values (see Goeree and Kushnir (2016) for the definition and references).[8] Since $\{(1-\alpha)x_i^n\}_{n=1,\ldots,N_i}$ is a decreasing sequence for $\alpha > 1$ its f-majorized sequence is a constant sequence $((1-\alpha)x_i^n)_+ = (1-\alpha)E(x_i)$. The second-best level of social surplus reduces then to

$$\mathcal{S}^{BIC}(\mathbf{1}) \;=\; \alpha \sum\nolimits_{i\in\mathcal{I}} E(x_i) + (1-\alpha)\min_{i\in\mathcal{I}}(E(x_i)). \tag{A.2}$$

The expression for $\mathcal{S}^{EPIC}(\mathbf{1})$ can be written similarly. The only differences is that the minimization takes place over $\lambda_i(\mathbf{x}) \geq 0$ that depends on the whole vector of agent types. Proposition B2 of Goeree and Kushnir (2016) then establishes that support function $\mathcal{S}^{EPIC}(\mathbf{1})$ has the same value. Hence, the second-best allocation can be implemented with some EPIC mechanism. ∎

Proof of Proposition 3. If the auctioneer can keep the object the support function (7) at unit weights equals

$$\mathcal{S}^{BIC}(\mathbf{1}) \;=\; \alpha \sum\nolimits_{j\in\mathcal{I}} E(x_j) + \inf_{\Lambda_i\geq 0} E_{\mathbf{x}} \max_{i\in\mathcal{I}}\big(-\alpha \sum\nolimits_{j\in\mathcal{I}} x_j, (1-\alpha)x_i - \frac{\Delta\Lambda_i(x_i)}{f_i(x_i)}\big). \tag{A.3}$$

Notice that the value of (A.3) is greater or equal to the value of (A.1) for all parameters Λ. If condition (8) is satisfied, however, the values of both support functions coincide for Λs delivering the minimum to (A.1), because $(1-\alpha)E(x_i) \geq -\alpha(\sum_{j\in\mathcal{I}} x_j)$ for at least some i. Hence, the same Λs deliver the minimum to (A.3). The same argument applies to $\mathcal{S}^{EPIC}(\mathbf{1})$. ∎

[8]The majorization procedure requires the calculation of the convex function that is "the closest" to a given one. Generally, the problem has no analytical solution. There have been developed, however, several efficient algorithms solving this problem. See Lucet (2010) for an excellent survey.

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