On Sufficiency of Dominant Strategy Implementation in Environments with Correlated Types

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Abstract

We show that for any mechanism there exists an equivalent dominant strategy incentive compatible mechanism for social choice environments with correlated types when agent's matrix of conditional probabilities satisfies the full rank condition.

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1. Introduction

The design of robust mechanisms that do not rely on the assumptions of a common prior and equilibrium play has recently attracted a lot of attention. The restrictions to robust mechanisms, however, can potentially limit the set of outcomes that the designer can achieve. In this paper we analyze whether for a given mechanism there exists an equivalent dominant strategy incentive compatible (DIC) mechanism that delivers the same interim expected utilities to all agents and generates the same social surplus.

The notion of mechanism equivalence based on interim expected utilities is introduced by Manelli and Vincent (2010) who show for standard single-unit auctions that for any Bayesian incentive compatible (BIC) mechanism there exists a DIC mechanism that delivers the same interim expected utilities to all agents. Gershkov et al. (2013) require additionally the mechanisms to generate the same social surplus and extend the BIC-DIC equivalence to social choice problems with independent, one-dimensional, private types and linear utilities (see also Goeree and Kushnir, 2015).

In this paper, we consider social choice environments with private values and discrete correlated types and further extend the equivalence to mechanisms that are not necessarily BIC. If each agent's matrix of conditional probabilities satisfies the full rank condition, for any mechanism there exists a DIC mechanism that delivers the same interim expected utilities to all agents and generates *at least the same* social surplus. Moreover, if there exists a social alternative that is *inferior* to all other alternatives for all agents an equivalent DIC always exists. We further explain how our results apply to environments with interdependent values that satisfy the single-crossing condition.

To provide intuition for our results note that the VCG mechanisms implement the maximum social surplus if agents have private values. Therefore, there exists a DIC mechanism that generates the highest possible social surplus. We then use the techniques of Cremer and McLean (1985, 1988) to tailor the VCG payments such that interim expected utilities match exactly the interim expected utilities of a given mechanism.

2. Model

Consider a set $\mathcal{I} = \{1, ..., I\}$ of agents and a set $\mathcal{K} = \{1, ..., K\}$ of social alternatives. Each agent has a private type x_i taken from some finite set $X_i = \{x_i^1, ..., x_i^{N_i}\}$.[1] Let $X = \times_{i \in \mathcal{I}} X_i$ and $X_{-i} = \times_{j \neq i} X_j$. Agent types are distributed according to distribution f. We denote the probability of other agent types x_{-i} given that agent i has type x_i as $f(x_{-i}|x_i)$ and assume that distribution f satisfies the spanning condition of Cremer and McLean (1985): for all $i \in \mathcal{I}$ there do not exist $\{\rho_i(x_i)\}_{x_i \in X_i}$, not all equal to zero, such that

$$\sum_{x_i \in X_i} \rho_i(x_i) f(x_{-i}|x_i) \; = \; 0 \text{ for all } x_{-i} \in X_{-i} \tag{1}$$

[1] Agents types can be multidimensional.

Equivalently, each agent's matrix of conditional probabilities has full rank. Agent's value for alternative k is given by $v_i^k(x_i)$ for some function $v_i^k : X_i \to R$.

We analyze direct mechanisms defined by allocation $\{q^k(\mathbf{x})\}_{k\in K}$ and transfers $\{t_i(\mathbf{x})\}_{i\in\mathcal{I}}$ where $\mathbf{x} = (x_1, ..., x_I)$ is a profile of agent reports. Allocation $q^k(\mathbf{x}) \geq 0$ specifies the probability that alternative k is chosen such that $\sum_{k\in\mathcal{K}} q^k(\mathbf{x}) = 1$ and $t_i(\mathbf{x})$ specifies the monetary transfer to agent i. Therefore, agent's utility from reporting x_i' given his true type x_i equals $u_i(x_i', x_i, x_{-i}) = \sum_{k\in\mathcal{K}} q_i^k(x_i', x_{-i})v_i^k(x_i) + t_i(x_i', x_{-i})$. We also denote agent's interim expected utility as $U_i(x_i'|x_i) = E_{x_{-i}|x_i} u_i(x_i', x_i, x_{-i})$. The social surplus equals

$$S = \sum_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x})\Big(\sum_{i\in\mathcal{I}}\sum_{k\in\mathcal{K}} q^k(\mathbf{x})v_i^k(x_i)\Big).$$

A direct mechanism $(\mathbf{q}, \mathbf{t})$ is DIC if $u_i(x_i, x_i, x_{-i}) \geq u_i(x_i', x_i, x_{-i})$ for any x_i', x_i and x_{-i}. We use the notion of mechanism equivalence based on Gershkov et al. (2013).

Definition. Two mechanisms are equivalent if and only if they deliver the same interim expected utilities for all agents and generate the same social surplus.

3. Results

This section presents our main results. Theorem 1 constructs for any given mechanism a DIC mechanism that delivers the same interim expected utilities to all agents and generates at least the same surplus. Theorem 2 provides a sufficient condition when the social surplus of some DIC mechanism can exactly match the social surplus of the given mechanism. Finally, we present an example illustrating why one cannot always match the social surplus exactly.

Theorem 1. *For any given mechanism there exists a DIC mechanism that delivers the same interim expected utility to all agents and generates at least the same social surplus.*

Proof. Consider some mechanism $(\tilde{\mathbf{q}}, \tilde{\mathbf{t}})$ that delivers interim expected utility $\widetilde{U}_i(x_i)$ to agent i with type x_i and generates social surplus $\widetilde{S}$. We denote $(\mathbf{q}^*, \mathbf{t}^*)$ be the pivot VCG mechanism (see Chapter 2 in Milgrom, 2004). The pivot VCG mechanism generates the highest possible social surplus S^* and satisfies DIC constraints. Let $U_i^*(x_i)$ be the interim expected utility of agent i with type x_i from participating in the pivot VCG mechanism. Let us denote

$$h_i(x_i) = \widetilde{U}_i(x_i) - U_i^*(x_i)$$

for each x_i and $i \in \mathcal{I}$. Given that the distribution function satisfies spanning condition (1) we employ the construction of Cremer and McLean (1985) to find $\{g_i(x_{-i})\}_{x_{-i}\in X_{-i}}$ such that

$$\sum_{x_{-i}\in X_{-i}} f(x_{-i}|x_i)g_i(x_{-i}) = h_i(x_i)$$

We now consider transfers $t_i(\mathbf{x}) = t_i^*(\mathbf{x}) + g_i(x_{-i})$. Since the additional term depends only on the types of the other agents the modified transfers together with the allocation rule of pivot VCG mechanism $\mathbf{q}^*$ still satisfy DIC constraints. At the same time, agents' expected utilities in the constructed mechanism $(\mathbf{q}^*, \mathbf{t})$ match exactly agents' expected utility of the given mechanism $(\tilde{\mathbf{q}}, \tilde{\mathbf{t}})$. ■

Remark 1. The constructed DIC mechanism is not necessarily ex post individually rational. This is in contrast to Gershkov et al. (2013) and Goeree and Kushnir (2015) who show that for any BIC and interim individually rational mechanism there exist an equivalent DIC and ex post individually rational mechanism when agents have independent, one-dimensional types, private values, and linear utilities.

Remark 2. Cremer and Mclean (1988) presents an example showing that one can extract the full social surplus with a BIC mechanism, but not with a DIC mechanism, when spanning condition (1) is violated. This example reaffirms that condition (1) is essential for our results.

Consider now environments in which there exists a social alternative k_0 that delivers the smallest utility to all agents independently on their types, e.g. not allocating the object in auction settings. To normalize we assume that $v_i^{k_0}(x_i) \equiv 0$ and $v_i^k(x_i) \geq 0$ for each $i \in \mathcal{I}$, $k \in \mathcal{K}$ and $x_i \in X_i$.

Theorem 2. *If there exists a social alternative that is inferior to all other alternatives for all agents independently of their types, then for any given mechanism there exist an equivalent DIC mechanism.*

Proof. Consider some direct mechanism $(\tilde{\mathbf{q}}, \tilde{\mathbf{t}})$ that generates social surplus $\widetilde{S}$. The pivot VCG mechanism $(\mathbf{q}^*, \mathbf{t}^*)$ generates the highest possible social surplus $S^* \geq \widetilde{S}$ and satisfies DIC constraints. Construct now a mechanism with allocation rule

$$q^{**k}(\mathbf{x}) = (\widetilde{S}/S^*)q^{*k}(\mathbf{x}) \text{ for } k \neq k_0, \qquad q^{**k_0}(\mathbf{x}) = 1 - \sum_{k \neq k_0} q^{**k}(\mathbf{x})$$

and transfer rule $t_i^{**}(\mathbf{x}) = (\widetilde{S}/S^*)t_i^*(\mathbf{x})$ for $\mathbf{x} \in X$. Note that mechanism $(\mathbf{q}^{**}, \mathbf{t}^{**})$ is DIC and generates the same social surplus as the given mechanism $(\tilde{\mathbf{q}}, \tilde{\mathbf{t}})$. The construction of Theorem 1 then establishes the existence of transfers that together with allocation rule $\mathbf{q}^{**}$ matches the interim expected utilities $\widetilde{U}_i(x_i)$, satisfies DIC constraints, and generates social surplus $\widetilde{S}$. ■

Remark 3. As a corollary of Theorem 2 we obtain an extension of BIC-DIC equivalence of Gershkov et al. (2013) to environments with correlated types satisfying the spanning condition of Cremer and McLean (1985) (with the difference highlighted by Remark 1).

We now illustrate how a violation of the condition of Theorem 2 leads to the existence of a mechanism, which is even BIC, that generates a smaller social surplus than any DIC mechanism.

Example. Consider a single-unit auction with two agents and two types $\underline{x} = 1$ and $\overline{x} = 2$. The distribution of types has the following matrix

$$f = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

with rows corresponding to agent 1's type and columns to agent 2's type, and the entries to the probability of vector of types (x_1, x_2) being realized. It is straightforward to check that the above distribution satisfies spanning condition (1). We assume that the object *has to be assigned to the agents* violating the condition of Theorem 2. Consider a symmetric allocation rule "allocate to the agent with the smallest value"

$$\mathbf{q} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

This allocation generates the smallest possible social surplus given that the object has to be assigned to an agent. Since $\mathbf{q}$ is not monotone it cannot be implemented in dominant strategies (e.g. Mookherjee and Reichelstein, 1992). Therefore, there is no DIC mechanism that generates the smallest social surplus. Allocation $\mathbf{q}$, however, can be implemented with BIC mechanism using the following symmetric transfer rule

$$\mathbf{t} = \begin{pmatrix} 0 & -\frac{5}{2} \\ 0 & -1 \end{pmatrix}.$$

4. Discussion

In the main text, we present our results for the environments with private values and discrete types to simplify the exposition. The full surplus extraction, however, does not rely on privacy of agent types (see McAfee and Reny, 1992; Krishna, 2009). In the environments with interdependencies a natural counterpart of dominant strategy incentive compatibility is ex post incentive compatibility (EPIC). Therefore, whenever EPIC mechanisms can generate the social surplus at least as high as any mechanism the results of Theorems 1 and 2 apply. For instance, if agent types can be ordered and values satisfy the single-crossing condition the generalized VCG mechanism is EPIC and generates the maximum social surplus (see Krishna, 2009).

If agent values do not satisfy single-crossing condition there can exist a mechanism, which is even BIC, that generates strictly higher social surplus than any EPIC mechanism (see Goeree and Kushnir (2015) for such an example when agent types are independently distributed). For these environments, however, we establish a partial equivalence: Assume a given mechanism generates the social surplus smaller than the maximum social surplus that can be generated with some EPIC mechanism. Then the construction of Theorem 1 still provides an EPIC mechanism that delivers the same interim expected utilities to all agents and generates at least the same social surplus.

We finally note that if agent types are single-dimensional our results can be extended to the environments with continuous types using the techniques of McAfee and Reny (1992).

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