INTERNATIONAL UNIVERSITY BREMEN
SCHOOL OF ENGINEERING AND SCIENCE
Foundations of Mathematics I

COMBINATORIAL GAME THEORY

NIMBERS

Project

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INTRODUCTION

Everyone knows what a game is. For someone it is only a way to spend a free time, for some it is a way of living and earning money. Implicitly games are associated with luck – usually we say that the winner was lucky, since in most of the cases we are not concerned about the things which hide under the moves in the game.

Nowadays there is a huge diversity of games. Some of them are not serious and does not involve intensive thinking and strategic moves. However, everyone would definitely agree that there are games, for instance – Chess, that are assumed to be serious, involving a lot of logics and being not a matter of luck, but of skills in reasoning, thinking ahead and experience a player has. In addition, outcome of the game is based on logics and strategy: a sequence of moves done in order to win. The certain sequence of moves which allowed one player to win the game – a winning strategy.

Since there are a lot of moves players can make, this leads us to the analysis of Combinatorial Game Theory. In this work we have chosen to analyze the simple game of Nimbers – the basis of all impartial games. Understanding the game of Nimbers leads to better understanding the Combinatorial Game Theory, revealing the winning strategies for other impartial games, applying basic mathematical terms in Games. Therefore this game will be analyzed in depth, related to the abstract mathematical terms (equivalence relation, classes, binary addition without carry etc.). The sequence of moves in order to win the game – winning strategy – will be introduced as well.

IMPARTIAL GAME

While analyzing the Game theory, we will always deal only with impartial games. Therefore it is necessary to distinguish an impartial game from the simple game. In addition to simple game properties, impartial game contains several unique properties.

Simple game is defined in this manner:

1. There are just two players.
2. There are several, usually finitely many, positions, and often a particular starting position.
3. There are clearly defined rules that specify the moves that either player can make from a given position to its options.
4. Players move alternately, in the game as a whole.
5. Both players know what is going on, i.e. there is complete information.
6. There are no chance moves such as rolling dice or shuffling cards.
7. In the normal pay convention a player unable to move, i.e. to make a legal move, loses.
8. The rules are such that play will always come to an end because some player will be unable to move. This is called the ending condition. So there are no games which are drawn by repetition of moves.

Unique property of impartial game is that we do not distinguish the players (e.g. player A, player B) the only thing that matters is who starts the game. In general case of the simple game we denote the possible number of moves in a game by G, then Gl denotes the moves of player A and Gr denotes the moves of player B.

Then the following relation holds:

$$G=(Gl|Gr)$$

In other words, G is the sum of all possible movements of players. However, when we deal with the impartial game, Gl=Gr. G=(Gl|Gr)=(Gr|Gl). We do not care about the players. The only thing that matters in impartial game is which player starts the game.

Hence, impartial game that we will deal with in this work is called Nimbers, or a nim-game. Arbitrary number of piles of beans (pebbles), each of arbitrary height *n*, is placed on the plane. One pile is called *a heap*, and in case the game consists of only one heap of height *n*, the game is said to have a value **n*.

Rules

The nim-game, just like the simple game, is played alternately by two players, where each of them can remove as many beans from only one heap as they want. Therefore, if the game consists of only one heap, the first player may end the game in one move (simply removing all the beans from the heap) and so win the game. In case there are several heaps, one cannot remove all the heaps at once, and the minimum number of moves needed to end the game is equal to the number of heaps.

The addition of games is defined in the following manner: adding two nim-games is the same as playing them simultaneously. Addition is based on the Nim-game rule that one player cannot make one move on two heaps at once – it has to remove the number of beans from only one heap. Moreover, since there is no rule on which heap a player is supposed to play, the arrangement of heaps does not matter at all. Then it is easy to see, that the addition of nim-games is both commutative and associative.

Since the Nim-game is an impartial game, the ending condition is when the consecutive player does not have a legal move to make. Therefore we can divide all the games according to which of the two players wins.

Types of Nim-games

Definition: We call a game a "0" game if the starting player loses.

We will provide two trivial cases of the "0" game. Later on, we will show how to distinguish "0" game more rigorously. There are some cases where the "0" game is obvious, e.g. if a game is started with no heaps of beans – no legal moves to make at all (*see Picture 1. a*). The one who starts will immediately lose. Another case is when the arrangement of heaps is symmetric, i.e. for each heap one can find another one of exactly the same height (*see Picture 1. b*).

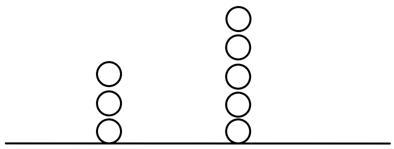
a) b)

Picture 1

Why these are "0" games and what moves are to be made in order to win will be introduced in the later on, when we will deal with the proof of existence of the winning strategy.

Definition: We call a game a “*” game when a starting player wins.

In order to show and analyze such games, we need some additional knowledge, such as binary addition without carry (XOR-addition). Therefore we will explain such games more thoroughly in the future. Yet, we will mention that if one plays a “*” game and knows the winning strategy, it can easily move to a “0” game and make the second player lose (*see Picture 2*).



Picture 2

WINNING STRATEGY

In a game every player is concerned about ending it successfully, i.e. winning it. In most of the cases, a winner is the player who has the highest experience and knowledge in the game, or by the probability theorem – the lucky one. However, they both will lose the game to the player who knows the winning strategy – the sequence of moves to be done in order to win.

Claim: Nim-game has a winning strategy.

Moreover, the one who knows it can determine the ending of the game from the initial conditions.

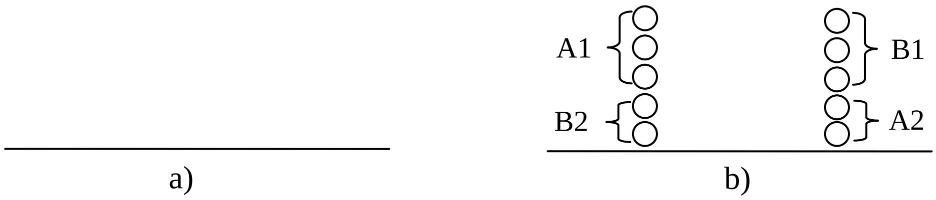
The “0” and “*” games will be used to illustrate the winning strategy for Nim-games. First of all we will show the winning strategy for the special case of “0” game – in this case heaps are arranged in symmetrical order. Later on we will move to winning strategy for “*” games and explained “0” game in details.

Tweedledum and Tweedledee

The given game is a “0” game, i.e. starting player always loses. This is obvious in a game when there are no heaps of beans, i.e. the player who starts the game has no legal moves to make and he loses instantly (*see Picture 3. a*).

Another case is when the arrangement of heaps is symmetric, i.e. for each heap one can find another one of exactly the same height. Then one uses the “*Tweedledee – Tweedledum*” strategy as the winning strategy: the second player must make exactly the same moves on another heap of the same height as the first player (the mirror effect). If the first player removes a number beans from an arbitrary heap, the second player must find another heap with the former height of previous heap remove exactly the same number of beans. Therefore, the first player will be always left with a symmetric arrangement of heaps. Finally, there will be no more heaps left and the first player will lose the game.

To illustrate the “*Tweedledee – Tweedledum*” strategy, consider two players. When the game starts, first player makes the first move *A1*. Following the “*Tweedledee – Tweedledum*” strategy, second player makes move *B1*, leaving first player with the symmetric arrangement after the move. Afterwards, *A2* move by first player follows. And second player makes move B2. Finally, first player has no legal moves to make, since there are no heaps on the plane, and he loses the game (*see Picture 3. b*).

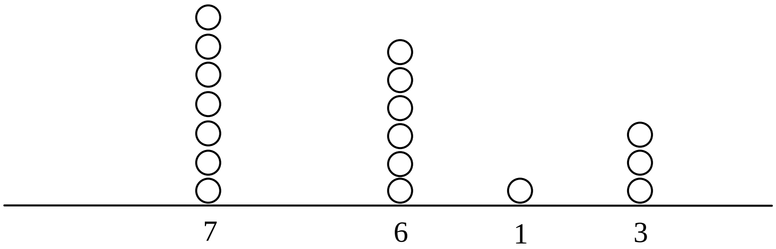


Picture 3

Binary addition without carry (XOR addition)

In this part the winning strategy for “*” and even more deeply for the “0” games will be analyzed using binary addition without carry also known as the XOR addition.

Firstly, an arbitrary “*” game should be analyzed. Consider such a case:



Picture 4

Here 4 nim heaps of the height 7, 6, 1 and 3 are given. The main idea of the winning strategy for this game is for the first player to create a “0” game after its move. This means, that first player always needs to push its opponent into the “0” game, i.e. second player will lose the game by definition of the “0” game, since he will be the first, who will start it. “0” game can be created using binary addition without carry (XOR addition) on the heaps of beans:

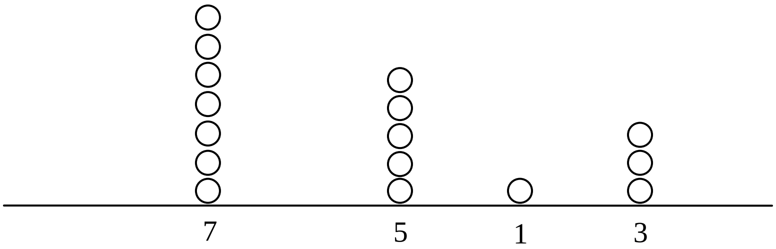
7 =	111
6 =	**110**
1 =	1
3 =	11
3 =	0**1**1

In the initial conditions the XOR sum of the nim heaps is 3. In order to win, first player must make such move, that the resulting XOR sum would be equal to 0. This is the winning strategy for “*” games. In order to do so, one must pick the highest 1 appearing in the XOR addition’s sum, i.e. the leftmost one, and then choose heap containing 1 in this particular position and adjust it in a way, that the sum would be equal to 0, i.e. “0” game. Pick nim heap of height 6 to be modified (as we see, it contains 1 in the second element of the binary form). We know that in XOR addition 1+1=0 and 0+1=1. Therefore one can change our chosen number in the way, that resulting sum would be equal to 0. Starting from the element determined by the highest 1 in the sum, we change the element in our chosen number by the inverse (1 ⇒ 0 and 0 ⇒ 1) if the correspondent element in the sum is equal to 1 and we leave it unchanged if the correspondent element of the sum is equal to 0.

In other words, player takes and adds a number of beans to the heap.

7 =	111		111	
6 =	**110**	⇒	**101**	**= 5 = 6 - 1**
1 =	1		1	
3 =	11		11	
3 =	0**1**1		000	

In fact we simply take 1 bean from the nim-heap of height 6. It results in a new game:



Picture 5

It has the sum equal to zero . Therefore second player has no choice but to make this game a "*" game once again, e .g it has to remove a number of beans and as a result, XOR sum will change and will not be equal to zero anymore. Then the first player uses the winning strategy again and makes this game's sum equal to zero. The game continues in this pattern until after first player's move there are no beans left ("0" game) and the second player loses.

Summary

Let's look again at the XOR addition – we always have two cases: it is equal to zero or it is not. Then the XOR addition is not equal to zero, then we have a "*" game, there the first player always wins changing the sum to zero after his move. This will result in a "0" game at the end (no beans on the plane) which XOR addition is also equal to zero. Therefore if game's XOR addition is equal to zero, this game is a "0" game.

Now we can name the winning strategy:

If XOR(G) is equal to zero, second player always wins. The winning strategy for the second player is always to move back to the "0" game (XOR addition is equals to zero).

If XOR(G) is not equal to zero, first player always wins. The winning strategy for the first player is always to move to a “0” game (XOR addition equals to zero).

We have shown that there is a winning strategy for all possible nim-games combinations. To complete our proof we must analyze the winning strategy and make sure that when we use XOR addition to get a “0” game (take or add beans) one will not result in the higher heap of beans (we will not add bens during our move):

$$\begin{array}{lr} & 0\mathit{1}010011 \\ \text{XOR} & \mathbf{1\mathit{1}010011} \\ & \underline{1\mathit{1}011101} \\ & 0\mathit{1}011101 \end{array}$$

in the sum, i.e. the element of the chosen number should be also equal to 1 and be in the same position. Then we modify the number in a way to get 0 in the sum, i.e. we change the number by changing its elements. In other words, when changing 0 and 1 to their inverses, we add or take beans from the heap. Therefore, there exists a possibility to result in the higher height than the initial height of the heap.

However, this cannot be true by the properties of binary numbers:

$$\begin{array}{rr} 2^n = & 111111 \\ \underline{2^0 =} & \underline{1} \\ 2^{n+1} = & 1000000 \end{array}$$

Or:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Therefore we cannot result in the higher heap than initial one.

EQUIVALENCE RELATION ON NIMBERS

Lemma 1: Adding a “0” game to another game does not change the outcome of the respective game.

Proof of Lemma 1:

Take an arbitrary game G and some “0” game O. We have to look at G + O:

- If G is a “0” game, the first player to begin will lose. If we add to G another zero game O, the move of the first player will still produce a winning game for the second one (either G or O). So by adding a “0” game to a “0” game we do not change the outcome.
- If G is a “*” game, the first player that begins will win the. If we add a “*” game O to G, the first player’s move will transform G in a “0” game in order for the second one to have only losing games. As in game G, again the first player will always win and thus adding a “0” game O will not change the final result. □

Definition: We define a relation “~” between two games G and H as:

$$G \sim H := G + H = 0$$

Theorem 1: “~”, the relation between two games G and H, is an equivalence relation.

Proof of theorem 1:

The relation “~” is an equivalence relation if and only if it satisfies the conditions below:

- Reflexivity: G ~ G

  By definition 1, G ~ G => G + G “=” “0” or G ~ G => G + G is a zero game.

  If we add two identical games we will have a zero game. The proof follows easily from the “*Tweedledee* - *Tweedledum*” strategy: if one player copies the same moves of the first player and has the same setup of the game, the player who will move first will be always in a loosing position. □

- Symmetry: G ~ H => H ~ G

  By definition, G ~ H => H ~ G means G + H “=” “0” => H + G “=” “0”

  If first player begins with a “0” game G + H, then second player has a winning strategy on the game G + H and since the games G + H and H + G are identical by the properties of addition, second player will also have a winning strategy on H + G which implies that H + G is a “0” game. □

- Transitivity: G ~ H and H ~ K => G ~ K

  Again by definition 1, G ~ H means G + H “=” “0”, H ~ K means H + K “=” “0”, and G ~ K means G + K “=” “0”.

  By the properties of the addition we can add the games G + H and H + K and because both are “0” games, the game G + H + H + K has to be a “0” game. Grouping the games H and H, from reflexivity we have H + H “=” “0”. By *Lemma 1*, G + H + H + K = G + K “=” “0” which implies that G ~ K. □

Addition of nimbers is well defined

We call a nimber equivalence class the set of all games G and H that satisfy the equivalence relation “~”.

Let G, G’, H, H’ be some games that satisfy the relations:

G ~ G’ and H ~ H’

If addition is well defined on this equivalence class then from:

G ~ G’ and H ~ H’ => G + H ~ G’ H’

because G + H ~ G’ + H’ <=> G + H + G’ + H’ = 0, which is true since G + G’ = 0 and H + H’ = 0 from the equivalence relations G ~ G’ and H ~ H’. □

OTHER THEOREMS

Having defined and proven the properties serving as a basis of the Nim-game, now we are going to analyze the game further and deeper in the global aspect and will introduce you to the three additional theorems which relate the Nim-game with any other impartial game. These three theorems cover a part of curriculum and therefore are very important and interesting. We will not provide the rigorous proofs for these three theorems, because this would require much additional knowledge and thus it is not the aim of this work: to provide large amount of theorems, rather to introduce the reader with the theory and the game of Nimbers and make him interested in it.

So, once we have proved that two nim-games are in an equivalence relation and we may determine the type of a game by using the XOR addition. Now we would like to relate an arbitrary nim-game with one nim-heap of certain height *n*.

Theorem 3: Every nim-game G is equivalent to a nim-heap of height *n* (**n*)

Proof of theorem 3: Let us take an arbitrary Nim-game G, consisting of several heaps. In order to find out whether it is a “0” game or a “*” game, one always has to calculate the binary value of the game (XOR value). This provides us with a sum, which is also a binary value. Now take a new heap of height equal to the binary sum of the former game. If we add these two games, we will definitely end up with the sum of zero, because 1’s and 0’s will appear in exactly the same places in both numbers. By definition, equivalence between games means that the sum of the games equals zero – this is exactly what we have. In general, no matter what Nim-game it is, one will always be able to calculate the binary sum of the game, and there will always be a certain heap of height equal to the sum of heaps in the chosen game. Thus, there always exists a number *n* such that a Nim-game is equivalent to a Nim-heap of height *n*. □

This theorem is of great use when playing the huge nim-game. According to this theorem, if one notices or calculates that in a game there are several heaps which together have the same binary value as another single heap, one can disregard them altogether and replace then by that single heap, since this part of the game will result in a “0” game. After some time and extensive calculations, one may already remember that, for example, *2 + *4 = *6, or *2 + *3 = *1 etc., and disregard such heaps at once. Therefore this theorem helps a lot when one has to play very complicated Nim-games.

Theorem 4: A nim-value can be determined using a “mex-rule”

The “*mex-rule*” (***M****inimal* ***EX****cluded*) allows a player to determine the value of a nim-game without using the binary addition. One simply has to write down all possible moves (options), i.e. how many pebbles one can remove at once, at look for the minimal value which is not possible to reach in one move. Then one has to calculate the values of options and then the final value is the same as value of the nim-game. Although this way of determining the value of a game is not very convenient, since one has to write down all the possible moves and it takes a lot of time to complete it. Theorem shows an interesting relation between the number of pebbles heaps contain and the binary sum of all the heaps.

Theorem 5: Any impartial game is equivalent to a nim-heap of height n

This theorem is of crucial importance because it allows a player playing any impartial game consider its value to be a nim-game value with some heaps of certain height.

CONCLUSION

With this work reader was shortly and briefly introduced to the Combinatorial Game Theory. We introduced one of the most universal and simplest impartial games, proved that there is a winning strategy and that, by using some mathematical knowledge and properties of Nim-game, one may determine the prospect and pattern of any impartial game. We have chosen to base our project on Nim-game, because it involves a lot of knowledge provided by Foundations of Mathematics course and a lot of intuition.

This theory, as many other mathematical theories, may not be easily applied in every day life; it provides one with idea how to apply our intuition and a lot of logical exercises, encourages to search for a pattern everywhere and to be successful at the end.

We hope that the reader will be interested in Combinatorial Game Theory and will delve into it on his own initiative, to find that Nim-game forms Abelian group, that there is another way to express the real numbers in Combinatorial Game Theory etc.

REFERENCES

1. Berlekamp Elwyn R., Conway John H., Guy Richard K. Winning Ways for Your Mathematical Plays. Natick, Massachusetts, USA: “A K Peters”, 2001
2. Conway John H. On Numbers and Games (second edition). Natick, Massachusetts, USA: “A K Peters”, 2001