Comparison of Black-Scholes and Heat equations

Algirdas Grybas

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1 Black-Scholes differential equation

We assume that the price of a single non-dividend-paying common stock S follows the geometric Brownian motion W_t with expected return μ and volatility σ:

$$dS = \mu S dt + \sigma S dW_t \tag{1}$$

Let $V = V(S, t)$ denote the value of a single European call option. Then, according to Ito's lemma,

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t \tag{2}$$

Since the two Wiener processes underlying the values of V and S are the same, one can create a riskless (i.e. without the stochastic component) portfolio by selling on call option and buying $\frac{\partial V}{\partial S}$ stocks. The change of portfolio value $\Pi = -V + \frac{\partial V}{\partial S} S$ is

$$\begin{aligned} d\Pi &= -dV + \frac{\partial V}{\partial S} dS \\ &= -\left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt - \sigma S \frac{\partial V}{\partial S} dW_t + \mu S \frac{\partial V}{\partial S} dt + \sigma S \frac{\partial V}{\partial S} dW_t \\ &= -\left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt \end{aligned} \tag{3}$$

The portfolio Π is riskless, so its return should be

$$d\Pi = r\Pi dt \tag{4}$$

where r is the risk-free return. Equating (3) and (4), we get the *Black-Scholes differential equation*

$$rV = \frac{\partial V}{\partial S} rS + \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}, \quad S > 0, \quad 0 \tag{5}$$

2 Boundary conditions

From the definition of the European call option there follows one of the basic conditions - the "payoff condition:

$$V(S,t) = \max(S-K,0), \quad S \geq 0 \tag{6}$$

If this lower bound was not satisfied, one could buy a call for V, immediately exercise the option buying the stock for K and gain riskless profit of $S-V-K>0$, violating the *no-arbitrage principle*.

Another, almost redundant condition considers the case when $S=0$:

$$V(0,t) = 0 \tag{7}$$

Finally, for large S the call value should approximate the stock price:

$$V(S,t) \sim S \text{ as } S \to \infty \tag{8}$$

3 Transformation to the Heat equation

Before we start transforming the Black-Scholes equation to the Heat equation, note that the latter looks a little like the heat equation on the infinite interval in that it has a first derivative with respect to time and the second derivative with respect to the other (space) variable. However, each time V is differentiated with respect to S, it is also multiplied by S, so the equation is not a constant coefficient equation. Second, there is a first derivative of V with respect to S in the equation and a zero-th order term rV in the equation. Finally, the sign on the second derivative is the opposite of the heat equation form, so the equation is of *backward parabolic* form.

We first transform (5) from a *backward-in-time* to *forward-in-time* by the change of variables $t = T - \frac{2\tau}{\sigma^2}$. Let $S = Ke^x$ and $V(S,t) = Kv(x,t)$. Then

$$\begin{aligned}
\tau &= \frac{\sigma^2}{2}(T-t) \\
x &= \ln(S/K) = \ln(S) - \ln(K)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\partial V}{\partial t} &= K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau} && (9)\\
\frac{\partial V}{\partial S} &= K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x} && (10)\\
\frac{\partial^2 V}{\partial S^2} &= -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right) = \frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2} && (11)
\end{aligned}$$

Plugging this into (5), we get

$$\begin{aligned}
0 &= -rKv + rS\frac{K}{S}\frac{\partial v}{\partial x} - \frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau} + \frac{\sigma^2 S^2}{2}\left(-\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}\right) \\
\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} &= -rv + r\frac{\partial v}{\partial x} + \frac{\sigma^2}{2}\left(-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) \\
\frac{\partial v}{\partial \tau} &= -\tilde{r}v + \tilde{r}\frac{\partial v}{\partial x} + \left(-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right), \text{ where } \tilde{r} = \frac{2r}{\sigma^2} \\
\frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial x^2} + (\tilde{r}-1)\frac{\partial v}{\partial x} - \tilde{r}v && (12)
\end{aligned}$$

Change the variables once again by letting $v(x,\tau) = e^{\alpha x+\beta\tau}u(x,\tau)$, where α and β are constants. Then

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \beta e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} \\
\frac{\partial v}{\partial x} &= \alpha e^{\alpha x+\beta\tau}u(x,\tau) + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} \\
\frac{\partial^2 v}{\partial x^2} &= \alpha^2 e^{\alpha x+\beta\tau}u(x,\tau) + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Plug these derivatives into (12) and dividing by $e^{\alpha x+\beta\tau}$ we get

$$\begin{aligned}
\beta u + u_\tau &= \alpha^2 u + 2\alpha u_x + u_{xx} + (\tilde{r}-1)(\alpha u + u_x) - \tilde{r}u \\
u_\tau &= u_{xx} + (2\alpha + (\tilde{r}-1))u_x + (\alpha^2 + (\tilde{r}-1)\alpha - \tilde{r} - \beta)u
\end{aligned}$$

Finally, let $\alpha = -(\tilde{r}-1)/2$ and $\beta = \alpha^2 + (\tilde{r}-1)\alpha - \tilde{r} = -(\tilde{r}+1)^2/4$ to get the Heat equation

$$u_\tau = u_{xx}, \quad -\infty < x < +\infty \tag{13}$$

with the initial condition

$$u(x,0) = e^{-\alpha x}v(x,0) = \max(e^{(\tilde{r}+1)/2)x} - e^{((\tilde{r}-1)/2)x}, 0) \tag{14}$$

since $V(S,t) = Kv(x,0)$ and so $v(x,0) = \max(e^x - 1, 0)$.

4 Call option pricing formula

The *fundamental solution* of the Heat equation is given by

$$\phi(x,\tau) = \frac{1}{(4\pi\tau)^{n/2}}e^{-\frac{x^2}{4\tau}} \tag{15}$$

In our case, $n = 1$ and so

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4\tau}}u(s,0)ds \tag{16}$$

Change the variables by letting $z = \frac{s-x}{\sqrt{2\tau}}$. Then $dz = \frac{-1}{\sqrt{2\tau}}dz$. Also, from (14) we see that $u > 0$ iff $x > 0$, so we can restrict the integration range to $z > -\frac{x}{\sqrt{2\tau}}$. Then

$$\begin{aligned}
u(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{\tilde{r}+1}{2}(x+z\sqrt{2\tau})}e^{-z^2/2}dz - \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{\tilde{r}-1}{2}(x+z\sqrt{2\tau})}e^{-z^2/2}dz \\
&=: I_1 - I_2
\end{aligned}$$

Complete the square for the exponent in I_1:

$$\begin{aligned}
\frac{\tilde{r}+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} &= -\frac{1}{2}\left(z^2 - \sqrt{2\tau}(\tilde{r}+1)z\right) + \frac{\tilde{r}+1}{2}x \\
&= -\frac{1}{2}\left(z^2 - \sqrt{2\tau}(\tilde{r}+1)z + \frac{\tau(\tilde{r}+1)^2}{2}\right) + \frac{\tilde{r}+1}{2}x + \frac{\tau(\tilde{r}+1)^2}{4} \\
&= -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \frac{\tau(\tilde{r}+1)^2}{4} \\
&=: -\frac{1}{2}y^2 + c
\end{aligned}$$

Note that c does not depend on z, so we can move it in front of the integral. Also note that $dy = dz$, so

$$I_1 = \frac{e^c}{\sqrt{2\pi}} \int\limits_{-x/\sqrt{2\pi}-\sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-y^2/2} dy \tag{17}$$

Finally, recall that the cumulative distribution function of a normally-distributed random variable is

$$\Phi(d) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{d} e^{-y^2/2} dy \tag{18}$$

so

$$I_1 = e^c \Phi(d_1), \text{ where } d_1 = x/\sqrt{2\pi} + \sqrt{\tau/2}(\tilde{r}+1) \tag{19}$$

Calculation of I_2 is analogous to that of I_1 and the result is the same as (19) with $\tilde{r}+1$ replaced by $\tilde{r}-1$.

Upon changing back all variables, we obtain the *European call option pricing formula*:

$$V(S,t) = S\Phi\left(\frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{S}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \tag{20}$$

References

[1] Hull, J. C. *Options, Futures, and Other Derivatives*, 2004

[2] Website of S. Dunbar at University of Nebraska Lincoln: http://www.math.unl.edu/~sdunbar1/Teaching/MathematicalFinance/Lessons/BlackScholes/Solution/solution.shtml