Proofs Practice Worksheet

1. An integer n is called *even* if there is some integer m such that $n = 2m$, and *odd* if there is some integer m such that $n = 2m + 1$. Using these definitions, prove the following:

   (a) The sum of any two even integers is even.
   (b) The sum of any two odd integers is even.
   (c) No integer is both even and odd.

2. Let $\sim$ be a (two-place) relation on a set X, and consider the following possible properties of such a relation:

   Reflexive: for every $x \in X$, $x \sim x$.
   Symmetric: whenever $x \sim y$, also $y \sim x$.
   Transitive: whenever $x \sim y$ and $y \sim z$, also $x \sim z$.
   Euclidean: whenever $x \sim y$ and $x \sim z$, also $y \sim z$.

   (a) Prove that every relation that is reflexive and Euclidean is also symmetric.
   (b) Prove that every relation that is symmetric and transitive is also Euclidean.
   (c) Prove that not every reflexive and symmetric relation is Euclidean.
   (d) Prove that not every transitive and Euclidean relation is symmetric.
   (e) What is wrong with the following argument? "Every symmetric and transitive relation must be reflexive. To see why, take $x \in X$ and suppose $(x, y) \in R$. Then, by symmetry, $(y, x) \in R$ too. So by transitivity, $(x, x) \in R$. Since x was chosen arbitrarily, this shows that R is reflexive."

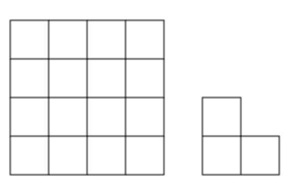
3. Let X be a finite set. A *weighting* on X is any function $f : X \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the non-negative real numbers. Given a weighting f on X and any subset $A \subseteq X$, define $f(A) = \sum_{x \in A} f(x)$. A *probability measure* on X is a weighting p on X such that $p(X) = 1$. Let p be a probability measure on X.

   (a) Prove that for all $x \in X$, $p(x) \leq 1$.
   (b) Prove that for all $A \subseteq X$, $p(X \setminus A) = 1 - p(A)$, where $X \setminus A = \{x \in X \ : \ x \notin A\}$.

   Given two functions $f, g : X \to \mathbb{R}$, define the *sum* $f + g : X \to \mathbb{R}$ by setting $(f + g)(x) = f(x) + g(x)$. Given a nonzero weighting f (i.e., a weighting with $f(X) > 0$), define the *normalization* $\bar{f} : X \to \mathbb{R}^+$ by $\bar{f}(x) = \frac{f(x)}{f(X)}$.

   (c) Show that the sum of two weightings is a weighting, but the sum of two probability measures is not a probability measure.
   (d) Show that the normalization of a (nonzero) weighting is a probability measure.
   (e) Let p and q be probability measures. Then $\overline{p + q}$ is also a probability measure (why?). Show that $\overline{p + q}$ assigns to each $x \in X$ the average of the values assigned by p and q.
   (f) Let p be a probability measure on X and let $f_1 : X \to \mathbb{R}^+$ be the constant function that always outputs 1. Show that the equality $\overline{p + f_1} = p$ does not hold in general, but does in certain cases.

4. The figure below shows, on the left, a surface made up of adjacent (same-sized) squares arranged in a 4×4 grid, and on the right, an "L-tile", which is composed of 3 such squares in the given pattern. An *L-tiling* of a surface made up of (same-sized) squares is a way of arranging L-tiles on top of it such that (1) each square of each L-tile is directly over some square of the underlying surface, and (2) every square of the underlying surface is covered by exactly one L-tile.

   (a) Consider the surface obtained by removing 1 corner square from the 4×4 surface. Prove that there is an L-tiling of this surface.
   (b) Prove that there is *no* L-tiling of the 4×4 surface, or of any $2^n \times 2^n$ surface for $n \geq 1$.
   (c) [BONUS] Prove that for every $n \geq 1$, there is an L-tiling of the surface obtained by removing 1 corner square from the $2^n \times 2^n$ surface.