The Two-Point Fano and Ideal Binary Clutters

Ahmad Abdi[(✉)] and Bertrand Guenin

Department of Combinatorics and Optimization,
University of Waterloo, Waterloo, Canada
{a3abdi,bguenin}@uwaterloo.ca

Abstract. Let $\mathbb{F}$ be a binary clutter. We prove that if $\mathbb{F}$ is non-ideal, then either $\mathbb{F}$ or its blocker $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5, \mathbb{LC}_7$ as a minor. $\mathbb{L}_7$ is the non-ideal clutter of the lines of the Fano plane, $\mathbb{O}_5$ is the non-ideal clutter of odd circuits of the complete graph K_5, and the *two-point Fano* $\mathbb{LC}_7$ is the ideal clutter whose sets are the lines, and their complements, of the Fano plane that contain exactly one of two fixed points. In fact, we prove the following stronger statement: if $\mathbb{F}$ is a minimally non-ideal binary clutter different from $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$, then through every element, either $\mathbb{F}$ or $b(\mathbb{F})$ has a two-point Fano minor.

1 Introduction

Let E be a finite set. A *clutter* $\mathbb{F}$ over *ground set* $E(\mathbb{F}) := E$ is a family of subsets of E, where no subset is contained in another. We say that $\mathbb{F}$ is *binary* if the symmetric difference of any odd number of sets in $\mathbb{F}$ contains a set of $\mathbb{F}$. We say that $\mathbb{F}$ is *ideal* if the polyhedron

$$Q(\mathbb{F}) := \left\{ x \in \mathbb{R}^E_+ : \sum (x_e : e \in C) \geq 1 \quad C \in \mathbb{F} \right\}$$

has only integral extreme points; otherwise it is *non-ideal*. When is a binary clutter ideal? We will be studying this question.

Let us describe some examples of ideal and non-ideal binary clutters. Given a graph G and distinct vertices s, t, the clutter of st-paths of G over the edge-set is binary. An immediate consequence of Menger's theorem [12], as well as Ford and Fulkerson's theorem [6], is that this binary clutter is ideal [3]. The clutter of *lines of the Fano plane*

$$\mathbb{L}_7 := \big\{ \{1,2,6\}, \{1,4,7\}, \{1,3,5\}, \{2,5,7\}, \{2,3,4\}, \{3,6,7\}, \{4,5,6\} \big\}$$

is binary, and it is non-ideal as $\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ is an extreme point of $Q(\mathbb{L}_7)$. (See Fig. 1.) The clutter of odd circuits of K_5 over its ten edges, denoted $\mathbb{O}_5$, is also binary, and it is non-ideal as $\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ is an extreme point of $Q(\mathbb{O}_5)$.

We say that two clutters are *isomorphic* if relabeling the ground set of one yields the other. There are two fundamental clutter operations that preserve being binary and ideal, let us describe them. The *blocker of* $\mathbb{F}$, denoted $b(\mathbb{F})$, is another clutter over the same ground set whose sets are the (inclusionwise)

F. Eisenbrand and J. Koenemann (Eds.): IPCO 2017, LNCS 10328, pp. 1–12, 2017.
DOI: 10.1007/978-3-319-59250-3_1

minimal sets in $\{B \subseteq E : B \cap C \neq \emptyset\ \forall C \in \mathbb{F}\}$. It is well-known that $b(b(\mathbb{F})) = \mathbb{F}$ [5]. We may therefore call $\mathbb{F}, b(\mathbb{F})$ a *blocking* pair. A clutter $\mathbb{F}$ is binary if, and only if, $|B \cap C|$ is odd for all $B \in b(\mathbb{F})$ and $C \in \mathbb{F}$ [9]. Hence, if $\mathbb{F}$ is binary, then so is $b(\mathbb{F})$. Lehman's Width-Length Inequality shows that if $\mathbb{F}$ is ideal, then so is $b(\mathbb{F})$ [10]. In particular, since $\mathbb{L}_7$ and $\mathbb{O}_5$ are non-ideal, then so are $b(\mathbb{L}_7) = \mathbb{L}_7$ and $b(\mathbb{O}_5)$. Let I, J be disjoint subsets of E. Denote by $\mathbb{F}\backslash I/J$ the clutter over $E - (I \cup J)$ of minimal sets of $\{C - J : C \in \mathbb{F}, C \cap I = \emptyset\}$.[1] We say that $\mathbb{F}\backslash I/J$, and any clutter isomorphic to it, is a *minor of* $\mathbb{F}$ obtained after *deleting* I and *contracting* J. If $I \cup J \neq \emptyset$, then $\mathbb{F}\backslash I/J$ is a *proper* minor of $\mathbb{F}$. It is well-known that $b(\mathbb{F}\backslash I/J) = b(\mathbb{F})/I\backslash J$ [16]. If a clutter is binary, then so is every minor of it, and if a clutter is ideal, then so is every minor of it [17].

Let $\mathbb{F}$ be a binary clutter. Regrouping what we discussed, if $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5$ as a minor, then it is non-ideal. Seymour [17] (p. 200) conjectures the converse is also true:

The flowing conjecture. Let $\mathbb{F}$ be a non-ideal binary clutter. Then $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5$ as a minor.

The *two-point Fano clutter*, denoted by $\mathbb{LC}_7$, is the clutter over ground set $\{1, \ldots, 7\}$ whose sets are the lines, and their complements, of the Fano plane that intersect $\{1, 4\}$ exactly once, i.e. $\mathbb{LC}_7$ consists of $\{1,2,6\}, \{1,3,5\}, \{2,3,4\}, \{2,5,7\}$ and $\{3,4,5,7\}, \{2,4,6,7\}, \{1,5,6,7\}, \{1,3,4,6\}$. Observe that changing the two points $1, 4$ yields an isomorphic clutter. It can be readily checked that $\mathbb{LC}_7$ is binary *and* ideal. In this paper, we prove the following weakening of the flowing conjecture:

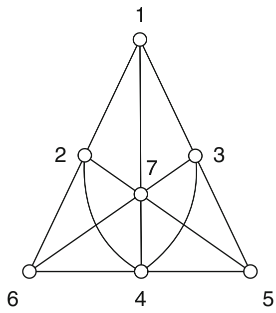


Fig. 1. The Fano plane

Theorem 1. *Let $\mathbb{F}$ be a non-ideal binary clutter. Then $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5, \mathbb{LC}_7$ as a minor.*

What makes this result attractive is its relatively simple proof. The techniques used in the proof give hope of resolving the flowing conjecture. An interesting feature of the proof is the interplay between the clutter $\mathbb{F}$ and its blocker $b(\mathbb{F})$; if we fail to find one of the desired minors in the clutter, we switch to the blocker and find a desired minor there. Theorem 1 is a consequence of a stronger statement stated in the next section.

2 Preliminaries and the Main Theorem

2.1 Minimally Non-ideal Binary Clutters

A clutter is *minimally non-ideal (mni)* if it is non-ideal and every proper minor of it is ideal. Notice that every non-ideal clutter has an mni minor, and if a clutter

[1] Given sets A, B we denote by $A - B$ the set $\{a \in A : a \notin B\}$ and, for element a, we write $A - a$ instead of $A - \{a\}$.

is mni, then so is its blocker. Justified by this observation, instead of working with non-ideal binary clutters, we will work with mni binary clutters. The three clutters $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ are mni, and the flowing conjecture predicts that these are the *only* mni binary clutters. We will need the following result of the authors:

Theorem 2 **([1]).** $\mathbb{L}_7, \mathbb{O}_5$ *are the only mni binary clutters with a set of size 3.*

We will also need the following intermediate result of Alfred Lehman on mni clutters, stated only for binary clutters. Let $\mathbb{F}$ be a clutter over ground set E. Denote by $\bar{\mathbb{F}}$ the clutter of minimum size sets of $\mathbb{F}$. Denote by $M(\mathbb{F})$ the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of the sets of $\mathbb{F}$. For an integer $r \geq 1$, a square $0-1$ matrix is r-*regular* if every row and every column has precisely r ones.

Theorem 3 **([2,11,15]).** *Let $\mathbb{F}$ be an mni binary clutter where $n := |E(\mathbb{F})|$, and let $\mathbb{K} := b(\mathbb{F})$. Then*

(1) $M(\bar{\mathbb{F}})$ and $M(\bar{\mathbb{K}})$ are square and non-singular matrices,
(2) $M(\bar{\mathbb{F}})$ is r-regular and $M(\bar{\mathbb{K}})$ is s-regular, for some integers $r \geq 3$ and $s \geq 3$ such that $rs - n$ is even and $rs - n \geq 2$,
(3) after possibly permuting the rows of $M(\bar{\mathbb{K}})$, we have that

$$M(\bar{\mathbb{F}})M(\bar{\mathbb{K}})^\top = J + (rs-n)I = M(\bar{\mathbb{K}})^\top M(\bar{\mathbb{F}}).$$

Here, J denotes the all-ones matrix, and I the identity matrix. Given a ground set E and a set $C \subseteq E$, denote by $\chi_C \subseteq \{0,1\}^E$ the incidence vector of C. We will make use of the following corollary:

Corollary 4. *Let $\mathbb{F}$ be an mni binary clutter. Then the following statements hold:*

(1) For $C_1, C_2 \in \bar{\mathbb{F}}$, the only sets of $\mathbb{F}$ contained in $C_1 \cup C_2$ are C_1, C_2 [7,8].
(2) Choose $C_1, C_2, C_3 \in \bar{\mathbb{F}}$ and $e \in E(\mathbb{F})$ such that $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{e\}$. If C, C' are sets of $\mathbb{F}$ such that $C \cup C' \subseteq C_1 \cup C_2 \cup C_3$ and $C \cap C' \subseteq \{e\}$, then $\{C, C'\} = \{C_i, C_j\}$ for some distinct $i, j \in \{1,2,3\}$.

Proof. **(2)** Denote by r the minimum size of a set in $\mathbb{F}$. As $\mathbb{F}$ is binary, $C_1 \triangle C_2 \triangle C_3 \triangle C \triangle C'$ contains another set C'' of $\mathbb{F}$. Notice that $C'' \cap C \subseteq \{e\}$ and $C'' \cap C' \subseteq \{e\}$. If k many of C, C', C'' contain e, then

$$3r - 3 = |(C_1 \cup C_2 \cup C_3) - e| \geq |(C \cup C' \cup C'') - e| = |C| + |C'| + |C''| - k \geq 3r - k,$$

implying in turn that $k = 3$ and equality must hold throughout. In particular, $C, C', C'' \in \bar{\mathbb{F}}$ and $\chi_{C_1} + \chi_{C_2} + \chi_{C_3} = \chi_C + \chi_{C'} + \chi_{C''}$, so as $M(\bar{\mathbb{F}})$ is non-singular by Theorem 3 (1), we get that $\{C_1, C_2, C_3\} = \{C, C', C''\}$. $\square$

2.2 Signed Matroids

All matroids considered in this paper are binary; we follow the notation used in Oxley [14]. Let M be a matroid over ground set E. Recall that a circuit is a minimal dependent set of M and a cocircuit is a minimal dependent set of the dual $M^\star$. A *cycle* is the symmetric difference of circuits, and a *cocycle* is the symmetric difference of cocircuits. It is well-known that a nonempty cycle is a disjoint union of circuits ([14], Theorem 9.1.2). Let $\Sigma \subseteq E$. The pair (M, Σ) is called a *signed matroid* over ground set E. An *odd circuit of* (M, Σ) is a circuit C of M such that $|C \cap \Sigma|$ is odd.

Proposition 5 ([9,13], also see [4]). *The clutter of odd circuits of a signed matroid is binary. Conversely, a binary clutter is the clutter of odd circuits of a signed matroid.*

A *representation* of a binary clutter $\mathbb{F}$ is a signed matroid whose clutter of odd circuits is $\mathbb{F}$. By the preceding proposition, every binary clutter has a representation. For instance, $\mathbb{L}_7$ is represented as $\big(F_7, E(F_7)\big)$, where F_7 is the Fano matroid. A *signature of* (M, Σ) is any subset of the form $\Sigma \triangle D$, where D is a cocycle of M; to *resign* is to replace (M, Σ) by $(M, \Sigma \triangle D)$. Notice that resigning does not change the family of odd cycles. We say that two signed matroids are *isomorphic* if one can be obtained from the other after a relabeling of the ground set and a resigning.

Remark 6. *Take an arbitrary element ω of F_7. Then $(F_7, E(F_7) - \omega)$ represents $\mathbb{LC}_7$.*

Proof. Suppose $E(F_7) = \{1, \ldots, 7\}$, and since F_7 is transitive, we may assume that $\omega = 7$. Consider the following representation of F_7,

$$\begin{pmatrix} 1\,0\,0\,0\,1\,1\,1 \\ 0\,1\,0\,1\,0\,1\,1 \\ 0\,0\,1\,1\,1\,0\,1 \end{pmatrix}$$

where the columns are labeled $1, \ldots, 7$ from left to right. Since $\{2, 3, 5, 6\}$ is a cocycle of F_7, $(F_7, \{1, \ldots, 6\})$ is isomorphic to $(F_7, \{1, \ldots, 6\} \triangle \{2, 3, 5, 6\}) = (F_7, \{1, 4\})$. It can be readily checked that the odd circuits of $(F_7, \{1, 4\})$ are precisely the sets of $\mathbb{LC}_7$, thereby proving the remark. $\square$

Proposition 7 ([9,13], also see [8]). *In a signed matroid, the clutter of minimal signatures is the blocker of the clutter of odd circuits.*

Let I, J be disjoint subsets of E. The *minor* $(M, \Sigma) \backslash I/J$ obtained after *deleting* I and *contracting* J is the signed matroid defined as follows: if J contains an odd circuit, then $(M, \Sigma) \backslash I/J := (M \backslash I/J, \emptyset)$, and if J does not contain an odd circuit, then there is a signature Σ' of (M, Σ) disjoint from J by the preceding proposition, and we let $(M, \Sigma) \backslash I/J := (M \backslash I/J, \Sigma' - I)$. Observe that minors are defined up to resigning.

Proposition 8 ([13], also see [4]). *Let $\mathbb{F}$ be a binary clutter represented as (M, Σ), and take disjoint $I, J \subseteq E(\mathbb{F})$. Then $\mathbb{F} \backslash I/J$ is represented as $(M, \Sigma) \backslash I/J$.*

2.3 Hubs and the Main Theorem

Let (M, Σ) be a signed matroid, and take $e \in E(M)$. An *e-hub of* (M, Σ) is a triple (C_1, C_2, C_3) satisfying the following conditions:

(h1) C_1, C_2, C_3 are odd circuits such that, for distinct $i, j \in \{1, 2, 3\}$, $C_i \cap C_j = \{e\}$,
(h2) for distinct $i, j \in \{1, 2, 3\}$, the only nonempty cycles contained in $C_i \cup C_j$ are $C_i, C_j, C_i \triangle C_j$,
(h3) a cycle contained in $C_1 \cup C_2 \cup C_3$ is odd if and only if it contains e.

A *strict* e-hub is an e-hub (C_1, C_2, C_3) such that the following holds:

(h4) if C, C' are odd cycles contained in $C_1 \cup C_2 \cup C_3$ such that $C \cap C' = \{e\}$, then for some distinct $i, j \in \{1, 2, 3\}$, $\{C, C'\} = \{C_i, C_j\}$.

Given $I \subseteq E$, denote by $M|I$ the minor $M \setminus (E - I)$, and by $(M, \Sigma)|I$ the minor $(M, \Sigma) \setminus (E - I)$. The following is the main result of the paper:

Theorem 9. *Let $\mathbb{F}, \mathbb{K}$ be a blocking pair of mni binary clutters over ground set E, neither of which has a set of size 3. Let (M, Σ) represent $\mathbb{F}$ and let (N, Γ) represent $\mathbb{K}$. Then, for a given $e \in E$, the following statements hold:*

(1) (M, Σ) has a strict e-hub (C_1, C_2, C_3) and (N, Γ) has a strict e-hub (B_1, B_2, B_3) where for $i, j \in \{1, 2, 3\}$,

$$|C_i \cap B_j| \begin{cases} \geq 3 & \text{if } i = j \\ = 1 & \text{if } i \neq j, \end{cases}$$

(2) either $M|(C_1 \cup C_2 \cup C_3)$ or $N|(B_1 \cup B_2 \cup B_3)$ is non-graphic,
(3) if $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic, then $(M, \Sigma)\backslash I/J \cong (F_7, E(F_7) - \omega)$ for some disjoint $I, J \subseteq E - e$, and similarly, if $N|(B_1 \cup B_2 \cup B_3)$ is non-graphic, then $(N, \Gamma)\backslash I/J \cong (F_7, E(F_7) - \omega)$ for some disjoint $I, J \subseteq E - e$.

Given this result, let us prove Theorem 1:

Proof (of Theorem 1). Let $\mathbb{F}$ be a non-ideal binary clutter, let $\mathbb{F}'$ be an mni minor of $\mathbb{F}$, and let $\mathbb{K}' := b(\mathbb{F}')$. If $\mathbb{F}'$ has a set of size 3, then by Theorem 2, $\mathbb{F}' \cong \mathbb{L}_7$ or $\mathbb{O}_5$. If $\mathbb{K}'$ has a set of size 3, then by Theorem 2, $\mathbb{K}' \cong \mathbb{L}_7$ or $\mathbb{O}_5$. Thus, if one of $\mathbb{F}', \mathbb{K}'$ has a set of size 3, then either $\mathbb{F}$ or $b(\mathbb{F})$ has one of $\mathbb{L}_7, \mathbb{O}_5$ as a minor. We may therefore assume that neither $\mathbb{F}'$ nor $\mathbb{K}'$ has a set of size 3. Let (M, Σ) represent $\mathbb{F}'$ and let (N, Γ) represent $\mathbb{K}'$, whose existence are guaranteed by Proposition 5. It then follows from Theorem 9 (2)–(3) that either (M, Σ) or (N, Γ) has an $(F_7, E(F_7) - \omega)$ minor. By Remark 6 and Proposition 8, we see that either $\mathbb{F}'$ or $\mathbb{K}'$ has an $\mathbb{LC}_7$ minor, implying in turn that either $\mathbb{F}$ or $b(\mathbb{F})$ has an $\mathbb{LC}_7$ minor, as required. □

In the remainder of this paper, we prove Theorem 9.

3 Proof of Theorem 9 Part (1)

Let $\mathbb{F}, \mathbb{K}$ be blocking mni binary clutters over ground set E, neither of which has a set of size 3. By Theorem 3, there are integers $r \geq 4$ and $s \geq 4$ such that $M(\bar{\mathbb{F}})$ is r-regular, $M(\bar{\mathbb{K}})$ is s-regular, and after possibly permuting the rows of $M(\bar{\mathbb{K}})$, $M(\bar{\mathbb{F}})M(\bar{\mathbb{K}})^\top = J + (rs-n)I = M(\bar{\mathbb{K}})^\top M(\bar{\mathbb{F}})$. Thus, there is a labeling $\bar{\mathbb{F}} = \{C_1, \ldots, C_n\}$ and $\bar{\mathbb{K}} = \{B_1, \ldots, B_n\}$ so that, for all $i, j \in \{1, \ldots, n\}$,

$$(\star) \qquad |C_i \cap B_j| = \begin{cases} rs - n + 1 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

and for all $g, h \in E$,

$$(\diamond) \qquad |\{i \in \{1, \ldots, n\} : g \in C_i, h \in B_i\}| = \begin{cases} rs - n + 1 & \text{if } g = h \\ 1 & \text{if } g \neq h. \end{cases}$$

Take an element $e \in E$. Since $rs - n \geq 2$, we may assume by $(\diamond)$ that $e \in C_i \cap B_i$ for $i \in \{1, 2, 3\}$. Recall that (M, Σ) represents $\mathbb{F}$ and that (N, Γ) represents $\mathbb{K}$. We will show that (C_1, C_2, C_3) is a strict e-hub of (M, Σ).

Claim 1. *C_1, C_2, C_3 are odd circuits of (M, Σ) such that, for distinct $i, j \in \{1, 2, 3\}$, $C_i \cap C_j = \{e\}$, i.e. (h1) holds.*

Proof of Claim. By definition, C_1, C_2, C_3 are odd circuits of (M, Σ). To see $C_1 \cap C_2 = \{e\}$, notice that if $f \in (C_1 \cap C_2) - e$, then $\{1, 2\} \subseteq \{i \in \{1, \ldots, n\} : f \in C_i, e \in B_i\}$, which cannot be the case as the latter set has size 1 by $(\diamond)$. Similarly, $C_2 \cap C_3 = C_3 \cap C_1 = \{e\}$. ♢

Claim 2. *For distinct $i, j \in \{1, 2, 3\}$, the only nonempty cycles of M contained in $C_i \cup C_j$ are $C_i, C_j, C_i \triangle C_j$, so (h2) holds.*

Proof of Claim. By symmetry, we may only analyze the cycles of M contained in $C_1 \cup C_2$. By Corollary 4 (1), the only odd circuits of (M, Σ) contained in $C_1 \cup C_2$ are C_1, C_2. We first show that C_1, C_2 are the only odd cycles of (M, Σ) in $C_1 \cup C_2$. Suppose otherwise. Let A be an odd cycle different from C_1, C_2. Write C as the disjoint union of circuits $A_1, \ldots, A_k$ for some $k \geq 2$. Since $|\Sigma \cap A| = \sum_{i=1}^k |\Sigma \cap A_i|$ and $|\Sigma \cap A|$ is odd, we may assume that $|\Sigma \cap A_1|$ is odd, so $A_1 \in \{C_1, C_2\}$, and we may assume that $A_1 = C_1$. But then $A_2 \subseteq C_2 - e$, a contradiction as both A_2, C_2 are circuits of M. Let C be a nonempty cycle of M contained in $C_1 \cup C_2$. If C is an odd cycle of (M, Σ), then as we just showed, $C \in \{C_1, C_2\}$. Otherwise, C is an even cycle, so $C \triangle C_1$ is an odd cycle, so $C \triangle C_1 \in \{C_1, C_2\}$, implying in turn that $C = C_1 \triangle C_2$, as required. ♢

Claim 3. *Every odd cycle of (M, Σ) contained in $C_1 \cup C_2 \cup C_3$ uses e, so (h3) holds.*

Proof of Claim. Since $s \geq 4$ and $M(\bar{\mathbb{K}})$ is s-regular, there is a $B \in \bar{\mathbb{K}} - \{B_1, B_2, B_3\}$ such that $e \in B$. Then, for each $i \in \{1,2,3\}$, $|B \cap C_i| = 1$ by $(\star)$, so $B \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. It follows from Proposition 7 that B is a signature of (M, Σ). Thus, if C is an odd cycle of (M, Σ) contained in $C_1 \cup C_2 \cup C_3$, then $|C \cap B|$ is odd and therefore nonzero, so $e \in C$. $\Diamond$

Claim 4. *If C, C' are odd cycles of (M, Σ) contained in $C_1 \cup C_2 \cup C_3$ such that $C \cap C' = \{e\}$, then for some distinct $i, j \in \{1,2,3\}$, $\{C, C'\} = \{C_i, C_j\}$, so (h4) holds.*

Proof of Claim. Let D, D' be odd circuits contained in C, C', respectively. It follows from Corollary 4 (2) that, for some distinct $i, j \in \{1,2,3\}$, $\{D, D'\} = \{C_i, C_j\}$. Since there is no even cycle contained in $(C_1 \cup C_2 \cup C_3) - (C_i \triangle C_j)$, it follows that $D = C$ and $D' = C'$, and the claim follows. $\Diamond$

Hence, (C_1, C_2, C_3) is a strict e-hub of (M, Σ). Similarly, (B_1, B_2, B_3) is a strict e-hub of (N, Γ). This finishes the proof of Theorem 9 part (1). $\square$

4 Hypergraphs, the Trifold, and Graphic Hubs

Let M be a binary matroid over ground set E. By definition, the cycles of M form a linear space modulo 2, so there is a $0-1$ matrix A such that the incidence vectors of the cycles in M are $\big\{x \in \{0,1\}^E : Ax \equiv \mathbf{0} \pmod 2\big\}$. The matrix A is referred to as a *representation of M*. Notice that elementary row operations modulo 2 applied to A yield another representation, and if $a \in \{0,1\}^E$ belongs to the row space of A modulo 2, then $\begin{pmatrix} A \\ a^\top \end{pmatrix}$ is also a representation.

A *hypergraphic representation of M* is a representation where every column has an even number of ones. If $a^\top$ is the sum of the rows of A modulo 2, then $\begin{pmatrix} A \\ a^\top \end{pmatrix}$ is a hypergraphic representation. In particular, a binary matroid always has a hypergraphic representation. A *hypergraph* is a pair $G = (V, E)$, where V is a finite set of *vertices* and E is a family of even subsets of V, called *edges*. Note that if A is a hypergraphic representation of M, then A may be thought of as a hypergraph whose vertices are labeled by the rows and whose edges are labeled by the columns. For instance, the Fano matroid F_7 may be represented as a hypergraph on vertices $\{1, \ldots, 4\}$ and edges $\big\{T \subseteq \{1, \ldots, 4\} : |T| \in \{2,4\}\big\}$. Denote by S_8 the binary matroid represented as the hypergraph displayed in Fig. 2, which has vertices $\{1, \ldots, 5\}$ and edges $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,3,4,5\}$. Label $\gamma := \{2,3,4,5\} \in E(S_8)$. A *trifold* is any signed matroid isomorphic to $\big(S_8, E(S_8) - \gamma\big)$.

Remark 10. *A trifold has an $(F_7, E(F_7))$ minor.*

Proof. Observe that $S_8/\gamma \cong F_7$, implying in turn that $(S_8, E(S_8) - \gamma)/\gamma \cong (F_7, E(F_7))$. $\square$

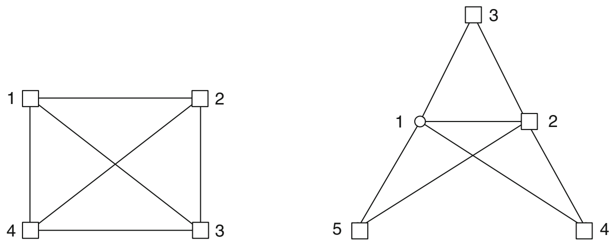


Fig. 2. The hypergraph on the left represents F_7, and the one on the right represents S_8. Line segments represent edges of size 2, and square vertices form the edges of size 4.

Given a hypergraph $G = (V, E)$ and $F \subseteq E$, let $\mathrm{odd}_G(F) := \triangle(e : e \in F) \subseteq V$. Observe that $\mathrm{odd}_G(F)$ is an even subset of V. We will make use of the following remark throughout the paper:

Remark 11. *Let M be a binary matroid over ground set $E \cup \{e\}$, where $M \setminus e$ is represented by the hypergraph $G = (V, E)$. If for some $F \subseteq E$, $F \cup \{e\}$ is a cycle of M, then the hypergraph on vertices V and edges $E \cup \{odd_G(F)\}$ represents M.*

Recall that a binary matroid is graphic if it can be represented by a graph. We will also need the following result, whose proof is straight-forward:

Proposition 12. *Take a signed matroid (M, Σ), $e \in E(M)$ and an e-hub (C_1, C_2, C_3). Then there is a signature Σ' such that $\Sigma' \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. Moreover, the following statements are equivalent:*

(i) $M|(C_1 \cup C_2 \cup C_3)$ *is graphic,*
(ii) $C_1, C_2, C_3, C_1 \triangle C_2 \triangle C_3$ *are the only odd cycles contained in* $C_1 \cup C_2 \cup C_3$.

5 Proof of Theorem 9 Part (2)

Let $\mathbb{F}, \mathbb{K}$ be blocking mni binary clutters over ground set E, neither of which has a set of size 3. Recall that (M, Σ) represents $\mathbb{F}$ and that (N, Γ) represents $\mathbb{K}$. Take an element $e \in E$. By Theorem 9 part (1), (M, Σ) has a (strict) e-hub (C_1, C_2, C_3) and (N, Γ) has a (strict) e-hub (B_1, B_2, B_3), where for $i \in \{1, 2, 3\}$, $|C_i \cap B_i| \geq 3$ and, for distinct $i, j \in \{1, 2, 3\}$, $C_i \cap B_j = \{e\}$. By Proposition 12, after a possible resigning of (M, Σ), we may assume that $\Sigma \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. Notice further that by Proposition 7, the odd circuits of (N, Γ) are (minimal) signatures of (M, Σ). We need to show that either $M|(C_1 \cup C_2 \cup C_3)$ or $N|(B_1 \cup B_2 \cup B_3)$ is non-graphic. Suppose otherwise. Since $N|(B_1 \cup B_2 \cup B_3)$ is graphic, it follows from Proposition 12 that B_1, B_2, B_3 are the only odd circuits of (N, Γ) contained in $B_1 \cup B_2 \cup B_3$. In other words, the only sets of $\mathbb{K}$ contained in $B_1 \cup B_2 \cup B_3$ are B_1, B_2, B_3.

Claim 1. *There is an odd circuit C of (M, Σ) such that $e \notin C$ and, for each $i \in \{1, 2, 3\}$, $C \cap B_i \subseteq C_i$.*

Proof of Claim. Let B be the union of $(B_1 \cup B_2 \cup B_3) - (C_1 \cup C_2 \cup C_3)$ and $\{e\}$. Since $B_1 \cap C_1 \neq \{e\}$, it follows that $B_1 \not\subseteq B$. Similarly, $B_2 \not\subseteq B$ and $B_3 \not\subseteq B$. Thus, since the only sets of $\mathbb{K}$ contained in $B_1 \cup B_2 \cup B_3$ are B_1, B_2, B_3, we get that B does not contain a set of $\mathbb{K} = b(\mathbb{F})$. In other words, there is a set $C \in \mathbb{F}$ such that $C \cap B = \emptyset$. By definition, C is an odd circuit of (M, Σ). Clearly, $e \notin C$. Consider the intersection $C \cap B_1$. Since $C \cap B = \emptyset$, it follows that $C \cap B_1 \subseteq C_1 \cup C_2 \cup C_3$. Moreover, as $B_1 \cap C_2 = B_1 \cap C_3 = \{e\}$, we see that $C \cap B_1 \subseteq C_1$. Similarly, $C \cap B_2 \subseteq C_2$ and $C \cap B_3 \subseteq C_3$. $\Diamond$

Since $e \notin C$, we get that $C \cap \Sigma \subseteq C - (C_1 \cup C_2 \cup C_3)$, and as C is odd, it follows that $C \not\subseteq C_1 \cup C_2 \cup C_3$.

Claim 2. $(M, \Sigma)|(C_1 \cup C_2 \cup C_3 \cup C)$ *has a trifold minor.*

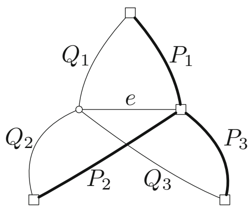
Proof Sketch. Let S be a minimal subset of $C - (C_1 \cup C_2 \cup C_3)$ such that (m1) $M|(C_1 \cup C_2 \cup C_3 \cup S)$ has a cycle containing S, and (m2) $|S \cap \Sigma|$ is odd. Note that S is well-defined, since $C - (C_1 \cup C_2 \cup C_3)$ satisfies both (m1)–(m2). Let

$$(M', \Sigma') := (M, \Sigma)|(C_1 \cup C_2 \cup C_3 \cup S).$$

The minimality of S implies that the elements of S are in series in M'. In particular, after a possible resigning, we may assume that $\Sigma' \cap (C_1 \cup C_2 \cup C_3 \cup S) = \{e, f\}$ for some element $f \in S$. Let

$$(M'', \{e, f\}) := (M', \Sigma')/(S - f).$$

Since B_1 is a signature for (M, Σ), and $B_1 \cap (C_1 \cup C_2 \cup C_3 \cup C) = B_1 \cap C_1$ by our choice of C, it follows that $B_1 \cap C_1$ is a signature for $(M'', \{e, f\})$. We have $M''\backslash f = M'/(S - f)\backslash f = M'\backslash S = M|(C_1 \cup C_2 \cup C_3)$, where the second equality follows from the fact that the elements of M' in S are in series. Since $M|(C_1 \cup C_2 \cup C_3)$ is graphic, $M''\backslash f$ may be represented as a graph $G = (V, C_1 \cup C_2 \cup C_3)$. It follows from (h2) that the circuits C_1, C_2, C_3 are pairwise vertex-disjoint except at the ends of $e = \{x, y\} \subseteq V$. By (m1), $M|(C_1 \cup C_2 \cup C_3 \cup S)$ has a cycle containing S, so M'' has a cycle $P \cup \{f\}$, for some $P \subseteq C_1 \cup C_2 \cup C_3$. By replacing P by $P \triangle C_1$, if necessary, we may assume that $e \notin P$. For each $i \in \{1, 2, 3\}$, let $P_i := P \cap C_i$ and $Q_i := C_i - (P_i \cup \{e\})$. After possibly rearranging the edges of G within each series class $C_i - e$, we may assume that each P_i is a path that starts from x. It follows from Remark 11 that M'' is represented as the hypergraph on vertices V and edges $C_1 \cup C_2 \cup C_3 \cup \{\mathrm{odd}_G(P)\}$. We may therefore label $f = \mathrm{odd}_G(P)$, and represent M'' with the following hypergraph



where f consists of the square vertices. Since $P \cup \{f\}$ is an odd cycle of $(M'', \{e, f\})$, it must contain an odd number of edges of the signature $B_1 \cap C_1$, implying in turn that each of P_1, Q_1 contains an odd number of edges of B_1, so $P_1 \neq \emptyset$ and $Q_1 \neq \emptyset$. Similarly, for each $i \in \{1, 2, 3\}$, $P_i \neq \emptyset$ and $Q_i \neq \emptyset$, so there are $p_i \in P_i$ and $q_i \in Q_i$. Since $\{e, p_1, p_2, p_3, q_1, q_2, q_3\}$ is a signature for $(M'', \{e, f\})$, we see that

$$(M'', \{e, f\}) \cong (M'', \{e, p_1, p_2, p_3, q_1, q_2, q_3\}).$$

Observe however that the right signed matroid has a trifold minor, obtained after contracting each $C_i - \{e, p_i, q_i\}$. As a result, $(M, \Sigma)|(C_1 \cup C_2 \cup C_3 \cup C)$ has a trifold minor. $\Diamond$

However, by Remark 10, a trifold has an $(F_7, E(F_7))$ minor, so (M, Σ) has an $(F_7, E(F_7))$ minor. As a consequence, Proposition 8 implies that $\mathbb{F}$ has an $\mathbb{L}_7$ minor. Since $\mathbb{F}$ is mni, we must have that $\mathbb{F} \cong \mathbb{L}_7$, but $\mathbb{F}$ has no set of size 3, a contradiction. This finishes the proof of Theorem 9 part (2). $\Box$

6 Non-graphic Strict Hubs

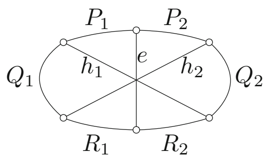
In this section, we prove the following result needed for Theorem 9 part (3):

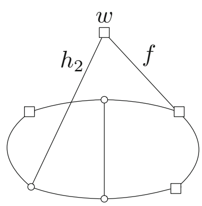
Proposition 13. *Take a signed matroid (M, Σ), $e \in E(M)$ and a strict e-hub (C_1, C_2, C_3) such that $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic. Then there exist $I \subseteq C_3 - e$ and distinct $g_1, g_2 \in (C_3 - I) - e$ where*

(1) $(C_1, C_2, C_3 - I)$ is an e-hub of $(M, \Sigma)/I$,
(2) $(M/I)|\big(C_1 \cup C_2 \cup \{g_i\}\big)$ has a circuit containing g_i, for each $i \in \{1, 2\}$,
(3) $(M/I)|\big(C_1 \cup C_2 \cup \{g_1, g_2\}\big)$ is non-graphic.

Proof Sketch. By Proposition 12, after a possible resigning, we may assume that $\Sigma \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. Let I be a maximal subset of $C_3 - e$ such that every cycle of $M|(C_1 \cup C_2 \cup I)$ is disjoint from I. Let $(M', \{e\}) := (M, \Sigma)|(C_1 \cup C_2 \cup C_3)/I$ and $C'_3 := C_3 - I$. Then (C_1, C_2, C'_3) is an e-hub of $(M', \{e\})$, and as $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic, it follows from Proposition 12 that M' is non-graphic. Moreover, the maximality of I implies that, for each $g \in C'_3 - e$, there is a cycle D_g of $M'|(C_1 \cup C_2 \cup \{g\})$ using g, where after possibly replacing D_g by $D_g \triangle C_1$, we may assume that $e \notin D_g$. Note that $D_g \triangle C_1 \triangle C_2$ is another cycle of $M'|(C_1 \cup C_2 \cup \{g\})$ that uses g and excludes e. For each such g, refer to $D_g - g$ and $(D_g \triangle C_1 \triangle C_2) - g$ as the *outer joins of* g. Notice that an outer join intersects both C_1, C_2. As $\triangle\big(D_g - g : g \in C'_3 - e\big)$ is either $C_1 - e$ or $C_2 - e$ by (h2), there exist $h_1, h_2 \in C'_3 - \{e\}$ and respective outer joins J_{h_1}, J_{h_2} that cross, that is, $J_{h_1} \cap J_{h_2} \neq \emptyset$, $J_{h_1} - J_{h_2} \neq \emptyset$, $J_{h_2} - J_{h_1} \neq \emptyset$ and $J_{h_1} \cup J_{h_2} \neq C_1 \triangle C_2$. If $M'|(C_1 \cup C_2 \cup \{h_1, h_2\})$ is non-graphic, then we are done. Otherwise, it may be represented as a graph $H = (V, C_1 \cup C_2 \cup \{h_1, h_2\})$, displayed below (left figure), where $C_1 = \{e\} \cup P_1 \cup Q_1 \cup R_1$, $C_2 = \{e\} \cup P_2 \cup Q_2 \cup R_2$, $J_{h_1} = P_1 \cup P_2 \cup Q_2$, and $J_{h_2} = P_1 \cup P_2 \cup Q_1$. Notice that $P_i, Q_i, R_i \neq \emptyset$ for each $i \in \{1, 2\}$. Let

$D_1 := \{e, h_1\} \cup P_1 \cup R_2$ and $D_2 := \{e, h_2\} \cup \{P_2, R_1\}$. For $i \in \{1, 2\}$, let D_i' be a cycle of M such that $D_i \subseteq D_i' \subseteq D_i \cup I$; as $D_i' \cap \Sigma = \{e\}$, D_i' is an odd cycle of (M, Σ). Note further, for $i \in \{1, 2\}$, that D_i' is different from C_1, C_2, C_3. Thus, since (C_1, C_2, C_3) is a strict e-hub of (M, Σ) and therefore satisfies (h4), we must have that $\{e\} \subsetneq D_1' \cap D_2'$. Because $D_1 \cap D_2 = \{e\}$, there is an element $f \in I$ such that $\{e, f\} \subseteq D_1' \cap D_2'$. Consider now the minor $(M, \Sigma)/(I - f)$; note that $D_1 \cup \{f\}$ and $D_2 \cup \{f\}$ are odd cycles of this signed matroid. We may represent $M/(I - f)$ as a hypergraph $G = (V \cup \{w\}, C_1 \cup C_2 \cup \{h_1, h_2, f\})$ obtained from H by adding a vertex w, displayed below (right figure), where the square vertices form the edge h_1. Now let $J := I \triangle \{f, h_2\}$. Observe that $(M/J)|(C_1 \cup C_2 \cup \{f, h_1\})$ is non-graphic, as it has an F_7 minor obtained after contracting $P_1 \cup R_2$ and contracting each of Q_1, R_1, P_2, Q_2 to a single edge. Thus, $J \subseteq C_3' - \{e\}$ and f, h_1 satisfy (3), and it can be readily checked that they also satisfy (1)–(2).





$\square$

7 A Sketch of the Proof of Theorem 9 Part (3)

Let $\mathbb{F}, \mathbb{K}$ be blocking mni clutters over ground set E, neither of which has a set of size 3, where (M, Σ) represents $\mathbb{F}$ and (N, Γ) represents $\mathbb{K}$. By Theorem 9 part (1), (M, Σ) has a strict e-hub (C_1, C_2, C_3) and (N, Γ) has a strict e-hub (B_1, B_2, B_3) such that for $i \in \{1, 2, 3\}$, $|C_i \cap B_i| \geq 3$ and, for distinct $i, j \in \{1, 2, 3\}$, $C_i \cap B_j = \{e\}$. Assume further that $M|(C_1 \cup C_2 \cup C_3)$ is non-graphic. We need to show that (M, Σ) has an $(F_7, E(F_7) - \omega)$ minor going through e. By Proposition 12, after a possible resigning, we may assume that $\Sigma \cap (C_1 \cup C_2 \cup C_3) = \{e\}$. By Proposition 13, there exist $I \subseteq C_3 - e$ and distinct $g_1, g_2 \in (C_3 - I) - e$ such that (1)–(3) hold. For each $i \in \{1, 2\}$, after possibly replacing D_i by $D_i \triangle C_1$, we may assume that $e \notin D_i$; as $(C_1, C_2, C_3 - I)$ is an e-hub of $(M, \Sigma)/I$, it follows from (h2) that $D_i \cap C_1 \neq \emptyset$ and $D_i \cap C_2 \neq \emptyset$. Notice that, for each $i \in \{1, 2\}$, $B_i \cap I = \emptyset$, so B_i is a signature of $(M, \Sigma)/I$.

Claim 1. *There exists an odd circuit C of $(M, \Sigma)/I$ such that $e \notin C$ and, for each $i \in \{1, 2\}$, $C \cap B_i \subseteq C_i$.*

Let $(M', \Sigma) := (M, \Sigma)/I$. Let S be a minimal subset of $C - (C_1 \cup C_2)$ such that (m1) $M'|(C_1 \cup C_2 \cup S)$ has a cycle containing S, and (m2) $|S \cap \Sigma|$ is odd. Note that S is well-defined as $C - (C_1 \cup C_2)$ satisfies (m1)–(m2). The minimality of S implies that $S \cap \{g_1, g_2\} = \emptyset$, and the elements of S are in series in $M'|(C_1 \cup C_2 \cup \{g_1, g_2\} \cup S)$. Thus, there exists a signature Σ' of (M', Σ) such that $\Sigma' \cap (C_1 \cup C_2 \cup \{g_1, g_2\} \cup S) = \{e, f\}$, for some $f \in S$. Consider the minor

$$(M'', \{e, f\}) := (M', \Sigma')|(C_1 \cup C_2 \cup \{g_1, g_2\} \cup S)/(S - f).$$

For each $i \in \{1, 2\}$, our choice of C implies that $B_i \cap S = \emptyset$, so $B_i \cap (C_1 \cup C_2 \cup \{g_1, g_2\}) = B_i \cap C_i$ is a signature of $(M'', \{e, f\})$.

Claim 2. *If $M''\backslash g_i$ is graphic for each $i \in \{1, 2\}$, then $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor.*

Assume that $M''\backslash g_i$ is graphic for each $i \in \{1, 2\}$. Then by the preceding claim, $(M'', \{e, f\})$ has an $(F_7, E(F_7))$ minor, implying in turn that (M, Σ) has an $(F_7, E(F_7))$ minor. So by Proposition 8, $\mathbb{F}$ has an $\mathbb{L}_7$ minor, and since $\mathbb{F}$ is mni, this means $\mathbb{F} \cong \mathbb{L}_7$, which cannot be as $\mathbb{F}$ has no set of size 3. Hence, by symmetry, we may assume that $M''\backslash g_2$ is non-graphic. Thus, there exists $I \subseteq C_1 \triangle C_2$ such that $M''\backslash g_2/I \cong F_7$. Then $(M'', \{e, f\}) \setminus g_2/I \cong (F_7, E(F_7) - \omega)$, and so (M, Σ) has an $(F_7, E(F_7) - \omega)$ minor going through e, as required. This finishes the proof of Theorem 9 part (3). $\square$

References

1. Abdi, A., Guenin, B.: The minimally non-ideal binary clutters with a triangle (Submitted)
2. Bridges, W.G., Ryser, H.J.: Combinatorial designs and related systems. J. Algebra **13**, 432–446 (1969)
3. Cornuéjols, G.: Combinatorial Optimization, Packing and Covering. SIAM, Philadelphia (2001)
4. Cornuéjols, G., Guenin, B.: Ideal binary clutters, connectivity, and a conjecture of Seymour. SIAM J. Discrete Math. **15**(3), 329–352 (2002)
5. Edmonds, J., Fulkerson, D.R.: Bottleneck extrema. J. Combin. Theory Ser. B **8**, 299–306 (1970)
6. Ford, L.R., Fulkerson, D.R.: Maximal flow through a network. Can. J. Math. **8**, 399–404 (1956)
7. Guenin, B.: A characterization of weakly bipartite graphs. J. Combin. Theory Ser. B **83**, 112–168 (2001)
8. Guenin, B.: Integral polyhedra related to even-cycle and even-cut matroids. Math. Oper. Res. **27**(4), 693–710 (2002)
9. Lehman, A.: A solution of the Shannon switching game. Soc. Ind. Appl. Math. **12**(4), 687–725 (1964)
10. Lehman, A.: On the width-length inequality. Math. Program. **17**(1), 403–417 (1979)
11. Lehman, A.: The width-length inequality and degenerate projective planes. In: DIMACS, vol. 1, pp. 101–105 (1990)
12. Menger, K.: Zur allgemeinen Kurventheorie. Fundamenta Mathematicae **10**, 96–115 (1927)
13. Novick, B., Sebő, A.: On combinatorial properties of binary spaces. In: Balas, E., Clausen, J. (eds.) IPCO 1995. LNCS, vol. 920, pp. 212–227. Springer, Heidelberg (1995). doi:10.1007/3-540-59408-6_53
14. Oxley, J.: Matroid Theory, 2nd edn. Oxford University Press, New York (2011)
15. Seymour, P.D.: On Lehman's width-length characterization. In: DIMACS, vol. 1, pp. 107–117 (1990)
16. Seymour, P.D.: The forbidden minors of binary matrices. J. Lond. Math. Soc. **2**(12), 356–360 (1976)
17. Seymour, P.D.: The matroids with the max-flow min-cut property. J. Combin. Theory Ser. B **23**, 189–222 (1977)