DELTAS, EXTENDED ODD HOLES AND THEIR BLOCKERS

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ABSTRACT. Let $\mathcal{C}$ be a clutter over ground set V where no element is contained in every member. We prove that if there is a $w \in \mathbb{R}^V_+$ such that every member has weight strictly more than half the capacity, then there must be a delta or the blocker of an extended odd hole minor. The proof of this result relies on a tool developed for finding extended odd hole minors.

1. INTRODUCTION

Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V, called *members*. We say that $\mathcal{C}$ is a *clutter* over *ground set* V if no member is contained in another [6]. A *cover* is a subset $B \subseteq V$ such that for all $C \in \mathcal{C}$, $B \cap C \neq \emptyset$. The *blocker of* $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set V whose members are the minimal covers of $\mathcal{C}$. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [8, 6]. Given disjoint subsets $I, J \subseteq V$, the *minor of* $\mathcal{C}$ obtained after *deleting* I and *contracting* J is the clutter over ground set $V - (I \cup J)$ whose members are

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}.$$

It can be readily checked that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [12]. We say that $\mathcal{C}$ is *ideal* if the polyhedron

$$\left\{x \in \mathbb{R}^V_+ : x(C) \geq 1 \ \forall C \in \mathcal{C}\right\}$$

is integral [3]. If a clutter is ideal, then so is every minor of it, essentially because in terms of the polyhedron, deleting e corresponds to projecting away x_e and contracting e corresponds to restricting x_e to 0 [13]. Alfred Lehman proved the following *width-length* characterization of ideal clutters:

Theorem 1 ([10], see also [7])**.** *Let $\mathcal{C}$ be a clutter over ground set V. Then $\mathcal{C}$ is ideal if, and only if, for all $w, \ell \in \mathbb{R}^V_+$,*

$$\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell.$$

In particular, a clutter is ideal if, and only if, its blocker is ideal.

Given a graph $G = (V, E)$ and distinct vertices $s, t \in V$, Moore and Shannon proved that the clutter over ground set E of st-paths satisfies the width-length inequality for unit widths and unit lengths [11], and Duffin generalized this by proving the width-length inequality for all non-negative widths and non-negative lengths [5]. Theorem 1 therefore implies that the clutter of st-paths of a graph is ideal. As the reader may expect, not every clutter is ideal. Lehman found three infinite classes of non-ideal clutters, a self-blocking class corresponding

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to the lines of a degenerate projective plane, a class corresponding to the edges of an odd hole, as well as the blocker of the latter [10]. Let us elaborate.

Take an integer $n \geq 3$. A *delta of dimension* n is the clutter over ground set $\{1, \ldots, n\}$ whose members are

$$\Delta_n := \big\{\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}\big\}.$$

As the reader can readily check, $b(\Delta_n) = \Delta_n$. Observe that Δ_n violates the width-length inequality for widths $w := (n-2, 1, \ldots, 1)$ and $\ell := (1, 1, \ldots, 1)$:

$$\min\{w(C) : C \in \Delta_n\} \cdot \min\{\ell(B) : B \in b(\Delta_n)\} = (n-1) \cdot 2 > (n-2) + (n-1) = w^\top \ell,$$

so Δ_n is non-ideal by Theorem 1. If n is odd and at least 5, an *extended odd hole of dimension* n is any clutter over ground set $\{1, \ldots, n\}$ whose *minimum cardinality* members are

$$\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{n-1,n\}, \{n,1\}.$$

Observe that every member has cardinality at least 2, and since n is odd, every minimal cover has cardinality at least $\frac{n+1}{2}$. As a result, an extended odd hole violates the width-length inequality for widths $w := (1, 1, \ldots, 1)$ and $\ell := (1, 1, \ldots, 1)$:

$$\min\{w(C) : C \text{ a member}\} \cdot \min\{\ell(B) : B \text{ a minimal cover}\} = 2 \cdot \left(\frac{n+1}{2}\right) > n = w^\top \ell,$$

so by Theorem 1, extended odd holes and their blockers are non-ideal.[1] The following immediate consequence of Theorem 1 is the common thread for why the deltas and blockers of extended odd holes are non-ideal:

Corollary 2. *Let $\mathcal{C}$ be a clutter over ground set V without a cover of cardinality one. If there is $w \in \mathbb{R}^V_+$ such that* $\min\{w(C) : C \in \mathcal{C}\} > \frac{\mathbf{1}^\top w}{2}$*, then $\mathcal{C}$ is non-ideal.*

We will prove the following strengthening of this consequence:

Theorem 3. *Let $\mathcal{C}$ be a clutter over ground set V without a cover of cardinality one. If there is a $w \in \mathbb{R}^V_+$ such that* $\min\{w(C) : C \in \mathcal{C}\} > \frac{\mathbf{1}^\top w}{2}$*, then $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor.*

Let us discuss an application of Theorem 3. A clutter is *binary* if the symmetric difference of any odd number of members contains a member [9]. For instance, given a graph $G = (V, E)$ and distinct vertices $s, t \in V$, the clutter over ground set E of st-paths of G is binary. However, a delta is not binary as $\{1,2\} \triangle \{1,3\} \triangle \{2, \ldots, n\}$ does not contain a member, and neither is an extended odd hole because $\{1,2\} \triangle \{2,3\} \triangle \{3,4\} = \{1,4\}$ does not contain a member. It is well-known that if a clutter is binary, then so is its blocker [9] and any minor of it [12]. Thus, an immediate consequence of Theorem 3 is the following:

Corollary 4. *Take an integer $n \geq 2$ and a binary clutter over n elements without a cover of cardinality one. Then the clutter has a member of cardinality at most $\frac{n}{2}$.*

[1] Extended odd holes are an extension of Lehman's original infinite class corresponding to odd holes.

(Although, this corollary can be proved directly with a probabilistic argument.) In §2, we develop a tool for finding extended odd hole minors, which is interesting in its own right. Using this tool, we prove Theorem 3 for unit widths in §3. Finally, we prove Theorem 3 in its generality in §4. We would like to thank Gérard Cornuéjols and Bertrand Guenin for helpful discussions.

2. Extended odd hole minors

We will need the following tool for finding delta minors:

Theorem 5 ([1]). *Let $\mathcal{C}$ be a clutter. If there exist an element u and distinct members C_1, C_2, C such that $u \in C_1 \cap C_2, u \notin C$ and $C_1 \cup C_2 \subseteq C \cup \{u\}$, then $\mathcal{C}$ has a delta minor using u.*

Given a simple graph, we will treat each edge as a vertex subset of cardinality two, and each circuit as a vertex subset. We will need the following remark:

Remark 6. *Let V be a finite set, and let $\mathcal{C}$ be a clutter over ground set V where*

$$\min\{|C| : C \in \mathcal{C}\} = 2$$

and the minimum cardinality members correspond to edges of a graph $G = (V, E)$. If G is not bipartite, then $\mathcal{C}$ has a Δ_3 or an extended odd hole minor.

Proof. Suppose that G is not bipartite. Let $C \subseteq V$ be an odd circuit of G with the smallest number of vertices. Our minimality assumption implies that C has no chords. If $|C| = 3$, then $\mathcal{C} \setminus (V - C) \cong \Delta_3$. Otherwise, $|C| \geq 5$ and $\mathcal{C} \setminus (V - C)$ is an extended odd hole, as required. □

Let $G = (V, E)$ be a connected graph. A *block* is a maximal vertex-induced subgraph of G that is 2-(vertex)-connected. The *block graph of* G is the graph X constructed as follows:

- X is a bipartite graph,
- the cut-vertices of G form one color class of X,
- the blocks of G form the other color class of X, and
- cut-vertex u and block B are adjacent in X if $u \in V(B)$.

It is well-known that X is a tree (see [4], Lemma 3.1.4). Observe that every cut-vertex of G has degree at least 2 in X, so the leaf vertices of X correspond to blocks of G, and we will refer to those blocks as *leaf blocks*. For each block of G, we will refer to its cut-vertices as *boundary* vertices and its other vertices as *interior* vertices. We are now ready for the main result of this section:

Theorem 7. *Let V be a finite set of cardinality at least 4. Let $\mathcal{C}$ be a clutter over ground set V where*

$$\min\{|C| : C \in \mathcal{C}\} = 2$$

and the minimum cardinality members correspond to the edges of a connected bipartite graph G over vertex set V whose color classes are R, B. If R contains a member, then $\mathcal{C}$ has a delta or an extended odd hole minor.

Proof. Let us proceed by induction on $|V| \geq 4$. For the base case $|V| = 4$, it can be readily checked that $\mathcal{C}$ has one of Δ_3, Δ_4 as a minor, so we are done. For the induction step, assume that $|V| \geq 5$. Let $C \subseteq V$ be a member of $\mathcal{C}$ contained in the color class R. As R is a stable set of G, it follows that $|C| \geq 3$. Let us refer to the vertices of G in C as *terminals*, and to the other vertices as *non-terminals*. So the terminals form a stable set.

Claim 1. *Let u be a non-terminal vertex of G. If the terminals of G belong to the same component of $G \setminus u$, then $\mathcal{C} \setminus u$ has a delta or an extended odd hole minor.*

Proof of Claim. Let W be the set of vertices of $G \setminus u$ that do not belong to the component containing the terminals. Consider the clutter $\mathcal{C} \setminus u \setminus W$; its minimum cardinality members correspond to the edges of the connected bipartite graph $G \setminus u \setminus W$. As $G \setminus u \setminus W$ has at least 4 vertices, and the member $C \in \mathcal{C} \setminus u \setminus W$ is still a subset of one of the color classes of $G \setminus u \setminus W$, the induction hypothesis implies that $\mathcal{C} \setminus u \setminus W$ has a delta or an extended odd hole minor, as claimed. $\Diamond$

We may therefore assume that every non-terminal vertex of G is a cut-vertex separating the terminals. In particular, G is not 2-connected. Let us then consider the blocks of G. Since every non-terminal is a cut-vertex, the interior vertices of each block are terminals. Let X be the block graph of G; recall that X is a tree.

Claim 2. *Every leaf block of G consists of precisely two vertices and an edge between them, where one vertex belongs to the boundary and is a non-terminal, and the other vertex belongs to the interior, is a terminal and has degree one in G.*

Proof of Claim. By definition, the leaf blocks are precisely those blocks with exactly one boundary vertex. Consider a leaf block B, viewed as a vertex-induced subgraph of G. Then B has a unique boundary vertex, and as each block has at least two vertices, B has at least one interior vertex. As the interior vertices are terminals and therefore form a stable set, B must have exactly one interior vertex and therefore exactly one edge, implying in turn that its boundary vertex is a non-terminal vertex. This also implies that the interior vertex has degree one in G. $\Diamond$

Claim 3. *If a non-terminal vertex is adjacent to at least two terminals, then $\mathcal{C}$ has a delta minor.*

Proof of Claim. Suppose that a non-terminal vertex u is adjacent to two terminals v and w. Then both $\{u, v\}$ and $\{u, w\}$ are members of $\mathcal{C}$, and $\{v, w\} \subseteq C$. Since u is not a terminal, u is not contained in C. It therefore follows from Theorem 5 that $\mathcal{C}$ has a delta minor. $\Diamond$

We may therefore assume that every non-terminal vertex is adjacent to at most one terminal. In particular, by Claim 2,

Claim 4. *Every non-terminal vertex belongs to at most one leaf block. That is, X does not have leaf edges that share a vertex.*

Using this, we prove the following:

Claim 5. *There exists a leaf block with boundary vertex u such that $G \setminus u$ has exactly two components. That is, X has a vertex u of degree two that is incident with a leaf edge.*

Proof of Claim. Pick a leaf vertex B of X, and among all the other leaf vertices, pick one B' that is farthest from B. Let u be the vertex of X adjacent to B'. Clearly u has degree at least two, as it lies the path joining B, B'. Notice that by Claim 4, the edge uB' is the unique leaf edge incident with u. This fact, together with our maximal choice of B', implies that u has degree exactly two, as required. $\Diamond$

Let B be the leaf block from Claim 5, let u be the boundary vertex, and let v be the interior vertex. Recall that u is a non-terminal, v is a terminal, and that v has degree one in G. Moreover, by Claim 5, $G \setminus u \setminus v$ is a connected bipartite graph. Since there are at least two terminals in $G \setminus u \setminus v$, and the terminals form a stable set, it follows that $G \setminus u \setminus v$ has at least 3 vertices.

Claim 6. *$\mathcal{C} \setminus u/v$ has a delta or extended odd hole minor.*

Proof of Claim. Let $\mathcal{C}' := \mathcal{C} \setminus u/v$. Observe that $C' := C - \{v\} \in \mathcal{C}'$. Since v has degree one in G, and u is its neighbor, it follows that $\min\{|D| : D \in \mathcal{C}'\} \geq 2$. Therefore, each edge of $G \setminus u \setminus v$ corresponds to a minimum cardinality member of $\mathcal{C}'$. In particular,

$$\min\{|D| : D \in \mathcal{C}'\} = 2.$$

If $|V - \{u, v\}| = 3$, then $\mathcal{C} \setminus u/v = \Delta_3$, so we are done. We may therefore assume that $|V - \{u, v\}| \geq 4$. Let G' be the graph over vertex set $V - \{u, v\}$ corresponding to the minimum cardinality members of $\mathcal{C}'$. Notice that $E(G \setminus u \setminus v) \subseteq E(G')$. In particular, G' is connected. If G' is not bipartite, then by Remark 6, $\mathcal{C}'$ has a Δ_3 or an extended odd hole minor, so we are done. Otherwise, G' is bipartite. Since C' is contained in one of the color classes of $G \setminus u \setminus v$, it must also be contained in one of the color classes of G'. It therefore follows from the induction hypothesis that $\mathcal{C}'$ has a delta or an extended odd hole minor, as required. $\Diamond$

This claim completes the induction step, thereby finishing the proof.[2] $\Box$

3. Theorem 3 for unit widths

For a clutter $\mathcal{C}$, denote by $\tau(\mathcal{C})$ the minimum cardinality of a cover.

Theorem 8. *Let $\mathcal{C}$ be a clutter over ground set V without a cover of cardinality one. If every member has cardinality at least $\frac{|V|+1}{2}$, then $\mathcal{C}$ has either a delta or the blocker of an extended odd hole minor.*

Proof. Observe that $|V| \geq 3$. We will proceed by induction on $|V|$. For the base case $|V| = 3$, observe that $\mathcal{C} \cong \Delta_3$, so we are done. For the induction step, assume that $|V| \geq 4$.

Claim 1. *If an element v does not appear in a cover of cardinality two, then $\mathcal{C} \setminus v$ has a delta or the blocker of an extended odd hole minor.*

[2] The careful reader will have noticed the resemblance between this proof and Berge's proof that a balanced hypergraph is 2-colorable [2].

Proof of Claim. Assume that v does not appear in a cover of cardinality two. Then $\tau(\mathcal{C} \setminus v) \geq 2$. As every member of $\mathcal{C} \setminus v$ has cardinality at least $\frac{|V|+1}{2} \geq \frac{|V-\{v\}|+1}{2}$, the induction hypothesis implies that $\mathcal{C} \setminus v$ has a delta or the blocker of an extended odd hole minor, as required. $\Diamond$

We may therefore assume that $\tau(\mathcal{C}) = 2$ and that every element appears in a minimum cover. Let $G = (V, E)$ be the graph whose edges correspond to the minimum covers of $\mathcal{C}$.

Claim 2. *If G is not bipartite, then $\mathcal{C}$ has a Δ_3 or the blocker of an extended odd hole minor.*

Proof of Claim. It follows from Remark 6 that $b(\mathcal{C})$ has a Δ_3 or an extended odd hole minor, as required. $\Diamond$

We may therefore assume that G is a bipartite graph. Take a component H of G where $V(H)$ is partitioned into color classes U_1, U_2 such that $|U_2| \geq |U_1|$.

Claim 3. *If $\tau(\mathcal{C} \setminus U_2/U_1) \geq 2$, then $\mathcal{C} \setminus U_2/U_1$ has a delta or the blocker of an extended odd hole minor.*

Proof of Claim. Suppose that $\tau(\mathcal{C} \setminus U_2/U_1) \geq 2$. Observe that every member of $\mathcal{C} \setminus U_2/U_1$ has at least

$$\frac{|V|+1}{2} - |U_1| \geq \frac{1}{2}(|V| - |U_1| - |U_2| + 1)$$

elements. It therefore follows from the induction hypothesis that $\mathcal{C} \setminus U_2/U_1$ has a delta or the blocker of an extended odd hole minor, as required. $\Diamond$

We may therefore assume that $\tau(\mathcal{C} \setminus U_2/U_1) \leq 1$.

- If $H = G$, then $U_1 \cup U_2 = V$. Then $|U_2| \geq \frac{|V|}{2}$ because $|U_2| \geq |U_1|$. Since every member has cardinality at least $\frac{|V|+1}{2}$, U_2 is a cover of $\mathcal{C}$.
- Otherwise, H is a proper component of G, so $b(\mathcal{C}) \setminus U_1/U_2$ has a member of cardinality at most 1.

Either way, there is a member B of $b(\mathcal{C})$ such that $B \cap U_1 = \emptyset$ and $|B - U_2| \leq 1$. Observe that as H is a component of G, the graph G has no edge with exactly one end in $U_1 \cup U_2$. In particular, $|B| \geq 3$, $|B \cap U_2| \geq 2$ and $|U_1 \cup U_2| \geq 3$.

Claim 4. *$b(\mathcal{C})$ has a delta or an extended odd hole minor.*

Proof of Claim. Let $b(\mathcal{C}')$ be the minor obtained from $b(\mathcal{C})$ after deleting $V - (U_1 \cup U_2 \cup B)$ and contracting $B - U_2$; so $b(\mathcal{C}')$ has ground set $U_1 \cup U_2$. Notice that $B' := B \cap U_2$ is a member of $b(\mathcal{C}')$. Since G has no edge with exactly one end in $U_1 \cup U_2$, each edge of H is a minimum cardinality member of $b(\mathcal{C}')$. Let G' be the graph over vertex set $U_1 \cup U_2$ whose edges are the minimum cardinality members of $b(\mathcal{C}')$. As $E(H) \subseteq E(G')$, G' is a connected graph. If G' is not bipartite, it then follows from Remark 6 that $b(\mathcal{C}')$ has a Δ_3 or an extended odd hole minor, so we are done. Otherwise, G' is bipartite. Observe that the color classes of G' are inevitably U_1 and U_2, so $|B'| \geq 3$ and $|U_1 \cup U_2| \geq 4$. Since $B' \subseteq U_2$, it follows from Theorem 7 that $b(\mathcal{C}')$ has a delta or an extended odd hole minor, as claimed. $\Diamond$

Thus, $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor, thereby completing the induction step. $\square$

4. Proof of Theorem 3

Let $\mathcal{C}$ be a clutter over ground set V, and take distinct elements $u, v \in V$. We say that u, v are *duplicates* if every member contains u if and only if it contains v. We say that u, v are *replicates* if no member contains both u and v, and for each $C \subseteq V - \{u, v\}$, $C \cup \{u\}$ is a member if and only if $C \cup \{v\}$ is a member. We leave the following as an easy exercise for the reader:

Remark 9. *Take a clutter $\mathcal{C}$ and distinct elements u, v. Then u, v are duplicates in $\mathcal{C}$ if, and only if, u, v are replicates in $b(\mathcal{C})$.*

Using this remark, we prove the following:

Remark 10. *Deltas and blockers of extended odd holes do not have duplicated elements.*

Proof. It follows from the definition that deltas do not have duplicated elements. To prove that blockers of extended odd holes do not have duplicated elements, it suffices by Remark 9 to show that extended odd holes do not have replicated elements. Pick an odd integer $n \geq 5$ and let $\mathcal{C}$ be an extended odd hole over ground set $[n]$ whose minimum cardinality members are $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}$. Suppose for a contradiction that $\mathcal{C}$ has replicates $i, j \in [n]$. After a possible relabeling, we may assume that $i = 1$. As $1, j$ do not appear in a member together, $j \in \{3, 4, \ldots, n-1\}$. However, as $1, j$ are replicates and $\{1,2\}, \{1,n\}$ are members, it follows that $\{j,2\}, \{j,n\}$ are members also, a contradiction as $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}$ are the only members of cardinality two. Thus, extended odd holes do not have replicated elements, as required. $\square$

Let $\mathcal{C}$ be a clutter over ground set V. To *duplicate an element* v is to introduce a new element $\bar{v}$, and replace $\mathcal{C}$ by the clutter over ground set $V \cup \{\bar{v}\}$ whose members are

$$\{C : v \notin C \in \mathcal{C}\} \cup \{C \cup \{\bar{v}\} : v \in C \in \mathcal{C}\}.$$

We are now ready to prove Theorem 3, stating that if $\tau(\mathcal{C}) \geq 2$ and there is a $w \in \mathbb{R}^V_+$ such that $\min\{w(C) : C \in \mathcal{C}\} > \frac{\mathbf{1}^\top w}{2}$, then $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor.

Proof of Theorem 3. After tweaking the widths, if necessary, we may assume that w has rational entries. Pick an appropriate integer $N \geq 1$ such that Nw_v is an integer, for each $v \in V$. Let $\mathcal{C}'$ be the clutter over ground set V' obtained from $\mathcal{C}$ as follows: for each $v \in V$,

- if $Nw_v = 0$ then contract v,
- otherwise, duplicate $Nw_v - 1$ times the element v.

Observe that $\tau(\mathcal{C}') \geq 2$, and as $\min\{N \cdot w(C) : C \in \mathcal{C}\} > \frac{N(\mathbf{1}^\top w)}{2}$, every member of $\mathcal{C}'$ has cardinality more than $\frac{|V'|}{2}$. It therefore follows from Theorem 8 that $\mathcal{C}'$ has a delta or the blocker of an extended odd hole minor. Since deltas and blockers of extended odd holes do not have duplicated elements by Remark 10, this minor gives rise to a delta or the blocker of an extended odd hole minor in $\mathcal{C}$, as required. $\square$

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