Idealness and 2-resistant sets

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May 8, 2019

Abstract

A subset of the unit hypercube \( \{0, 1\}^n \) is cube-ideal if its convex hull is described by hypercube and generalized set covering inequalities. In this note, we study sets \( S \subseteq \{0, 1\}^n \) such that, for any subset \( X \subseteq \{0, 1\}^n \) of cardinality at most 2, \( S \cup X \) is cube-ideal.

1 Introduction

Take an integer \( n \geq 1 \). Denote by \( \{0, 1\}^n \) the extreme points of the \( n \)-dimensional unit hypercube \([0, 1]^n\). A sub-hypercube of \( \{0, 1\}^n \) is a subset of the form

\[
\{ x \in \{0, 1\}^n : x_i = 0 \quad i \in I, x_j = 1 \quad j \in J \} \quad I, J \subseteq \{1, \ldots, n\}, I \cap J = \emptyset;
\]

its rank is \( n - |I| - |J| \). For a coordinate \( i \in [n] := \{1, \ldots, n\} \), we refer to \( x_i \geq 0 \) and \( x_i \leq 1 \) as hypercube inequalities. Generalized set covering inequalities are inequalities of the form

\[
\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad I, J \subseteq [n], I \cap J = \emptyset,
\]

which are precisely the inequalities that cut off sub-hypercubes of \( \{0, 1\}^n \). Interpreted as clause satisfaction inequalities for the Boolean satisfiability problem, generalized set covering inequalities are prevalent in the literature. Also referred to as cropping inequalities \([7, 13]\), these inequalities have surfaced as cocycle inequalities valid for cycle polytopes of binary matroids \([5]\), as set covering inequalities \( (J = \emptyset) \) for various set covering problems \([6, 11, 8]\), and as cover inequalities \( (I = \emptyset) \) for the knapsack problem \([4, 12, 16]\).

Take a set \( S \subseteq \{0, 1\}^n \). \( S \) is cube-ideal if its convex hull, denoted \( \text{conv}(S) \), can be described by hypercube and generalized set covering inequalities. This notion was introduced and studied in \([1]\). Cube-ideal sets form a rich class of objects: Basic classes include the cycle space of a graph \([15]\) and the up-monotone set associated with an ideal clutter (see \([1]\)). Ideal clutters arise from \( T \)-joins and \( T \)-cuts in grafts (Edmonds-Johnson \([10]\)), from dijoins and dicuts in digraphs (Lucchesi-Younger \([14]\)) and other combinatorial structures. In this note, we introduce a new class of cube-ideal sets that is geometric in nature. We need a few definitions first.

Given points \( a, b \in \{0, 1\}^n \), the distance between \( a \) and \( b \), denoted \( \text{dist}(a, b) \), is the number of coordinates \( a \) and \( b \) differ on. Denote by \( G_n \) the skeleton graph of \([0, 1]^n\), whose vertices are the points in \( \{0, 1\}^n \), where two
vertices \( a, b \in \{0, 1\}^n \) are adjacent if \( \text{dist}(a, b) = 1 \). For a subset \( X \subseteq \{0, 1\}^n \), denote by \( G_n[X] \) the subgraph of \( G_n \) induced on vertices \( X \).

Given \( S \subseteq \{0, 1\}^n \), we refer to the points in \( S \) as \emph{feasible} and to the points in \( \overline{S} := \{0, 1\}^n - S \) as \emph{infeasible}. The connected components of \( G_n[S] \) are \emph{feasible components}, while the components of \( G_n[\overline{S}] \) are \emph{infeasible components}.

**Theorem 1** ([2]). Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \). If each infeasible component is a sub-hypercube or has maximum degree at most two, then \( S \) is cube-ideal.

The various basic classes of cube-ideal sets suggest that finding a structure theorem for cube-ideal sets is a daunting task. In this note, however, we provide a structure theorem for cube-ideal sets \( S \subseteq \{0, 1\}^n \) that remain cube-ideal even after adding one or two points to \( S \).

**Theorem 2.** Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \) where, for every subset \( X \subseteq \{0, 1\}^n \) of cardinality at most two, \( S \cup X \) is cube-ideal. Then every infeasible component is a sub-hypercube or has maximum degree at most two.

To prove this theorem, it will be more convenient to work with the more concrete concept of 2-resistance. We define and study 2-resistance in §2, and then prove Theorem 2 as well as other applications in §3. In the latter section we will also introduce and discuss the concept of \emph{k-resistance} for integers \( k \geq 1 \).

## 2 A characterization of 2-resistant sets

Take an integer \( n \geq 1 \) and a set \( S \subseteq \{0, 1\}^n \). Take a coordinate \( i \in [n] \). The set obtained from \( S \cap \{x : x_i = 0\} \) after dropping coordinate \( i \) is called the \emph{0-restriction of \( S \) over coordinate \( i \)}, and the set obtained from \( S \cap \{x : x_i = 1\} \) after dropping coordinate \( i \) is called the \emph{1-restriction of \( S \) over coordinate \( i \)}. A \emph{restriction} of \( S \) is a set obtained after a series of 0- and 1-restrictions. The \emph{projection of \( S \) over coordinate \( i \)} is the set obtained from \( S \) after dropping coordinate \( i \). A \emph{minor of \( S \)} is what is obtained after a series of restrictions and projections. A minor is \emph{proper} if at least one operation is applied. Denote by \( e_i \) the \( i \)th unit vector. To \emph{twist coordinate} \( i \in [n] \) is to replace \( S \) by

\[
S \triangle e_i := \{x \triangle e_i : x \in S\},
\]

where the second \( \triangle \) denotes coordinate-wise addition modulo 2. \( S' \subseteq \{0, 1\}^n \) is \emph{isomorphic} to \( S \), written as \( S' \cong S \), if \( S' \) is obtained from \( S \) after relabeling and twisting some coordinates.

Let \( P_3 := \{110, 101, 011\} \subseteq \{0, 1\}^3 \) and \( S_3 := \{110, 101, 011, 111\} \subseteq \{0, 1\}^3 \), as displayed in Figure 1. We say that \( S \) is \emph{2-resistant} if, for every subset \( X \subseteq \{0, 1\}^n \) of cardinality at most two, \( S \cup X \) has no \( P_3, S_3 \) isomorphic minor.\(^1\) The following is straightforward:

**Remark 3.** If a set is 2-resistant, then so is every minor of it.

\(^1\)Going forward, the prefix “isomorphic” will be omitted from “isomorphic minor” and “isomorphic restriction”. 

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How is 2-resistance relevant? Notice that
\[
\text{conv}(P_3) = \{ x \in [0,1]^3 : x_1 + x_2 + x_3 = 2 \} \quad \text{and} \quad \text{conv}(S_3) = \{ x \in [0,1]^3 : x_1 + x_2 + x_3 \geq 2 \},
\]
implying in turn that $P_3, S_3$ are not cube-ideal. In fact, up to isomorphism, $P_3, S_3$ are the only non-cube-ideal sets of dimension at most 3.

**Remark 4 ([1]).** If a set is cube-ideal, then so is every minor of it.

As a consequence, a cube-ideal set has no $P_3, S_3$ minor. In particular, if $S \cup X$ is cube-ideal for every set $X$ of cardinality at most two, then $S$ must be 2-resistant. Moving forward we need the following lemma:

**Lemma 5 ([2]).** Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$, where for all $x \in \{0,1\}^n$ and distinct $i, j \in [n]$, the following statement holds:

\[
\text{if } x, x \triangle e_i, x \triangle e_j \in S \text{ then } x \triangle e_i \triangle e_j \in S.
\]

Then every feasible component of $S$ is a sub-hypercube.

We are now ready to prove the following characterization of 2-resistant sets:

**Theorem 6.** Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:

(i) $S$ is 2-resistant,

(ii) $S$ has no restriction $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$,

(iii) $S$ has no minor $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$,

(iv) every infeasible component of $S$ is a sub-hypercube or has maximum degree at most two.

**Proof.** (i) $\Rightarrow$ (ii): Observe that $F$ is not 2-resistant, because $F \cup \{101, 011\}$ is either $P_3$ or $S_3$. Thus, a 2-resistant set has no $F$ restriction by Remark 3. (ii) $\Rightarrow$ (iv): Assume that $S$ has no $F$ restriction.
Claim 1. Let $x$ be an infeasible point with at least three infeasible neighbors. If $x \triangle e_i, x \triangle e_j$ are infeasible for some distinct $i, j \in [n]$, then $x \triangle e_i \triangle e_j$ is also infeasible.

Proof of Claim. Suppose for a contradiction that $x \triangle e_i \triangle e_j$ is feasible. Since $x$ has at least three infeasible neighbors, there is a coordinate $k \in [n] - \{i, j\}$ such that $x \triangle e_k$ is infeasible. Then the 3-dimensional restriction of $S$ containing $x \triangle e_i, x \triangle e_j, x \triangle e_k$ is a set $F \subseteq \{0, 1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$, a contradiction.

Claim 2. Let $x$ be an infeasible point with at least three infeasible neighbors. Let $k \geq 3$ be the number of infeasible neighbors of $x$. Then the sub-hypercube of rank $k$ containing $x$ and its infeasible neighbors is infeasible.

Proof of Claim. After a possible twisting and relabeling, if necessary, we may assume that $x = 0$ and its infeasible neighbors are $e_1, \ldots, e_k$. We need to show that for all subsets $I \subseteq [k]$, $\sum_{i \in I} e_i \in S$. We will proceed by induction on $|I| \geq 0$. The base cases $|I| \in \{0, 1\}$ hold by assumption, and the case $|I| = 2$ follows from Claim 1. For the induction step, assume that $|I| \geq 3$. After a possible relabeling, if necessary, we may assume that $I = [\ell]$. Let $y := \sum_{i=1}^{\ell-2} e_i$. By the induction hypothesis, $y$ and its three neighbors $y \triangle e_{\ell-2}, y \triangle e_{\ell-1}, y \triangle e_\ell$ are infeasible. It therefore follows from Claim 1 that $y \triangle e_{\ell-1} \triangle e_\ell = \sum_{i=1}^{\ell} e_i$ is infeasible, thereby completing the induction step.

Let $K$ be an infeasible component, and let $k$ be the maximum number of infeasible neighbors of a point in $K$. If $k \leq 2$, then $K$ has maximum degree at most two. Otherwise, $k \geq 3$. It then follows from Claim 2 that $K$ contains a sub-hypercube of rank $k$. Our maximal choice of $k$ in turn implies that $K$ is in fact the sub-hypercube of rank $k$. Thus, every infeasible component is a sub-hypercube or has maximum degree at most two. (iv) \Rightarrow (iii): Assume that every infeasible component is a sub-hypercube or has maximum degree at most two.

Claim 3. If $S'$ is a minor of $S$, then every infeasible component of $S'$ is a sub-hypercube or has maximum degree at most two.

Proof of Claim. It suffices to prove this for single restrictions and single projections. The claim clearly holds for single restrictions. As for projections, assume that $S'$ is obtained from $S$ after projecting away coordinate $n$. Let $K' \subseteq \{0, 1\}^{n-1}$ be an infeasible component of $S'$. Clearly, $\{(x, 0), (x, 1) : x \in K'\} \subseteq \{0, 1\}^n$ is connected in $G_n$ and infeasible for $S$, so it is contained in an infeasible component $K$ of $S$. If $K$ has maximum degree at most two, then so does $\{(x, 0), (x, 1) : x \in K'\}$, implying in turn that $K'$ has maximum degree at most two. Otherwise, $K$ is a sub-hypercube. In this case, as $K'$ is an infeasible component of $S'$, it must be that $K = \{(x, 0), (x, 1) : x \in K'\}$, implying in turn that $K'$ is a sub-hypercube. Thus, $K'$ is a sub-hypercube or has maximum degree at most two, as claimed.

Thus, since the infeasible component of $F$ containing 000 is neither a sub-hypercube or of maximum degree at most two, $S$ does not have an $F$ minor. (iii) \Rightarrow (i): Assume that $S$ is not 2-resistant. Then there is a subset $X \subseteq \{0, 1\}^n$ of cardinality at most two such that $S \cup X$ has a $P_3, S_3$ minor. Thus there is a subset $Y \subseteq \{0, 1\}^3$
of cardinality at most two such that $S$ has a $P_3 - Y, S_3 - Y$ minor. After relabeling the coordinates, if necessary, we see that both $P_3 - Y, S_3 - Y$ are the desired minor.

3 Consequences of Theorem 6

The first application of Theorem 6 is Theorem 2:

Proof of Theorem 2. Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$ where, for every subset $X \subseteq \{0, 1\}^n$ of cardinality at most two, $S \cup X$ is cube-ideal. In particular, $S$ is 2-resistant, so by Theorem 6, every infeasible component of $S$ is a sub-hypercube or has maximum degree at most two, as required.

Using Theorem 1 we get the following immediate consequence:

Corollary 7. A 2-resistant set is cube-ideal.

For the third application, we need another concept. Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. $S$ is polar if either it contains antipodal points, or all of its points agree on a coordinate:

$$\{x, 1-x\} \subseteq S \text{ for some } x \in \{0, 1\}^n \text{ or } S \subseteq \{x: x_i = a\} \text{ for some } i \in [n] \text{ and } a \in \{0, 1\}.$$ $S$ is strictly polar if every restriction of it, including $S$ itself, is polar. Introduced and studied in [1], strict polarity is a notion closely tied with cube-idealness as the authors used the two notions to reformulate the $\tau = 2$ Conjecture of Cornuéjols, Guenin and Margot [9]. We will characterize when 2-resistant sets are strictly polar.

To this end, consider the sets

$$R_{1,1} := \{000, 110, 101, 011\} \subseteq \{0, 1\}^3$$
$$R_{2,1} := \{0000, 1110, 1001, 0101, 0011, 1101, 1011, 0111\} \subseteq \{0, 1\}^4$$
$$R_5 := \{00000, 10000, 11000, 11100, 11110, 01110, 00110, 00010\}$$
$$\cup \{01001, 01101, 00101, 10101, 10111, 10011, 11011, 01011\} \subseteq \{0, 1\}^5,$$
a

as displayed in Figure 3. Notice that these sets are 2-resistant, as every infeasible component has maximum degree at most two, and non-polar. (These three sets are part of an infinite class $\{R_{k,1} : k \geq 1\}$ of non-polar sets, introduced and studied in [2], and “correspond” to an infinite class $\{Q_{k,1} : k \geq 1\}$ of ideal minimally non-packing clutters [9].) We will prove the following characterization:

Theorem 8. A 2-resistant set is strictly polar if, and only if, it has no $R_{1,1}, R_{2,1}, R_5$ restriction.

$S$ is strictly non-polar if it is not polar, but every proper restriction is polar. Notice that a set is strictly polar if, and only if, it has no strictly non-polar set as a restriction. Examples of strictly non-polar sets include the sets $R_{1,1}, R_{2,1}, R_5$ [3]. As an application of Theorem 6, we will prove that up to isomorphism, these three sets are the only 2-resistant strictly non-polar sets, thereby proving Theorem 8. We will need the following result:
Theorem 9 ([3]). Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only strictly non-polar sets where every infeasible point has at most two infeasible neighbors.

We will also need the following two lemmas:

**Lemma 10.** Take an integer $n \geq 5$ and a set $S \subseteq \{0,1\}^n$, where every infeasible point has at most two infeasible neighbors. Then $|S| \geq 2^{n-1}$.

**Proof.** Let us proceed by induction on $n \geq 5$. The base case is the crux of the proof, as the induction step is straightforward. Assume that $n = 5$. For $i,j \in \{0,1\}$, let $S_{ij} \subseteq \{0,1\}^3$ be the restriction of $S$ obtained after $i$-restricting coordinate 4 and $j$-restricting coordinate 5. We may assume that $|S_{00}| + |S_{10}| \leq 7$ and $|S_{00}| \leq 3$. After a possible twisting of coordinates 1, 2, 3, we may assume that $\{000, 111\} \subseteq S_{00} \subseteq \{000, 111, 110\}$. This implies that $\{001, 101, 011\} \subseteq S_{10}$. Since $|S_{00}| + |S_{10}| \leq 7$, we get that $S_{00} = \{000, 111, 110\}$ and therefore $S_{10} = \{001, 101, 011, 110\}$. Since every infeasible point of $S$ has at most two infeasible neighbors, it follows that $\{100, 010, 001, 101, 011\} \subseteq S_{01}$ and $\{000, 100, 010\} \subseteq S_{11}$, implying in turn that $|S_{01}| + |S_{11}| \geq 8$. In fact, as every infeasible point of $S$ has at most two infeasible neighbors, $|S_{01}| + |S_{11}| > 8$, so $|S| \geq 7 + 9 = 16 = 2^4$. This proves the base case. For the induction step, assume that $n \geq 6$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the $i$-restriction of $S$ over coordinate $n$; note that every infeasible component of $S_i$ has maximum degree at most two. By the induction hypothesis, $|S| = |S_0| + |S_1| \geq 2^{n-2} + 2^{n-2} = 2^{n-1}$, thereby completing the induction step.

**Lemma 11.** Take an integer $n \geq 5$ and a nonempty set $S \subseteq \{0,1\}^n$, where every infeasible component is a sub-hypercube or has maximum degree at most two. If $S$ has no $R_{1,1}$ restriction and one of its infeasible components is a sub-hypercube of rank at least 3, then

- $|S| \geq 2^{n-1}$, and
- If $|S| = 2^{n-1}$, then $S$ is either a sub-hypercube of rank $n - 1$ or the union of antipodal sub-hypercubes of rank $n - 2$. 

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Figure 3: The 2-resistant strictly non-polar sets.
Proof: We will prove this by induction on \( n \geq 5 \). The base case \( n = 5 \) is clear. For the induction step, assume that \( n \geq 6 \). For \( i \in \{0, 1\} \), let \( S_i \subseteq \{0, 1\}^{n-1} \) be the \( i \)-restriction of \( S \) over coordinate \( n \). If one of \( S_0, S_1 \) is empty, then the other one must be \( \{0, 1\}^{n-1} \), so \( S \) is a sub-hypercube of rank \( n - 1 \) and the induction step is complete. We may therefore assume that \( S_0, S_1 \) are nonempty.

Assume in the first case that \( S \) has an infeasible sub-hypercube of rank \( \geq 4 \) active in, say, direction \( e_n \); that is, the infeasible sub-hypercube intersects both \( S \cap \{x : x_n = 0\} \) and \( S \cap \{x : x_n = 1\} \). Then both \( S_0, S_1 \) have infeasible sub-hypercubes of rank \( \geq 3 \). Thus by the induction hypothesis, \( |S_0| \geq 2^{n-2} \) and \( |S_1| \geq 2^{n-2} \), implying in turn that \( |S| = |S_0| + |S_1| \geq 2^{n-1} \). Assume next that \( |S| = 2^{n-1} \). Then \( |S_0| = |S_1| = 2^{n-2} \). By the induction hypothesis, one of the following cases holds:

- **\( S_0 \)** is a sub-hypercube of rank \( n - 2 \geq 4 \): In this case, we may assume that \( S \cap \{x : x_n = 0\} = \{x : x_{n-1} = x_n = 0\} \). Since every infeasible component of \( S \) is a sub-hypercube or has maximum degree at most two, the sub-hypercube \( \{x : x_{n-1} = 0, x_n = 1\} \) is either totally feasible or totally infeasible. Since \( |S_1| = 2^{n-2} \), it follows that \( S \cap \{x : x_n = 1\} \) is either
  \[
  \{x : x_{n-1} = 0, x_n = 1\} \quad \text{or} \quad \{x : x_{n-1} = x_n = 1\}.
  \]

Thus, \( S \) is either a sub-hypercube of rank \( n - 1 \) or the union of antipodal sub-hypercubes of rank \( n - 2 \).

- **\( S_1 \)** is the union of two antipodal sub-hypercubes of rank \( n - 3 \geq 3 \): In this case, we may assume that \( S \cap \{x : x_n = 0\} = \{x : x_{n-2} = x_{n-1} = x_n = 0\} \). Since every infeasible component of \( S \) is a sub-hypercube or has maximum degree at most two, and \( |S_1| = 2^{n-2} \), it follows that \( S \cap \{x : x_n = 1\} \) is either
  \[
  \{x : x_{n-2} = x_{n-1}, x_n = 1\} \quad \text{or} \quad \{x : x_{n-2} + x_{n-1} = 1, x_n = 1\}.
  \]

However, since \( S \) has no \( R_{1,1} \) restriction, the latter is not possible. Thus, \( S = \{x : x_{n-2} = x_{n-1}\} \), so \( S \) is the union of antipodal sub-hypercubes of rank \( n - 2 \).

Thus, \( S \) is either a sub-hypercube of rank \( n - 1 \) or the union of antipodal sub-hypercubes of rank \( n - 2 \), thereby completing the induction step in this case.

Assume in the remaining case that every infeasible component of \( S \) has maximum degree at most two or is a cube (i.e. a sub-hypercube of rank 3). By assumption, one of the infeasible components is a cube, which we may assume is contained in \( S_0 \). By the induction hypothesis, \( |S_0| \geq 2^{n-2} \) and if equality holds, then \( S_0 \) is either a sub-hypercube of rank \( n - 2 \) or the union of antipodal sub-hypercubes of rank \( n - 3 \). If \( S_1 \) has an infeasible component that is a cube, then the induction hypothesis implies that \( |S_1| \geq 2^{n-2} \), and if not, \( S_1 \) has maximum degree at most two, so by Lemma 10, \( |S_1| \geq 2^{n-2} \). Either way, \( |S_1| \geq 2^{n-2} \), so \( |S| = |S_0| + |S_1| \geq 2^{n-1} \). We claim that equality does not hold. Suppose for a contradiction that \( |S| = 2^{n-1} \). Then \( |S_0| = |S_1| = 2^{n-2} \). So \( S_0 \) is either a sub-hypercube of rank \( n - 2 \geq 4 \) or the union of antipodal sub-hypercubes of rank \( n - 3 \geq 3 \). As \( S \) has no infeasible sub-hypercube of rank \( \geq 4 \), it follows that \( n = 6 \) and \( S_0 \) is the union of antipodal cubes, say

\[
S \cap \{x : x_6 = 0\} = \{x : x_4 = x_5, x_6 = 0\},
\]
and so
\[ S \cap \{ x : x_6 = 1 \} = \{ x : x_4 + x_5 = 1, x_6 = 1 \} \]
as \(|S_1| = 2^{n-2}\). But then \(S\) has an \(R_{1,1}\) restriction, a contradiction to our assumption. This completes the induction step.

We are now ready the following characterization:

\textbf{Theorem 12.} Up to isomorphism, \(R_{1,1}, R_{2,1}, R_5\) are the only 2-resistant strictly non-polar sets.

\textit{Proof.} We know that \(R_{1,1}, R_{2,1}, R_5\) are strictly non-polar sets, and since their infeasible components have maximum degree at most two, they are 2-resistant by Theorem 6. To prove that they are up to isomorphism the only 2-resistant strictly non-polar sets, pick an integer \(n \geq 1\) and a 2-resistant set \(S \subseteq \{0,1\}^n\) without an \(R_{1,1}, R_{2,1}, R_5\) restriction. It suffices to show that \(S\) is polar. By Theorem 6, every infeasible component is a sub-hypercube or has maximum degree at most two. If \(S\) has maximum degree at most two, then by Theorem 9, \(S\) is polar. Otherwise, \(S\) has an infeasible sub-hypercube of rank at least 3. If \(n = 4\) or \(S = \emptyset\), then \(S\) is clearly polar. Otherwise, \(n \geq 5\) and \(S \neq \emptyset\). By Lemma 11, \(|S| \geq 2^{n-1}\); if equality holds, then \(S\) is either a sub-hypercube or the union of antipodal sub-hypercubes, so \(S\) is clearly polar. Otherwise, \(|S| > 2^{n-1}\), implying in particular that there are antipodal feasible points, so \(S\) is polar, as required. \(\square\)

Theorem 8 follows as an immediate consequence.

For the fourth and final application of Theorem 6, take an integer \(k \geq 1\). We say that \(S\) is \(k\)-resistant if \(S \cup X\) has no \(P_3, S_3\) minor, for every subset \(X \subseteq \{0,1\}^n\) of cardinality at most \(k\).

\textbf{Theorem 13.} Take an integer \(n \geq 1\) and a set \(S \subseteq \{0,1\}^n\). Then for any integer \(k \geq 3\), \(S\) is \(k\)-resistant if, and only if, every infeasible component of \(S\) has maximum degree at most two.

\textit{Proof.} \((\Rightarrow)\) If every infeasible component of \(S\) has maximum degree at most two, then by Theorem 6, \(S\) is 2- and therefore \(k\)-resistant. \((\Rightarrow)\) Assume that \(S\) is \(k\)-resistant. In particular, \(S\) is 2-resistant by Theorem 6, so every infeasible component of \(S\) is a sub-hypercube or has maximum degree at most two. Notice however that \(S\) cannot have an infeasible component that is a sub-hypercube of rank at least 3, for if not, then \(S \cup X\) would have a \(P_3\) restriction for some \(X \subseteq \{0,1\}^n - S\) of cardinality 3, which is not possible as \(S\) is 3-resistant. Thus, every infeasible component of \(S\) has maximum degree at most two. \(\square\)

In particular, \(R_{1,1}, R_{2,1}, R_5\) are \(k\)-resistant for any integer \(k \geq 3\), so

\textbf{Corollary 14.} For an integer \(k \geq 3\), a \(k\)-resistant set is strictly polar if, and only if, it has no \(R_{1,1}, R_{2,1}, R_5\) restriction.

What about 1-resistant sets? It turns out that these sets, simply referred to as \(k\)-resistant sets, form a very rich class of cube-ideal sets and are much more complex than \(k\)-resistant sets for any integer \(k \geq 2\). These sets are studied in detail in [2].

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Acknowledgements

This work was supported in parts by ONR grant 00014-18-12129, NSF grant CMMI-1560828, and NSERC PDF grant 516584-2018.

References


