Is Mathematics the Language of Nature?

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Abstract

1 Introduction.

The thesis that 'the language of nature is mathematics' is deeply etched into the collective consciousness of modern physics. It occupies an equally fundamental role in many philosophers' ways of thinking about the workings of nature. Indeed, for most thinkers who have been raised with a modern scientific temperament, the thesis has become a commonplace; repeated so often, and by intellectuals of such stature, that one wonders how any sort of discourse in physics could be possible that did not presuppose it.

Most would say that the thesis goes back at least to Pythagoras. See, for instance, Aristotle's Metaphysics [1] xii. 6; 1080 b 16:

> *"The Pythagoreans say that there is but one number, the mathematical, but things of sense are not separated from this, for they are composed of it; indeed, they construct the whole heaven out of numbers ..."*

Centuries later, we find Kepler saying (see [8]):

> *"The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics."*

Consider also the following quote from Galileo's *Assayer* (cited on p. 64 of [9]):

> *"Philosophy is written in this grand book of the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and to read the alphabet of which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures,*

without which it is humanly impossible to understand a single word of it; without these, one wanders in a dark labyrinth."

Finally, from no less than Gauss we have (see [11]):

"God is a mathematician."

No doubt, Pythagoras, Kepler, Galileo and Gauss all had different things in mind when they praised mathematics in their various ways. At a very minimum, each seemed to think that all the processes of nature could be completely described in mathematical terms. But in addition to this, I think that each also entertained a slightly more ambitious view, according to which the natural world *itself* (and not just the processes that compose it) is isomorphic to some giant mathematical structure.[1] The temporal evolution of this structure is then governed, in turn, by precise mathematical equations.

There are a few different ways of making this more ambitious idea totally explicit. One could think of the world[2] *at any particular time* as given by some giant mathematical structure. One could then postulate that the various giant mathematical structures at different times can be connected together in some lawlike, mathematical way. This gives us a first way in which we can think of the natural world as isomorphic to some giant mathematical structure – specifically, a giant mathematical structure that changes over time.

Alternatively, one could just think of *trans-temporal* nature – i.e., the world as it exists *through* time, and not just the world at a *particular* time – as given by an even more enormous mathematical structure. Traditionally, this has involved thinking of the world as a giant, four-dimensional structure. This gives us a second way of thinking of the natural world as isomorphic to some mathematical structure.

For the sake of simplicity, I shall focus on this later formulation of the idea that I have attributed to Pythagoras, Kepler, Galileo, and Gauss.[3] This thesis – *that trans-temporal reality is isomorphic to some enormous mathematical structure* – is what I shall mean by the claim that 'the language of nature is mathematics'.[4]

In spite of its widespread acceptance, there are a few reservations one might have about this thesis. First, one might worry about to what extent 'higher order' phenomena must necessarily

[1] In fact, for the Pythagoreans, the world is not merely *isomorphic* to such a mathematical structure – it is *identical* with such a structure.

[2] By the 'world' I here mean one particular world – generally the actual world – as opposed to the collection of all worlds, both possible and actual.

[3] By focusing on this second alternative, I avoid issues about the nature of the lawlike connections that connect the giant time-slices of nature.

[4] My purpose here is not a detailed defense of the historical claim that the Pythagoreans, Galileo, Kepler and Gauss held one particular version of this view, as opposed to another. I simply wish to suggest that the thesis in question has, in one form or another, exerted a great influence on Western science. It is the thesis itself, and not its pedigree or historical development, that will be the real object of interest in the present paper.

be mathematical in nature. For instance – does it really follow from the fact that the language of micro-physics is mathematics, that human interactions or political parties can be described and understood in purely mathematical ways too? Anyone with anti-reductionist sympathies would surely be uncomfortable here.

One might also worry that there are facts about sense-experience that necessarily resist mathematical analysis. I know what the color red looks like, but is it really possible that all there is to know about redness can be captured in some mathematical structure? When one thinks of the world as given by some giant mathematical structure, it is surely conceded that many phenomenological properties of the world (for example, most secondary qualities) will have been abstracted away. But if so much is abstracted away, what right do we then have to say that the language of nature is mathematics?

For a further objection, recall that Henri Bergson argued in [3] that there were properties of objects that were in principle incapable of mathematical analysis (and in fact, incapable of any sort of *conceptual* analysis.) Bergson felt that the motion of a body was an example of this, insofar as any mathematical representation of the body that left us with nothing more than a union of time-slices necessarily abstracted away the 'mobility' of the body. Again, this might be taken to challenge the notion that an exhaustive discussion of nature can occur in purely mathematical terms.

Finally, one might worry that any mathematical representation of reality must make idealized assumptions that need not be true. Objects will be treated as point particles, or continua, or wavefunctions in mathematically convenient ways. Is it really obvious that these idealizations are harmless? If they are not harmless, or if we do not know whether they are harmless, our ability to confidently declare that the language of nature is mathematics might also be thought to be compromised.

Each of these objections deserves careful analysis. However, for the sake of the present paper, I wish to put all of them to the side. Let us forget about 'higher-order' states of affairs, phenomenological properties and secondary qualities. Let us not worry about whether a mathematical representation can truly capture the 'mobility' or 'duration' of objections in motion. Furthermore, let us blindly assume that the world is made of nothing more than point particles floating around in space. Given all these concessions, I now ask: *must the language of the world be mathematics?* That will be the central question of this paper.

Spelt out in more detail, my question is as follows: let us assume that the world is nothing more than a collection of point particles moving around in space, and that the only things we care about are the spatio-temporal trajectories of these particles. Does it follow that the world must be isomorphic to some gigantic mathematical structure? I would like to argue that the answer is No.

In the next section, I will give my argument for this claim. But first I would like to consider an argument that one can imagine a mathematician giving *against* my main thesis – that is, an argument the mathematician might give in *support* of the claim that the world must be isomorphic to some giant mathematical structure.

The argument goes something like this: label all the points of space with the elements of some set S, and associate with each point particle P a function $f_P : \mathbb{R} \to P$ representing the trajectory of the particle through time in the obvious way.[5] Also, associate with the set S some other sort of mathematical structure that encodes the spatial relations that hold amongst the different elements of S; for instance, a metric, or a topology. For the sake of definiteness, let us assume that a metric ρ has been associated with S. We may then mathematically represent the world by the structure $<S, \rho, F>$, where $F = \{f_P : P \text{ is an existing point particle}\}$. The mathematical structure $< S, \rho, F >$ is then 'isomorphic' to the physical world, in the sense that every possible relation between particles and/or points of space may be captured by some purely mathematical statement about the structure $< S, \rho, F >$. It then follows that, in the sense I have described, mathematics may be taken to be the language of nature.

But let us consider this proposal more closely. The mathematician begins by asking us to 'label the points of space with the elements of some set S'. What does he mean by this? Presumably, he means that we should give a name to each point in space, and then throw all these names into a big set. But how do we know that this is possible? We can certainly imagine giving names to a finite handful of points in space. But what is supposed to happen after that? How do we know that all the points of space can be 'enumerated' in the way that the mathematician will require? And how do we know that we won't run out of names?

Certainly, one can invoke abstract principles such as Zorn's lemma to try and answer such questions. But these principles are purely mathematical principles, and the only way they can have any bearing on the question of whether or not one can associate a unique name with each point of space, is if one assumes from the beginning that space is some sort of mathematical structure. In this way, such fancy maneuvers simply beg the question.

Let me distinguish my difficulty with the mathematician's demand that we 'label the points of space with the elements of some set S' from a couple of difficulties which which it could be confused. My objection is *not* that, for all we know, the points of space may form a 'class', rather than a 'set'. Certainly, that would be one way in which the mathematician's demand could turn out to be unreasonable. But this objection presupposes that one *can* successfully label all the points of space with the elements of some sort of structure S – but that S happens to be a class, rather than a set. My point is that it is not obvious why *any* attempt to label the points of space with *any* sort of mathematical structure S should be successful.

[5] It is assumed here that time is structured like the reals $\mathbb{R}$. Of course, one can formulate a more abstract conception of the trajectory of a particle that does not depend on this assumption, but because nothing that follows is affected by this, I do not opt for such generality here.

Nor is my objection that there may not be any 'constructive' way of labelling the points of space with elements of some set S. My objection is not that the mathematician has given us no explicit algorithm for carrying out his demand, but rather that the mathematician has given us no reason for thinking that his demand could be carried out in *any* way, constructive or otherwise.

There is no reason to think that the mathematician's demand that we 'label the points of space with the elements of some set S' can be met, *without* presupposing that space *already* has some sort of mathematical structure. Certainly, the mathematician's assertion that 'one can label the points of space with the elements of some set S' is perfectly acceptable as an expression of faith in the mathematical structure of the universe. It cannot, however, count as a *justification* of such faith. In fact, it is precisely this sort of faith that I shall try to undermine in the next section.

2 The Main Argument.

In this section, I shall argue that it is possible that the world is not isomorphic to any mathematical structure. My argument will require two premises - one about the nature of possibility, and another about the nature of mathematics. The premises are as follows:

Modal Premise: Every logically consistent state of affairs is possible.

Mathematical Premise: The referent of the term $\mathbb{R}$ is the same in all possible worlds.

The second premise is a little less familiar than the first, and so I shall discuss it briefly before commencing my main argument.

The terms $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ refer respectively to the set of natural numbers, the set of rational numbers, and the set of reals (i.e., the set of Dedekind cuts of the rational numbers.)[6] Let us imagine someone in a possible world who has constructed (or obtained by non-constructive means) some real number r. Imagine this person (truthfully) asserting that r is a real number. The content of the Mathematical Premise is that any other person in any other possible world will be able to recognize this claim as both meaningful and true. The meaningfulness and truth of claims of the form '$r \in \mathbb{R}$' is independent of any particular possible world. So, if a claim of the form '$r \in \mathbb{R}$' is meaningful and true in *some* possible world, then it is *necessarily* meaningful and true.

[6]By a 'Dedekind cut' of the set of rationals $\mathbb{Q}$, we mean a subset S of $\mathbb{Q}$ such that:

(i) $S \neq \emptyset$ and $S \neq \mathbb{Q}$,

(ii) $\forall p, q \in \mathbb{Q}[q \in S \text{ and } p < q \rightarrow p \in S]$, and

(iii) S contains no largest element.

For a discussion of Dedekind cuts, see [12].

In addition, we assume that the set of natural numbers in the transworld referent of $\mathbb{R}$ are all standard – i.e., that the transworld referent of $\mathbb{R}$ contains no nonstandard natural numbers. We take this to be a 'subpremise' of the Mathematical Premise.

Of course, either of the two premises just listed can be denied. In a later section (specifically, §4), I will discuss the extent to which the conclusions of the present section can be avoided by denying either the Modal Premise or the Mathematical Premise. For now, however, let us press on, asumming both the Modal and Mathematical Premises to be correct.

We will need to use a theorem that comes from set theory. Fix a model $V \models ZFC$. Let $\mathbb{R}^V$, $\mathbb{Q}^V$ and $\mathbb{N}^V$ denote the set of reals, rationals, and natural numbers contained in V. In addition, if T is a a set of sentences, define $Con(T)$ to be the proposition that says that T is consistent. One may then prove the following theorem:

Main Theorem: $Con(ZFC) \rightarrow Con(ZFC+$there is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$.)

The sentence 'there is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$' is to be thought of as an existential quantification over the conjunction of an infinite set of sentences, specifically:

$$\text{`there is a Dedekind cut of } \mathbb{Q}^V \text{ not contained in } \mathbb{R}^V\text{'} \leftrightarrow (\exists x)\ [s(x) \wedge \bigwedge_{r \in \mathbb{R}^V} t_r(x)]$$

where $s(x)$ is the formula 'x is a Dedekind cut of $\mathbb{Q}^V$', and for each $r \in \mathbb{R}^V$, t_r is the formula '$x \neq r$'.

The sentence 'x is a Dedekind cut of $\mathbb{Q}^V$' should, in turn, be thought of as the infinitary sentence:

$$(\forall d)[(d \in x \rightarrow \bigvee_{q \in \mathbb{Q}^V} d = q) \wedge \Phi(x)],$$

where $\Phi(x)$ consists of the usual conditions for a Dedekind cut, given in the most recent footnote.

Once we introduce infinite conjunctions and disjunctions, we need to re-specify what is meant by 'consistency'. To say that a set of sentences is consistent is to say that no contradiction may be deduced from them. But what are the rules of inference relative to which we are to make this judgment?

To the usual logical axioms and rules of inference of the predicate calculus, we adjoin the logical axiom:

$$(\bigwedge_{\alpha<\beta} \Phi_\alpha) \rightarrow \Phi_\gamma \quad \text{(for each } \beta \text{ and } \gamma < \beta)$$

and the rule of inference:

$$\text{if } \Phi_\alpha \text{ for all } \alpha < \beta \text{, infer } \bigwedge_{\alpha<\beta} \Phi_\alpha.$$

We must then allow for the possibility of proofs of arbitrary (infinite) length. For further discussions of proofs in infinitary logic, see, for example, Dickman [5], and §5 of Chapter III of Barwise [2].

To say that a set of sentences S involving infinitary connectives is consistent is then just to say that no sentence of the form $\Phi \wedge \neg\Phi$ can be derived from S, using the logical axioms and rules of inference just described.[7]

The Main Theorem tell us that if ZFC is consistent, then so is ZFC together with the assumption that there is a Dedekind cut of $\mathbb{Q}^V$ not in $\mathbb{R}^V$. The proof of this claim proceeds by a largely standard set theoretic forcing argument, and will be omitted in this version of the paper.

Fix a model $V \models ZFC$ for which $\mathbb{R}^V$ is the unique trans-world referent of $\mathbb{R}$ referred to in the Mathematical Premise. Consider now the following proposition (call it X):

> X: 'Space-time has a 4-dimensional Euclidean geometry, and there exists a point particle P whose trajectory $\vec{f}(t)$ satisfies $f_x(0) = r$, where f_x is the x co-ordinate of $\vec{f}$, and r is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$.'

One might worry that, at this stage, a claim like $f_x(0) = r$, as it appears in X, is meaningless. Specfically, one might worry that, even having said that space-time has a metric structure isomorphic to $\mathbb{R}^4$, one still has to specify an origin and a set of orthogonal vectors $\hat{x}, \hat{y}, \hat{z}$ with which to define a co-ordinate system, before a claim like $f_x(0) = r$ even makes sense. These details are easily filled in. Let us specify in X that space-time is to contain at least four uniquely identifiable and simultaneous events E, E_1, E_2 and E_3. Then one can specify that E is to represent the origin of our co-ordinate system, and use the vectors from E to E_1, E to E_2, and E to E_3 (assuming that they are not coplanar, and that no pair is collinear), appropriately normalized, to define an orthonormal co-ordinate system $\hat{x}, \hat{y}$ and $\hat{z}$. One can then specify that $f_x(0) = r$ relative to this co-ordinate system.

In addition, one might similarly wonder whether at this stage a claim like 'r is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$' has any meaning, given that all we know about r is that it is a point in a 4-dimensional Euclidean space. Let us therefore explain what we take the meaning of 'r is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$' to be. When we say that space-time has a 4-dimensional Euclidean geometry, we mean that space-time is given by a structure $\bar{\mathbb{R}}^4$, where $\bar{\mathbb{R}}$ is the set of Dedekind cuts, with the usual metric, of some ordered field $\mathbb{F}$. Now there exists a unique embedding $i : \mathbb{Q} \to \mathbb{F}$ of the (standard) rationals $\mathbb{Q} = \mathbb{Q}^V$ into $\mathbb{F}$. We then say that 'r is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$' just in case $i^{-1}(r \cap i(\mathbb{Q}^V)) \notin \mathbb{R}^V$. Note that the structure $\bar{\mathbb{R}}$ is *rigid*, i.e., admits no automorphisms other than the identity, and so the truth value of 'r is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$' is unambiguously determined by the criterion just outlined.

[7] The reader versed in such things will note that I have just described the infinitary logic $\mathcal{L}_{\infty,\omega}$.

Having explained the meaning of X, we then use the Main Theorem to conclude that the proposition X is logically consistent. Therefore, using the Modal Premise, X describes a possible state of affairs. Let w be a possible world in which X is true.

I claim that there there is no mathematical structure that can be used to represent the trajectory of P in w, without demanding that the reference of $\mathbb{R}$ in w be different from $\mathbb{R}^V$; i.e., without demanding that there is a Dedekind cut of $\mathbb{Q}$ not to be found in $\mathbb{R}^V$. In other words, the existence of a mathematical structure representing the possible world w runs afoul of the Mathematical Premise. Let us consider why.

Obviously, we cannot represent space-time with the mathematical structure $\mathbb{R}^4$, because then the trajectory of P would have to be represented by some $\vec{f}: \mathbb{R} \to \mathbb{R}^3$, where $\vec{f}(t)$ is the position of the particle at time t. This would force us to conclude that $f_x(0) \in \mathbb{R}$, which would be a contradiction. But it does not follow from this fact alone that the trajectory of P is mathematically ineffable. Perhaps we can use some unconventional mathematical trick to represent the trajectory of P. It does not follow from the fact that *one* way of representing P fails, that *all* ways must also fail. Perhaps, if we allow ourselves to exercise some mathematical imagination, what appears to be mathematically ineffable might not be mathematically ineffable after all.

I shall argue, however, that if the trajectory of the point particle P can be mathematically represented in *any* way – by *any* sort of mathematical structure – then one can construct, from such a structure, a Dedekind cut of $\mathbb{Q}^V$ not in $\mathbb{R}^V$. This will then contradict the Mathematical Premise, and we will be able to conclude that the language of nature need not be mathematics. Let us see why.

A basic requirement of any mathematical representation of the trajectory of a particle is that, for any proposition p about the relative positions of objects involving natural language, there should exist some purely mathematical claim M_p such that p holds if and only if M_p holds. As an example of this, for any $q \in \mathbb{Q}^V$, let the proposition p_q be as follows:

$$p_q: \quad \text{‘at time } t = 0\text{, the particle } P \text{ is to the left of the plane determined by } x = q.\text{’}$$

This is not a purely mathematical claim, as it involves the ordinary language expression ‘to the left of’. If one takes trajectories to be represented by functions $f: \mathbb{R}^V \to \mathbb{R}^V$, however, then one can construct an equivalent, purely mathematical claim M_q as follows:

$$M_q: \text{ ‘}f_x(0) < q, \text{ where } f_x \text{ is the } x\text{-coordinate of } f.\text{’}$$

We will then have that p_q is true iff M_q is true. It is a requirement of any mathematical representation of the trajectory of a particle that, for each p_q, there is some purely mathematical proposition M_q with this property.

Imagine that, using some sort of unusual mathematical machinery, we are able to construct a mathematical representation of the particle P. Using this machinery, we must be able to construct purely mathematical propositions M_q (presumably different from those constructed above) such that p_q is true iff M_q is true. (We do not place any restriction on the M_q – in particular, we do not even require that the M_q be first-order for what follows.) Now define the following subset S of the rationals $\mathbb{Q}^V$:

$$x \in S \text{ iff } x \in \mathbb{Q}^V \text{ and } M_x \text{ is true.}$$

Then S is just the Dedekind cut of $\mathbb{Q}^V$ corresponding to the real $r \notin \mathbb{R}^V$. So regardless of which type of mathematical structure we use to define the trajectory of P, the real $r \notin \mathbb{R}^V$ will be definable in turn from sentences concerning that mathematical structure.

What can we conclude from this? If S is any mathematical structure from which the trajectory of the point particle P can be defined, then S can be used to define a real – more specifically, a Dedekind cut of $\mathbb{Q}^V$ – not in $\mathbb{R}^V$. So if the trajectory of the particle P is mathematically representable, then the reference of $\mathbb{R}$ cannot be $\mathbb{R}^V$ – it must include some $r \notin \mathbb{R}^V$. To put it differently – if we have any mathematical machinery with which we can represent the trajectory of P, then we have the mathematical machinery with which we can show that there are reals that are not in $\mathbb{R}^V$. Regardless of what one takes the trans-world referent of $\mathbb{R}^V$ to be, one can construct a possible world containing a point particle P such that, if P can be mathematically represented in some way, then one can construct a real not in $\mathbb{R}^V$. It follows from the Mathematical Premise that the trajectory of such a particle *cannot* be represented by *any* mathematical structure. So the world w is *not* isomorphic to a giant mathematical structure, and the language of nature need not be mathematics. This concludes my main argument.

In the next two sections (§§3–4) I shall discuss some objections to my main argument. In §3 I shall focus on objections that concede the Modal and Mathematical Premises, but that argue that my conclusion somehow fails to follow. In §4 I will then discuss the obstacles involved in denying either the Mathematical or Modal premise.

3 Objections and Replies.

In this section, I wish to consider several objections to the validity of the argument just given. The first objection really just amounts to a suspicion that something has gone wrong:

Objection 1: You appear to have used mathematics to construct something mathematically ineffable. There is something paradoxical about this - something must have gone wrong somewhere in your argument.

Reply: Contrary to appearances, no such thing has been done. The main mathematical step in my argument is simply directed towards showing:

$$Con(ZFC) \to Con(ZFC + \text{`there is a Dedekind cut of } \mathbb{Q}^V \text{ not contained in } \mathbb{R}^V.\text{'})$$

This is, first and foremost, a claim about sentences, and does not involve the construction of anything mathematically ineffable in any way. For instance, this mathematical result does not involve the explicit construction of a model of ZFC satisfying 'there is a Dedekind cut of $\mathbb{Q}^V$ not contained in $\mathbb{R}^V$.' In fact, such a construction would be impossible, as it would violate Gödel's Second Incompleteness Theorem.

The real 'constructive' work in the argument of the previous section is done by the Modal Premise, according to which every logically consistent state of affairs is possible. From this, we are able to move from a consistent set of sentences to a possible world in which those sentences are true. This is not a 'mathematical construction' in any sense of the word – if anything, it is a 'philosophical construction'. It is from this 'philosophical construction' that something mathematically ineffable can be found. The mathematically ineffable trajectory has therefore been 'philosophically', and not 'mathematically', constructed.

Let us turn to the next objections:

Objection 2: Your techniques prove too much. In particular, they show that mathematical truths cannot be necessary truths. To see why, let w_1 be a possible world in which space-time has the Euclidean structure $(\mathbb{R}^V)^4$. Using the Main Theorem together with the Modal Premise, construct a possible world w_2 in which space-time has the Euclidean structure $(\mathbb{R}^*)^4$, where $\mathbb{R}^V \subset \mathbb{R}^*$, and in which $\mathbb{R}^*$ contains a real number not contained in $\mathbb{R}^V$. Consider the proposition Y defined as follows:

$$\text{Y}: \text{ `}\exists x \in \mathbb{R}\ [x \notin \mathbb{R}^V]\text{'}$$

Then in w_1, Y is false, but in w_2, Y is true. Mathematical truths are no longer necessary truths.

Reply: One option here is to go along with the objection, and conclude that mathematical truths are not necessary truths. I will not take this route for two reasons: first – I think that there are independent reasons for wanting mathematical truths to be necessary, and second – we shall see in the next section that someone who does not believe in the necessity of mathematical truths is unlikely to go along with the Mathematical Premise. We must consequently face this objection head on.

I do *not* think that the techniques of the previous section show that mathematical truths are not necessary truths. Let us consider the possible world w_2 described in the objection. If one held that the 4-dimensional Euclidean geometry of space time *had* to be mathematically representable (i.e., had to be isomorphic to some fixed mathematical structure), then it could indeed be inferred that such a structure would have to have the form $(\mathbb{R}^*)^4$, where $\mathbb{R} \subset \mathbb{R}^*$. This would seem to entail the existence of real numbers not in $\mathbb{R}^V$. One could then take this to mean that the sentence Y was true in w_2, even though it is obviously false in w_1.

However, this reasoning is based on the assumption that the 4-dimensional Euclidean geometry of space-time *has* to be fully mathematically representable. This is precisely the view I want to reject. If one does *not* insist that the 4-dimensional Euclidean geometry of space-time must be mathematically representable, then one can say that, in w_2, there is a portion of space-time (specifically, the portion not contained in $(\mathbb{R}^V)^4$) that is mathematically ineffable. This does not mean that the sentence Y is true in w_2 – to the contrary, insofar as one focuses on the portion of space time that *does* admit a mathematical description, Y will be false. That is, insofar as in the sentence

$$\text{Y}: \text{ `}\exists x \in \mathbb{R}\ [x \notin \mathbb{R}^V]\text{'}$$

one takes the existential quantifier to range over the mathematically describable points of space time, Y is false. Of course, one is free to let the existential quantifier range over the 'points of space-time in general', in such a way that the mathematically ineffable points are included; but in that case, Y is no longer a purely *mathematical* assertion – it is a hybrid mathematical/ ordinary language assertion – and therefore no tension is to be found with the doctrine that the truths of mathematics must be necessary.

In brief, it is only insofar as one clings to the idea that all physical structures must be mathematically representable that one can reach the uncomfortable conclusions outlined in the objection. Once one rejects this idea, as I wish to do, the uncomfortable conclusions outlined in the objection are no longer forced upon us.

Objection 3: Your argument exploits the fact that ZFC is unable to fix the reference of $\mathbb{R}$. However, perhaps the trouble here lies with ZFC. If you want to insist that the reference of $\mathbb{R}$ is fixed, perhaps we need to add axioms to ZFC to guarantee this – for instance, perhaps we should consider adding an axiom that specifically tells us what the reference of the term $\mathbb{R}$ is to be. Instead of pointing out an expressive limitation of mathematics itself, your argument does nothing more than highlight a problem with ZFC.

Reply: The problem with this suggestion is that there is *no* way of extending ZFC to a larger first-order theory ZFC^* in such a way that ZFC^* fixes the reference of $\mathbb{R}$. To see this, let c be a constant symbol unused in ZFC^*, and let T be the set of sentences consisting of ZFC^*, '$c \in \mathbb{R}$', and '$c \neq r$' for each $r \in \mathbb{R}$. Then the set of sentences T is consistent, because each finite subset of T is consistent. So any model of ZFC^* will contain a real number not in $\mathbb{R}$. Consequently, ZFC^* does not fix the reference of $\mathbb{R}$. The inability to fix the reference of $\mathbb{R}$ is not a peculiarity of ZFC, but applies to any first-order set theory quite generally. New axioms cannot save us.

In reply to this, it could be suggested that a more radical departure from conventional wisdom in the foundations of mathematics is needed. For instance, perhaps the foundations of mathematics need to be stated in a higher order logic, or infinitary logic, for which we do not have the usual compactness, completeness, and incompleteness results on which the previous arguments depend. On one level, of course, I have nothing with which to defend myself against this specific line of attack – if my opponent wants to radically rewrite the foundations of mathematics, then I have no desire to stop him.

Nevertheless, it is worth noting that there may be good reasons for thinking of first order logic as the appropriate logic in which to seek the foundations of mathematics. Consider, for instance, second-order logic, in which quantifiers of the form $\forall X \subseteq Y$ and $\exists X \subseteq Y$ are taken as basic. One might worry that it is precisely the job of the foundations of mathematics to explain what the relationship '$\subseteq$' consists in, rather than taking it as unanalyzable. One might express similar qualms about infintary connectives. Such concerns might lead one to think that only first order logic can provide the logical framework for the foundations of mathematics, and that higher order logics cannot fulfill this role.[8] If this is right, then there is little room in which to argue that my result highlights nothing more than an inadequacy in the current foundations of mathematics.

4 Two more ways out.

How easy is it to avoid the conclusion of my main argument by denying either of the Modal or Mathematical Premises?

Let us consider the Modal Premise, which states that every logically consistent state of affairs is possible. Certainly there are various ways in which one could weaken this claim in order to avoid the conclusion of my main argument. For instance, one could maintain that, in full generality, the Modal Premise is false, and should be replaced by the following claim:

Weak Modal Premise: A logically consistent set of sentences is possible just in case it describes a possibility that can be represented mathematically.

One would therefore be able to avoid the conclusion that there might be mathematically ineffable structures in the world. How reasonable an approach is this?

One concern is that the weak modal premise forces onto us an artificial restriction on what states of affairs are to count as possible. Let us imagine someone who, in order to convince us that it is impossible for evil to triumph over good, tells us that a logically consistent state of affairs is

[8] Whether this conclusion is nothing other than the expression of a philosophical and mathematical prejudice is an open question that I cannot hope to address here.

possible just in case it involves evil triumphing over good. If he cannot justify why, on *independent grounds*, we should accept his restriction on the nature of possibility, then he cannot hope to be taken seriously. How does advancing the Weak Modal Premise above differ from this?

One might suggest that there *are* independent grounds for thinking that we ought to restrict the Modal Premise to cover only situations that are mathematically representable, and deny the possibility of situations that cannot be mathematically represented. In the first section of this paper, I have presented one such argument, and shown it to be unpersuasive. Alternatively, one might suggest that mathematics just *is* the science of all possible structures, and therefore declare that the restriction is reasonable. But this later justification of the Weak Modal Premise is also unpersuasive – if mathematics really *is* the science of all possible structures, then one should not *need* to add provisos to the Modal Premise in order to block the existence of mathematically ineffable structures – one should be able to get any relevant provisos, and block the existence of mathematically ineffable structures, for 'free'. What my argument shows is that one *cannot* blindly assume that mathematics is the science of all possible structures. To try and salvage the hypothesis that mathematics is the science of all possible structures by restricting the world of possible structures to those which we already know are mathematically representable, is no different from the awkward maneuvers described in the above paragraph concerning the triumping of good over evil.

It is unreasonable to try and block the possible existence of mathematically ineffable structures by moving from the Modal Premise to the Weak Modal Premise. Such a move would be blatantly ad hoc. But, of course, to say that something is ad hoc is not to argue that it is false. There is no *logical* problem with the Weak Modal Premise. However, the arbitrariness of the restrictions involved will, I think, make the Weak Modal Premise an unattractive alternative to the Modal Premise.

What about the Mathematical Premise? One might think that surely we are at greater liberty to deny *this* assumption. I think, however, that the obstacles surrounding the denial of the Mathematical Premise are actually more substantial that those surrounding the denial of the Modal Premise.

The main problem with denying the Mathematical premise is that it forces one to give up the idea that mathematical truth is a species of necessary truth. To see why, assume that the Mathematical Premise is false. Let w_1 and w_2 be possible worlds in which $\mathbb{R}^{w_1} \neq \mathbb{R}^{w_2}$, where $\mathbb{R}^{w_1}$ and $\mathbb{R}^{w_1}$ are the referents of the term 'the reals' in w_1 and w_2. Assume, without loss of generality, that $\exists x[x \in \mathbb{R}^{w_2}$ and $x \notin \mathbb{R}^{w_1}]$.

Consider the proposition Z defined as follows:

$$\text{Z} : \text{`}\exists x \in \mathbb{R}\ [x \notin \mathbb{R}^{w_1}]\text{'}$$

Then in w_1, Z is false, but in w_2, Z is true. So mathematical truths are no longer necessary truths.

One might think it unfair that, in our sentence Z, we have made explicit reference to a possible world w_1. I do not think this is anything to be worried about, as we can just think of $\mathbb{R}^{w_1}$ as the name of a set. However, it is worth discussing ways in which this difficulty may be able to be avoided.

Using the method of set theoretic forcing, one can 'add reals' in such a way that the value of the continuum is changed. (For references on the method of forcing, see [6] or [7].) Specifically, we begin with the following theorems (proofs for which can be found in the references just given):

Theorem 1: $Con(ZFC) \rightarrow Con(ZFC + 2^{\aleph_0} = \aleph_1)$.

Theorem 2: $Con(ZFC) \rightarrow Con(ZFC + 2^{\aleph_0} = \aleph_2)$.

What these theorems show is that it is possible for the definite description $\mathbb{R}$ to refer to sets $\mathbb{R}^*$ and $\mathbb{R}^{**}$ respectively, such that the following holds:

(i) the set $\mathbb{R}^*$ has no uncountable subset S such that $card(S) \neq card(\mathbb{R}^*)$, and

(ii) the set $\mathbb{R}^{**}$ has an uncountable subset S such that $card(S) \neq card(\mathbb{R}^{**})$.

Consider now the proposition Y:

$$\text{Y}: \text{ `}\exists S \subset \mathbb{R}\ [S \text{ is uncountable, and } card(S) \neq card(\mathbb{R})]\text{'}$$

If $\mathbb{R}$ refers to $\mathbb{R}^*$, then Y is false, but if $\mathbb{R}$ refers to $\mathbb{R}^{**}$, then Y is true. This is an example of a mathematical assertion not explicitly or implicity making reference to any possible world, which can perhaps be contingently true, but not necessarily true.

Of course, it is perfectly consistent to maintain that, although the term $\mathbb{R}$ can have different references in different possible worlds, it cannot refer to both $\mathbb{R}^*$ in one possible world, and $\mathbb{R}^{**}$ in another. (More generally, in order to avoid the sort of problem I am trying to make here, one would have to insist that the various possible references of the term $\mathbb{R}$ all be 'elementarily equivalent'; see [4] for an explanation of this terminology.) But in that case my first counterexample to the necessity of mathematical truth given by the proposition Z still holds. In order to deny the Mathematical Premise, one therefore must either (i) accept that there can be mathematical truths that are not necessary, or (ii) argue that the various references of the term $\mathbb{R}$ are all elementarily equivalent, *and* that there is something illegitimate about the mathematical proposition Z. Both of these options are possible; the first, however, parts substantially with conventional philosophical wisdom about mathematical truth, while the second cries out for a detailed technical account, yet to be provided, of the possible references of mathematical terms. Neither of these options are absurd, but the pursuit of either option demands, I think, substantial and new developements in the philosophy of mathematics. Neither option offers an easy and obvious 'safe haven' from my main argument.

5 Consequences for physics.

Physics tries to make sense of the world using mathematics. In fact, the trend of post-Aristotelian physics strongly suggests that all there is to be known about the world can be expressed in purely mathematical language. What implications do my arguments presented thus far have for this doctrine?

One might take my main argument to suggest that any attempt to understand physical reality solely by mean of equations could well be doomed to failure.[9] Alternatively, one might take my main result to indicate that, insofar as one focuses on laws that can be mathematically formulated, there may well be no 'final' set of laws that decisively 'gets the whole universe right'. Perhaps one might even take my result to suggest that a 'qualitative', rather than 'quantitative' physics may be necessary.

But these conclusions are too strong. While the possibilities they describe are *bona fide* possibilities, it would be an exaggeration to suggest that my main argument supports them.

Certainly, my argument demonstrates that a blind optimism about the expressive power of mathematics, according to which it goes without saying that the world may be taken to be isomorphic to some giant mathematical structure, is misguided. I take myself to have shown that such optimism is unfounded. But the ideas in the previous paragraph are surely too *pessimistic*, insofar as they suggest that the tools of mathematics may be inadequate, on their own, for discussing reality in any *comprehensive* way.

I think that both the optimistic and pessimistic views just described are unfounded. Giant mathematical structures may not necessarily capture all that there is to be known about reality – but it does not follow that mathematics on its own is impotent in any comprehensive description of reality. In particular, it does not follow that a comprehensive description of reality using nothing but the tools of mathematics is impossible. There is surely a middle ground here. In this section, I wish to describe such a middle ground, and suggest that my arguments only demand a retreat from the blindly optimistic view of the expressive power of mathematics to such a middle ground; rather than a full swing from the blindly optimistic view of the expressive power of mathematics to an equally blindly pessimistic view.

[9] There are several reasons for which such a project could be doomed to failure. One reason, for instance, might be that the equations describing the world are too complicated to be grasped by the human mind. Another reason might be that the world can only be described by a *non-recursive* set of equations. My emphasis here, however, is on an even more serious sense of failure, according to which mathematical physics may well be doomed to failure not because of limitations on the human mind, but because of theoretical limitations on the expressive power of mathematics itself.

In order to support this thesis, I shall need to distinguish different ways of *using* mathematics. Let us imagine a Greek geometer who traces a set of lines through the sand, and then uses the axioms of Euclidean geometry to argue that some angle is equal in magnitude to some other angle. In the process, he will make claims like: *'this is a straight line, therefore ... '*. Of course, if pressed, he will admit that no straight lines can be found in what he has traced in the sand. But for his purposes, this does not matter; the various angles that he has drawn will be, roughly speaking, equal. Of course, our geometer will admit that if the angles in question were measured with precise instruments, they would probably turn out to be unequal. But this should not worry him – if he is modest enough in what sorts of inferences he draws, mathematics will not lead him to falsehoods. So, for instance, if he were to infer that a measurement of the third decimal places of the two angles he had drawn would show no discrepancy, then surely he would be in trouble. But if he refuses to draw this sort of conclusion, then he will be fine.

By restricting his inferences in this way, the geometer can use mathematics with great success in deducing all sorts of properties of the objects he has created in the sand.[10] I shall call this an *inferentially restrictive* use of mathematics. I shall oppose it to an *inferentially permissive* use of mathematics, according to which all inferences – including those about the agreement of third, fourth, and one hundredth significant figures – are permitted.

My example might suggest that inferentially restrictive uses of mathematics are mainly at home when reasoning with idealizations and approximations. I do not think, however, that this is so. Inferentially restrictive uses of mathematics are also to be found when reasoning with inconsistent or non-existent mathematical objects. For instance, in physics, one often reasons with delta functions or path measures of certain sorts. Such mathematical objects can be shown not to exist. Nevertheless, physicists manage to reason quite successfully with them, by avoiding certain types of inferences that they know will get them into trouble. In this way, many non-rigorous mathematical methods in physics may simply be viewed as *inferentially restrictive* uses of mathematics.[11] This has nothing to do with approximations and idealizations – it is *not* because physicists think that Schrödinger's equation is only approximate that they freely use delta functions or path measures in quantum mechanics. Rather, in many cases (and especially in the case of path measures) the physicist simply feels that he has no alternative but to use mathematically questionable concepts. He gets by with the hope that if he is careful enough with his inferences, he can avoid many types of trouble. Thus, inferential restrictiveness can rear its head when dealing with descriptions of reality that are in no way intended to be approximate.

[10]It is crucial for my purposes that we take the geometer to be trying to draw conclusions about the specific shapes he has drawn in the sand, as opposed to conclusions about 'ideal triangles', or abstract entities of some sort. Certainly, it is possible only to worry about the later sorts of objects, but it is equally possible only to worry about the former, which is what I shall assume is the case here.

[11]This is not to say that *all* non-rigorous mathematical methods in physics are of this form, however.

Any sort of reasoning with non-existent or inconsistent mathematical objects will necessarily be inferentially restrictive. Reasoning with such objects therefore involves a fundamentally different *use* of mathematics than the orthodox use, insofar as it cannot be taken for granted that mathematically valid arguments take true premises to true conclusions. When reasoning with non-existent or inconsistent mathematical objects, one will sometimes have to simply abstain from a certain pattern of inference, not so much because there is some clearly recognizable error in it, but rather because it is clear that, if such inferences *were* allowed, a contradiction, or something else undesirable, would follow.[12]

This sort of use of mathematics allows us to define a middle ground between the earlier described optimism (according to which reality is necessarily isomorphic to some sort of mathematical structure), and pessimism (according to which mathematics may well be useless to us, on its own, in any comprehensive description of reality).

To see how, note that if the world is isomorphic to a giant, existing mathematical structure, then, given certain general facts about the world, one can use the techniques of mathematics to deduce more specific facts about the world. As long as the world is isomorphic to a giant, existing mathematical structure, *any* mathematical inference will take us from true premises to a true conclusion. In such a case, one can employ mathematics in an inferentially permissive way when doing physics.

However, we now know that we *cannot* take for granted that the world is isomorphic to any actual mathematical structure. My suggestion is that this does not mean that we may have to give up on the project of using mathematics to completely describe the world. All that follows, I would like to suggest, is that we may have to give up on the project of using mathematics in an *inferentially permissive* way to completely describe the world. We may be able to describe the universe *completely* in purely mathematical terms; but not necessarily in an *inferentially permissive* way.

In fact, the examples of delta functions and path integrals provide a nice illustration of this point. Delta functions and path integrals are examples of inconsistent mathematical objects that are used to move from certain claims about the physical world to other claims about the physical world. As long as one obeys the relevant inferential restrictions associated with them, the consistency of the set of claims about the physical world itself is in no way threatened. The possibility I wish to raise is that perhaps the use of such inconsistent concepts is necessary in physics. [13]

[12]In fact, I think that the case of the geometer reasoning about lines in the sand may also be construed a type of reasoning with inconsistent mathematical objects. In this case, the geometer recognizes that his lines in the sand have all their obvious physical properties – they have width, unclear boundaries, etc. – while at the same time declaring that they are the sorts of things of which Euclid's axioms hold. These are inconsistent demands. Because of this, I actually find it more natural to think of the geometer's inferentially restrictive use of mathematics as rooted in the presence of inconsistency, rather than being rooted in the usefulness of approximations.

[13]This possibility, however, should not be confused with the possibility raised by others (see, for example, Priest's [10]) that the set of truths about the world itself may be inconsistent.

These considerations give us a middle ground between (i) the blindly optimistic view about the representational scope of mathematics, according to which reality is necessarily isomorphic to some sort of mathematical structure, and (ii) the pessimistic view about the the representational scope of mathematics, according to which mathematics may well be useless to us, on its own, in providing us with a complete description of reality.

The possibility I have described is that a complete mathematical description of reality may need to proceed in *inferentially restrictive* terms. More specifically, I have suggested that any mathematical description of reality may need to make use of inconsistent mathematical entities. This is weaker than the optimistic view, according to which one *must* be able to come up with a complete description of reality in inferentially permissive terms. But it is stronger than the pessimistic view, insofar as it does not abandon the idea of completely describing the world in mathematical terms. The arguments offered in the earlier section of the paper refute the optimistic view, but this does not make the pessimistic view inevitable. The possibility of a mathematical description of reality in inferentially restrictive terms is what saves the day.

References

[1] Aristotle., (1988) *Metaphysics*, Clarendon.

[2] Barwise. J., (1975) 'Admissible Sets and Structures', Springer-Verlag, Berlin.

[3] Bergson, H., (1903) 'Introduction to Metaphysics', reprinted in H. Bergson, (1946), *The Creative Mind*, Citadel Press.

[4] Chang, C., and Kiesler, J., (1990) *Model Theory*, Elsevier Science.

[5] Dickman, M., (1975) 'Large Infinitary Languages', North-Holland, Amsterdam.

[6] Jech, T., (1997) *Set Theory*, Springer-Verlag.

[7] Kunen, K., (1980) *Set Theory*, Elsevier Science Publishing Co.

[8] Caspar, M., ed. (1937) *Johannes Kepler Gesammelte Werke*, Munich.

[9] Machamer, P., (ed.) (1998) *The Cambridge Companion to Galileo*, Cambridge.

[10] Priest, G., (2000) *'Could everything be True?'*, Australasian Journal of Philosophy, 78, 189-95.

[11] Rassias, G., (1991), *The mathematical heritage of C. F. Gauss*, Singapore.

[12] Rudin, W., (1964), *Principles of Mathematical Analysis*, Mc-Graw Hill.