Problem Set on Optimal Control

1. Solve the following problems, paying particular attention to the appropriate transversality conditions.

(a) Fixed end points

$$\max_u \int_0^1 -u(t)^2\,dt\,, \quad \text{subject to } \dot{y}(t) = y(t) + u(t)\,, \quad \text{y(0)=1, y(1)=0.}$$

(b) Constrained end point

$$\max_u \int_0^1 -u(t)^2\,dt\,, \quad \text{subject to } \dot{y}(t) = y(t) + u(t)\,, \text{ y(0)=1, } \quad y(1) \geq 2\,.$$

(c) T free, but $y(T)$ fixed

$$\max_u \int_0^T -1dt\,, \quad \text{subject to } \dot{y}(t) = y(t) + u(t)\,, \quad \text{y(0)=5, y(T)=11, } u \in [-1,1].$$

2. Characterize the solution to the following investment problem:

$$\max_c \int_o^T e^{-\rho t} U(c(t))dt$$

subject to $\dot{k}(t) = rk(t) - c(t)$, $k(0)=k_0>0$, $k(T) \geq 0$, and where $\lim_{c\to 0} u^{/}(c) = \infty$, and $u^{//}(c) < 0$.

3. Find the path of $x(t)$ that maximizes

$$V = \int_{t_0}^{\infty} e^{-\rho t} \ln x(t)dt\,,$$

subject to $\dot{m}(t) = \beta m(t) - x(t)$, $m(t_0) = m_0 > 0$, $\lim_{t\to\infty} m(t) = 0$. Assume that $\rho > \beta$. (Why must one make this assumption?)

4. (*Resource extraction*). There is a resource stock, $x(t)$ with $x(0)$ known. Let $p(t)$ be the price of the stock, let $c(x(t))$ be the cost of extraction when the remaining stock is $x(t)$, and let $q(t)$ be the extraction rate. A firm maximizes the present value of profits over an infinite horizon, with an interest rate of r.

(a) Assume that extraction costs are constant at c. Show that, for any given $p(0) > c$, the price of the resource rises over time at the rate r.

(b) Assume $c'(x(t)) < 0$. Derive an expression for the evolution of the price (it may include an integral term) and compare your answer with part (a).

(c) How do we produce a time path for price without considering demand? What does your solution tell you about the rate of extraction?

(d) Over the last 20 years, there has been no tendency for the price of oil to rise, despite continued rapid rates of extraction. How does your answer to part (b) shed light on the possible explanations for the failure of the price of oil to rise?

5. (*Resource extraction*). Let a renewable resource stock evolve according to

$$\dot{x}(t) = f(x(t)) - q(t) \, ,$$

where $f(x(t))$ is a growth function and $q(t)$ is the extraction rate. Given extraction costs $c(x(t))$ and a constant interest rate of r, solve the profit maximization problem for a representative competitive firm. Provide sufficient conditions for the existence of a unique steady state.

6. (*Resource extraction*). Continuing the renewable resource stock problem of Exercise 5, assume now that $f(x(t)) = x(t)(1 - x(t))$, and that the extraction rate is given by

$$q(t) = 2x(t)^{1/2} n(t)^{1/2} ,$$

where $n(t)$ is labor effort with a constant cost of w per unit. Assume further that this resource is fish that are sold at a fixed price p. The optimization problem is to choose $n(t)$ to maximize discounted profits.

(a) Derive the necessary conditions and interpret them verbally.

(b) Draw the phase diagram for this problem, with the resource stock and the shadow price of the stock on the axes.

(c) Show that if p/w is large enough, that the resource stock will be driven to zero, while if p/w is low enough, there is a steady state with a positive stock.

7. (*The Ramsey-Cass-Koopmans model*) Suppose family utility is given by

$$\int_0^\infty e^{-\rho t} U(c(t)) dt$$

where U is an increasing strictly concave function. The household budget constraint is

$$\dot{k}(t) = r(t)k(t) + w(t) - c(t) .$$

Households choose $c(t)$ to maximize discounted lifetime utility. Derive the steady state (done in class as an example). Draw the phase diagram, and analyze the stability of the system both graphically and algebraically.

8. (*The Ramsey-Cass-Koopmans model with a pollution externality*). Suppose family utility is given by

$$\int_0^\infty \big(U(c(t)) - V(y(t))\big)e^{-\rho t}dt$$

where U is an increasing strictly concave function and V is an increasing strictly convex function. The term $V(y(t))$ is the disutility of pollution associated with production.

(a) Decentralized economy. The household budget constraint is

$$\dot{k}(t) = r(t)k(t) + w(t) - c(t)\,.$$

Households choose $c(t)$ to maximize family utility, and ignore the effects of $k(t)$ on $y(t)$. Derive the steady state of the competitive economy.

(b) Command economy. The social planner's resource constraint is

$$\dot{k}(t) = f(k(t)) - c(t)\,,$$

and the planner maximizes the utility function subject to the constraint, taking into account the fact that $y(t) = f(k(t))$. Derive the steady state conditions and compare your answer with part (a).

9. (*The Ramsey-Cass-Koopmans model with leisure*). Consider a competitive economy populated by identical infinitely-lived individuals whose utility functions are given by

$$\int_0^\infty \big(U(c(t)) + V(T - h(t))\big)e^{-\rho t}dt\,,$$

where U and V are increasing strictly concave functions. $c(t)$ is consumption, T is the number of hours in the day, $h(t)$ is the number of hours spent working, so that T–$h(t)$ is the number of hours of leisure per day. The marginal utilities of consumption and leisure are positive and diminishing. An individual's budget constraint is given by

$$\dot{k}(t) = r(t)k(t) + w(t)h(t) + z(t) - c(t) - \tau w(t)h(t)\,,$$

where $k(t)$ is capital holdings, $r(t)$ is the rental rate on capital, $w(t)$ is the hourly wage, $z(t)$ is a transfer payment from the government, and t is the tax rate on labor income.

(a) Derive the optimality conditions for the individual.

The per capita net production function for this economy is $y(t) = f(k(t), h(t))$, where f is strictly concave in each input, homogeneous of degree one, and the derivatives satisfy $f_{hk} > 0$ and $f_{hh}f_{kk} - (f_{hk})^2 > 0$.

(b) Assume that markets are competitive and that the government maintains a balanced budget. Find and interpret the three conditions that determine steady-state per capita consumption, capital and hours as a function of τ.

(c) Find expressions for the effect of an increase in τ on the steady-state values of c, k, and h. Interpret these results.

10. (*R&D spillovers*). Let $\alpha_i = q_i \,/\, Q_i$ denote the quality $q_i(t)$ of a firm's product relative to the average quality of products $Q(t)$ in the industry. Average quality, $Q(t)$, grows at the constant exponential rate β. Assume the firm's profits (gross of research expenditures) are proportional to rela-

tive quality: $\pi_i(t) = \theta\alpha_i(t)$. A firm increases its quality $q_i(t)$ by spending on R&D. The equation of motion for firm i's quality is

$$\dot{q}_i(t) = \gamma q_i(t)^{1-\varepsilon} Q(t)^{\varepsilon} R_i(t)^b,$$

where ε<1 and *b*<1. The parameter ε measures the degree of spillovers between firms. When ε=1 the rise in firm *i*'s quality depends only on average quality, while when *ε*=0 it depends only on firm *i*'s quality.

(a) Write down the profit maximizing optimal control problem for a firm facing an infinite planning horizon and an interest rate *r*. (Hint, express the constraint in terms of relative quality).

(b) Characterize the evolution of relative quality and R&D.

(c) Derive expressions for the steady-state values of R&D and relative quality.

(d) What is the speed of convergence to the steady state in the symmetric equilibrium? (Hint: log-linearize and note that in the symmetric equilibrium $\beta = \gamma\alpha^{-\varepsilon}R^b$). What are the necessary conditions for stability?