

Problem Set on Differential Equations

1. Solve the following differential equations

(a) $\dot{x}(t) = e^{-t} - 2x(t)$, $x(0) = 3/4$.

(b) $\dot{x}(t) = te^{-2t} - 2x(t)$, $x(1) = 0$.

(c) $\dot{x}(t)e^t = t + (1 - x(t))e^t$, $\dot{x}(0) = 0$.

2. (*An asset market model*). Let $p(t)$ denote the price of equity, let $d(t)$ denote the dividend paid at time t , and let r denote the yield on a risk free bond.

(a) What equation of motion yields the following forward solution for the price of equity:

$$p(t) = \lim_{T \rightarrow \infty} p(T)e^{-r(T-t)} + \int_t^{\infty} d(s)e^{-r(s-t)} ds ?$$

(b) Explain the economic intuition behind this equation of motion. (c) What assumption about the forward solution implies that the price of equity is equal to the present value of current and future dividends? (d) What are the economic justifications for this assumption?

3. (*Solving general linear equations*). Consider equations of the form, $\dot{x}(t) + p(t)x(t) = g(t)$. (a) If $g(t)$ is identically zero, show that the solution is

$$x(t) = A \exp\left(-\int_0^t p(s) ds\right),$$

where A is a constant determined by boundary conditions. (b) If $g(t)$ is not identically zero, assume a solution of the form

$$x(t) = A(t) \exp\left(-\int_0^t p(s) ds\right),$$

where A is now a function of t . Show that $A(t)$ must satisfy the condition

$$\dot{A}(t) = g(t) \exp\left(\int_0^t p(s) ds\right).$$

(c) Find $A(t)$ from this expression and then use your answer to write an expression for $x(t)$. (d) Using the formulae just obtained, solve

$$\dot{x}(t) + \frac{x(t)}{\sqrt{t}} = 3.$$

4. (*Stability of nonlinear equations*). For each of the following differential equations, analyze the global stability of the steady state:

$$(a) \dot{x}(t) = b(x(t) - a)^2; \quad (b) \dot{x}(t) = -b(x(t) - a)^2;$$

$$(c) \dot{x}(t) = b(x(t) - a)^3; \quad (d) \dot{x}(t) = -b(x(t) - a)^3$$

5. (*Log-linearization*). Consider the nonlinear capital-stock equation

$$\dot{k}(t) = sf(k(t)) - \delta k(t).$$

A common analytical approach is log-linearize the equation. To do so, one substitutes $y(t) = \ln k(t)$, and then linearizes around the steady state value

of y . (a) Log-linearize this equation. (b) Interpret the result. Why might log-linearization be preferable to linearization?

6. (*Population growth*). a) Suppose $\dot{x} = x(\alpha + \beta x)$. Derive an explicit solution for x and show that it becomes infinite in finite time.

b) For the Gompertz growth equation,

$$\dot{x}(t) = \beta x(t)(\alpha - \ln x(t)),$$

(i) Solve the equation subject to $x(0)=x_0$.

(ii) Sketch the graph and its associated phase diagram. Derive the steady states and establish their stability or instability.

7. (*R&D driven growth*). A well-known empirical regularity in industrial economics is that firm R&D is more or less proportional to size. Making use of this regularity write down a simple model of R&D-driven growth in a firm's market share, s , that incorporates the following features: (i) market share is bounded between 0 and 1; (ii) if a firm does no R&D it will lose market share due to the R&D efforts of other firms; (iii) there are n firms; (iv) all firms have the same R&D ability. Solve (if possible) and characterize the solution of the model. What is (are) the steady state(s)?

8. (*Stability of a linear system*). Solve and assess the stability of the following differential equations:

(a) $\dot{x}(t) = x(t) + y(t)$ and $\dot{y}(t) = 4x(t) + y(t)$;

(b) $\dot{x}(t) = -3x(t) + \sqrt{2}y(t)$ and $\dot{y}(t) = \sqrt{2}x(t) + -2y(t)$;

$$(c) \quad \dot{x}(t) = -\frac{1}{2}x(t) + y(t) \quad \text{and} \quad \dot{y}(t) = -x(t) - \frac{1}{2}y(t).$$

9. (*Stability of a nonlinear system*). The following system has two steady states:

$$\dot{x}(t) = x(t)^2 + y(t)$$

$$\dot{y}(t) = x(t) - y(t) + 1$$

- a) Construct the phase diagram for this system to fully characterize the system's behavior.
- b) Find the roots of the linearized system and verify your graphical characterization of the local properties of the system.

10. (Solow model with human capital). Mankiw, Romer and Weil [“A Contribution to the Empirics of Economic Growth.” *Quarterly Journal of Economics*, 107(2):407-437] analyze the following version of the Solow model:

$$y(t) = k(t)^\alpha h(t)^\beta,$$

$$\dot{k}(t) = s_k y(t) - (n + g + \delta)k(t),$$

$$\dot{h}(t) = s_h y(t) - (n + g + \delta)h(t).$$

where y is output per effective unit of labor, h is human capital per effective unit of labor, k is physical capital per effective unit of labor, and s_h and s_k are the savings rates for physical and human capital. g is the rate of technical change, n of population growth and δ the depreciation rate.

- a) How do the parameters of the model affect the steady state income level, y^* ?
- b) Draw the phase diagram for this model and analyze the stability of the steady state(s).
- c) Mankiw, Romer and Weil point out that the Solow model makes quantitative predictions about the speed of convergence to the steady state. Specifically, a log linear approximation around the steady state yields

$$\frac{d \ln(y(t))}{dt} = (n + g + \delta)(1 - \alpha - \beta)(\ln(y^*) - \ln(y(t))).$$

Derive this expression formally, and interpret it.