Lecture Notes on Differential Equations

Peter Thompson

Carnegie Mellon University

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A differential equation is an equation of the form

$$\dot{x}(t) = \frac{dx(t)}{dt} = f(x,y,t)\,,$$

usually with an associated boundary condition, such as

$$x(0) = x_0\,.$$

The solution to the differential equation,

$$x(t) = g(y,t,x_0)\,,$$

contains no differential in x.

The techniques for solving such equations can a fill a year's course. In this part of the course, we study some basic types, with special emphasis on

- economic interpretation
- types of equations common in the study of economic growth
- characterization of arbitrary equations
- stability of systems

EXAMPLE 0.1 (*Capital accumulation by a firm*). The capital stock of a firm evolves according to the linear equation of motion

$$\dot{k}(t) = i(t) - \delta k(t)\,,$$

net
investment

gross investment

depreciation

where $i(t)$ is the rate of investment at time t and δ is the instantaneous rate of deprecia-
tion. To find the capital stock at any time t given an initial stock $k(0)=k_0$ requires that
we solve the differential equation.   •

EXAMPLE 0.2 (*Capital accumulation by a country*). Let GDP per capita be given by the
intensive-form production function

$$y(t) = f(k(t)),$$

and let investment satisfy the national income identity for a closed economy,

$$i(t) = sf(k(t)).$$

The equation for motion of capital is then

$$\dot{k}(t) = sf(k(t)) - \delta k(t),$$

a nonlinear differential equation.   •

EXAMPLE 0.3 (*Labor market matching*). Let L denote the number of workers in the labor
force, $u(t)$ the unemployment rate, and $v(t)$ the vacancy rate (expressed as a fraction of
L). Workers and vacancies are assumed to find each other by a random matching process
whereby the total number of matches made in an interval of time Δt is given by the
matching function

$$M\Delta t = M(uL, vL)\Delta t.$$

The matching function is increasing and concave in each of its arguments, and homoge-
neous of degree one. Let $\theta = v/u$. The rate at which vacancies are filled, expressed as a
fraction of the unemployed is

$$\begin{aligned}
\frac{M}{uL}\Delta t &= \frac{M(uL, vL)}{uL}\Delta t \\
&= \frac{vL \cdot M(\theta, 1)}{uL}\Delta t \\
&= \theta M\left(\theta^{-1}, 1\right)\Delta t \\
&= \theta m(\theta)\Delta t,
\end{aligned}$$

where $m(\theta) = M\left(\theta^{-1}, 1\right)$. The flow into unemployment occurs at the rate λ. In the interval Δt, the number of workers who become unemployed is therefore $\lambda(1-u)L\Delta t$. Thus, the change in the number of unemployed is

$$\Delta(uL) = \lambda((1-u)L\Delta t - \theta m(\theta)uL\Delta t .$$

Divide by $L\Delta t$ and let $\Delta t \to 0$, yielding

$$\dot{u}(t) = \lambda(1-u(t)) - \theta m(\theta)u(t) .$$

Solving this linear differential equation for the steady-state unemployment rate is easy. Set $\dot{u}(t) = 0$, to obtain

$$\overline{u} = \frac{\lambda}{\lambda + \theta m(\theta)} .$$

However, obtaining the value $u(t)$ given some $u(0)$ requires solving the differential equation. •

1. Linear Differential Equations

To begin, you will need to know how to solve a particular type of differential equation, known as a **constant coefficient first-order** equation. These take the form

$$\frac{dx(t)}{dt} = \dot{x}(t) = a(t) + bx(t) .$$

FIRST-ORDER

$a(t)$ is an additive
component of growth

CONSTANT COEFFICIENT.
b is the exponential growth rate

All differential equations have two types of solutions, **forward solutions** and **backward solutions**.

Backward Solutions

If we know a past value of $x(t)$, say x_0, and the past values of a, $a(s)$, $s \in [0,t]$, we can use them to find the current value of $x(t)$:

$$x(t) = x_0 e^{bt} + \int_o^t a(s) e^{b(t-s)} ds \ .$$

Current
value

sum of components of $x(t)$, all adjusted
for exponential growth

We can verify this is the solution by differentiating with respect to t:

$$\dot{x}(t) = bx_0 e^{bt} + a(t) + \int_0^t a(s) b e^{b(t-s)} ds$$

These two terms come from
applying **Leibnitz' Rule** of differ-
entiation.

$$= b\left[x_o(t) e^{bt} + \int_0^t a(s) e^{b(t-s)} ds \right] + a(t)$$

$$= bx(t) + a(t) \ .$$

There is an intuitive interpretation to the backward solution. The current value of x can be decomposed into the sum of the contribution of the initial value, which is x_0 compounded at the rate b, and all the individual increments $a(s)$, each of which is also compounded at the rate b for the interval $t-s$.

EXERCISE 1.1 *Verify that (1.4) is a solution to (1.1).*

EXAMPLE 1.1 (*Capital accumulation by the firm*). Recall from Example 0.1 the equation of motion for the capital stock of a firm:

$$\dot{k}(t) = i(t) - \delta k(t) \ .$$

The backward solution is

$$k(t) = k(0) e^{-\delta t} + \int_0^t i(s) e^{-\delta(t-s)} ds \ ,$$

which states that the current capital stock equals the initial capital stock plus the entire time path of investment, both adjusted for depreciation. The initial capital stock has depreciated at the rate δ for the interval of time t, while each addition to the capital stock from investment at time s has depreciated for the amount of time $t{-}s$. •

EXAMPLE 1.2 (*Labor market matching*). From Example 0.3, we have

$$\dot{u}(t) = \lambda(1 - u(t)) - \theta m(\theta) u(t)$$
$$= \lambda - (\lambda + \theta m(\theta)) u(t), \qquad u(0){=}u_0.$$

If $m(\theta)$ were a constant, this would be a straightforward linear differential equation. But $\theta{=}v(t)/u(t)$. However, when we study matching models later in the course, we will find that θ is solved by a firm optimization problem, and the solution is a constant that does not depend on $u(t)$! Hence, anticipating this result, let $\theta m(\theta)$ be treated as a constant here. Then, the backward solution is

$$u(t) = u_0 e^{-(\lambda+\theta m(\theta))t} + \int_0^t \lambda e^{-(\lambda+\theta m(\theta))(t-s)} ds$$

$$= u_0 e^{-(\lambda+\theta m(\theta))t} + \frac{\lambda e^{-(\lambda+\theta m(\theta))t}}{\lambda + \theta m(\theta)}\bigg|_0^t$$

$$= u_0 e^{-(\lambda+\theta m(\theta))t} + \frac{\lambda}{\lambda + \theta m(\theta)}\left(1 - e^{-(\lambda+\theta m(\theta))t}\right),$$

which is a weighted average of the initial unemployment rate and the steady-state unemployment rate derived in Example 0.3. Note that $u(t) \to \lambda/(\lambda + \theta m(\theta)) = \bar{u}$ as $t \to \infty$. •
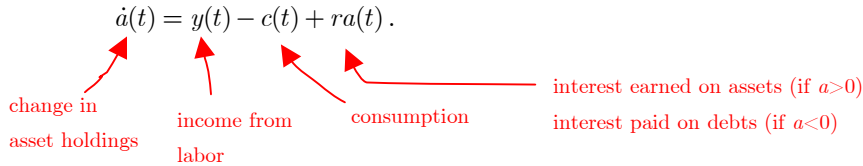
Forward Solutions

If we know a future value of $x(t)$, say $x(T)$, for some $T{>}t$, and the future values of a, $a(s)$, $s \in [t,T]$, we can use them to find the current value $x(t)$:

$$x(t) = x(T)e^{-b(T-t)} - \int_t^T a(s)e^{-b(s-t)} ds.$$

By differentiating, you can verify that this is a solution.

EXAMPLE 1.3 (*Dynamic Budget Constraint*). An infinitely-lived family has a point in time budget constraint

$$\dot{a}(t) = y(t) - c(t) + ra(t) .$$

change in asset holdings

income from labor

consumption

interest earned on assets (if $a>0$)
interest paid on debts (if $a<0$)

Given initial asset holdings $a(0)$, the backward solution is

$$a(t) = a(0)e^{rt} + \int_0^t \big(y(s) - c(s)\big) e^{r(t-s)} ds .$$

The forward solution is of particular economic interest. Consider a family thinking about how much it can optimally consume over its infinite lifetime. The backward solution really gives no information about this. If more consumption is preferred to less, then the family will just consume more than it earns and increase its debt without bound. Logic would tell us that we must put a bound on how much the family can go into debt. The forward solution allows us to do so. Given a time horizon T, the forward solution is

$$a(t) = a(T)e^{-r(T-t)} - \int_t^T \big(y(s) - c(s)\big) e^{-r(s-t)} ds ,$$

which states that assets at time t are equal to the terminal assets discounted back to t, minus any savings between t and T that contributed to the size of the terminal assets.

Now, the family is infinitely-lived, so let $T \to \infty$:

$$a(t) = \lim_{T \to \infty} a(T)e^{-r(T-t)} - \int_t^\infty \big(y(s) - c(s)\big) e^{-r(s-t)} ds \tag{1.1}$$

It is straightforward to see that the optimal and feasible strategy must have $\lim_{t\to\infty} a(t) = 0$. If $\lim_{t\to\infty} a(t) < 0$, the family has incurred debt it never intends to pay, and no one would be willing to finance this borrowing. Hence $\lim_{t\to\infty} a(t) < 0$ is not feasible. The practice is called a Ponzi scheme, named for a fraudulent financier. On the other hand, if $\lim_{t\to\infty} a(t) > 0$, the family continues to hold assets it never intends to use.

As these assets could be used to increase consumption, $\lim_{t\to\infty} a(t) > 0$ cannot be optimal. Thus, an economic solution, as opposed to simply a mathematical solution to the budget constraint problem, allows us to impose *a priori* the condition $\lim_{t\to\infty} a(t) = 0$. But this condition implies

$$\lim_{T\to\infty} a(T)e^{-r(T-t)} = 0 \,. \tag{1.2}$$

Substituting (1.1) into (1.2), we have

$$\int_t^\infty c(s)e^{-r(s-t)}ds = a(t) + \int_t^\infty y(s)e^{-r(s-t)}ds \,,$$

which has the nice interpretation that the discounted present value of the family's lifetime consumption is equal to the sum of its initial wealth and the discounted present value of its lifetime labor income. This is the solution to the family's budget constraint when it is behaving optimally ($\lim_{t\to\infty} a(t) \leq 0$) and feasibly ($\lim_{t\to\infty} a(t) \geq 0$). •

EXERCISE 1.2. *Solve the following differential equations*

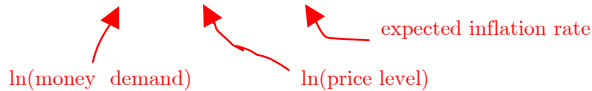(a) $\dot{x}(t) = e^{-t} - 2x(t)$, $\quad x(0) = 3/4$.

(b) $\dot{x}(t) = te^{-2t} - 2x(t)$, $\quad x(1) = 0$.

(c) $\dot{x}(t)e^t = t + (1 - x(t))e^t$, $\quad \dot{x}(0) = 0$.

If $x(t)$ is a continuous function of time (i.e. it does not have any jumps), the backward and forward solutions are merely alternative ways of representing the same solution (although one may allow us to define constraints more readily, as in Example 1.2). The choice between them depends only on the information you have – the future or past values of $a(s)$ and a future or initial value for x. However, if $x(t)$ can make a discrete jump at any given point in time, these expressions will not be equal, and we must use the logic of the model to which the equations apply to decide between the forward and backward solutions. In economic and financial problems, the variable of interest frequently is able to jump at a point in time, and so the distinction between forward and backward solu-

tions is often an important matter. We will illustrate this idea with a classic model about hyperinflation.

EXAMPLE 1.4 (*Hyperinflation*). Cagan (1956) developed a model of hyperinflation described by the following equations:

$$m(t) - p(t) = -\alpha\pi(t)\,, \tag{1.3}$$

ln(money demand)   ln(price level)   expected inflation rate

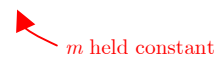$$\dot{\pi}(t) = \gamma\left(\dot{p}(t) - \pi(t)\right). \tag{1.4}$$

Equation (1.3) states the demand for real money balances depends negatively on the expected inflation rate; α is the semi-elasticity of real money demand with respect to inflation expectations. Equation (1.4) was the first statement of the theory of adaptive expectations. The change in expected inflation is proportional to the current mistake made in expectations. The parameter γ is the speed of adjustment of expectations.

Assume $m(t)$ is constant except possibly for a one-time jump. Then, differentiating (1.3) gives $\dot{p}(t) = \alpha\dot{\pi}(t)$. Using this in (1.4) and substituting into (1.3), we get

$$\dot{p}(t) = \frac{\gamma}{1-\alpha\gamma}\left(m(t) - p(t)\right). \tag{1.5}$$

Suppose now there is a once and for all rise in m, beginning from a position where $m(t){=}p(t)$ and $\dot{p}(t) = 0$. The quantity theory of money leads one to expect that a positive jump in m would induce $\dot{p}(t) > 0$. But this will only be true in (1.5) if $\alpha\gamma{<}1$. This is Cagan's famous condition for monetary stability. The stability condition states that a sensitive money demand is consistent with monetary stability only if expectations adapt sufficiently slowly.

Adaptive expectations are backward looking, so one way to think about $p(t)$ is that it can be obtained from the backward solution to the differential equation.

$$\dot{p}(t) = -\frac{\gamma}{1-\alpha\gamma}\,p(t) + \frac{\gamma}{1-\alpha\gamma}\,\bar{m}$$

m held constant

The backward solution to this equation is

$$p(t) = p(0)e^{-\gamma t/(1-\alpha\gamma)} + \frac{\gamma\bar{m}}{1-\alpha\gamma}\int_0^t e^{-\gamma(t-s)/(1-\alpha\gamma)}ds$$

$$= \bar{m} + \big(p(0) - \bar{m}\big)e^{-\gamma t/(1-\alpha\gamma)}. \tag{1.6}$$

Now, for arbitrary $p(0)$, we require that the last term vanish as $t \to \infty$ (the alternative is that it explodes). But this requires that $-\gamma/(1-\alpha\gamma)<0$, or $\alpha\gamma<1$.

The stability condition is clearly violated by rational expectations. With no uncertainty, rational expectations implies that expected inflation must equal actual inflation, or $\pi(t) = \dot{p}(t)$. That is, absent uncertainty rational expectations is equivalent to perfect foresight. Given Cagan's equation (1.6), rational expectations implies $\gamma \to \infty$ (expectations are not differentiable - $\dot{\pi}(t) = 0$ at all times; $\pi(t)$ jumps whenever $\dot{p}(t)$ jumps). Now, letting $\gamma \to \infty$ in (1.12), we get

$$p(t) = \bar{m} + \big(p(0) - \bar{m}\big)e^{t/\alpha},$$

which explodes whenever $p(0) \neq \bar{m}$.

The solution to this problem was provided by Sargent and Wallace (1973). One key implication of the backward solution is that $p(t)$ is continuous. But why shouldn't it jump? In fact, if agents are forward looking, as in rational expectations, shouldn't the solution also be forward looking? The forward solution to our problem (with $\gamma \to \infty$) is

$$p(t) = \lim_{T\to\infty} p(T)e^{-(T-t)/\alpha} + \frac{1}{\alpha}\int_t^\infty m(s)e^{-(s-t)\alpha}ds.$$

now we allow m to vary over time.

We need a terminal condition to pin down $p(t)$. Sargent and Wallace proposed a terminal condition that price remain bounded as $t \to \infty$. Why? First, because we don't observe exploding prices all the time, and second because this is the sort of condition required by an underlying individual optimization problem (consider the conditions imposed on assets in Example 1.2). Now, if $p(T)$ remains bounded, then it must be the case that $\lim_{T\to\infty} p(T)e^{-(T-t)/\alpha} = 0$. Hence, the solution to the rational expectations model is

$$p(t) = \frac{1}{\alpha}\int_t^\infty m(s)e^{-(s-t)/\alpha}ds.$$

and if $m(t) = \bar{m}$ for all t, we have $p(t) = \bar{m}$. Rational expectations requires that the price today jump to accommodate whatever people think the path of m will be. Such jumps are not possible in a backward solution, because the backward solution assumes that, once $p(0)$ is given, its subsequent path is always differentiable (and you can't be differentiable if you're not continuous!). The lesson here is that different economic assumptions (i.e. adaptive versus rational expectations) impose very different **boundary conditions** on the differential equation, and this can have important consequences for the way a model behaves.   ●

> EXERCISE 1.3 (An asset market model). *Let $p(t)$ denote the price of equity, let $d(t)$ denote the dividend paid at time t, and let r denote the yield on a risk free bond. (a) What equation of motion yields the following forward solution for the price of equity:*
>
> $$p(t) = \lim_{T \to \infty} p(T)e^{-r(T-t)} + \int_t^{\infty} d(s)e^{-r(s-t)}ds \ ?$$
>
> (b) *Explain the economic intuition behind this equation of motion.* (c) *What assumption about the forward solution implies that the price of equity is equal to the present value of current and future dividends?* (d) *What are the economic justifications for this assumption?*

General Linear Differential Equations

So far we have discussed differential equations of the form

$$\dot{x}(t) = a(t) + bx(t).$$

We would also like to be able to solve more complicated equations of the form

$$\dot{x}(t) + p(t)x(t) = g(t).$$

The following exercise will lead you to discover the technique for solving this **variable coefficient** problem.

EXERCISE 1.4 (Solving general linear equations). *Consider equations of the form,*
$\dot{x}(t) + p(t)x(t) = g(t)$. (a) *If $g(t)$ is identically zero, show that the solution is*

$$x(t) = A \exp\left(-\int_0^t p(s)ds\right),$$

where A is a constant determined by boundary conditions. (b) *If $g(t)$ is not identically zero, assume a solution of the form*

$$x(t) = A(t) \exp\left(-\int_0^t p(s)ds\right),$$

where A is now a function of t. Show that $A(t)$ must satisfy the condition

$$\dot{A}(t) = g(t) \exp\left(\int_0^t p(s)ds\right).$$

(c) *Find $A(t)$ from this expression and then use your answer to write an expression for $x(t)$.* (d) *Using the formulae just obtained, solve*

$$\dot{x}(t) + \frac{x(t)}{\sqrt{t}} = 3.$$

2. Nonlinear Differential Equations

A single linear differential equation can always be solved (we may not be able to evaluate the integral, but then the solution stated with the integral expression is the solution). However, most economic problems are nonlinear:

$$\dot{x}(t) = a(x,t) - f(x,t).$$

Special types of nonlinear equations have known explicit solutions. However, most economic problems that exploit these special cases tend to be quite contrived. For this reason, and also to save time, we will look at only a couple of examples, a little later in this section.

There are no generally applicable techniques for solving nonlinear differential equations, and many of them cannot be solved. Analysis of nonlinear problems in economics is greatly facilitated by the following assumptions:

- The problem is **autonomous**, which is to say that $a(\cdot)$ and $f(\cdot)$ do not depend on time except through the value of $x(t)$. That is, the problem has the form

$$\dot{x}(t) = a(x(t)) - f(x(t)).$$

- The problem is such that $x(t) \to \overline{x}$, a constant value. That is, the problem approaches a **steady state**.

With these assumptions, a graphical analysis is straightforward in many cases.

EXAMPLE 2.1 (*Solow growth model*). Output, y depends on the capital stock, k, according to the production function $y=f(k)$. Given a constant investment rate, s, and a depreciation rate, δ, the evolution of capital is given by

$$\dot{k}(t) = sf(k(t)) - \delta k(t) \tag{2.1}$$

We make the usual neoclassical assumptions about the production function: $f'(k(t)) > 0$, $f''(k(t)) < 0$, $f'(0) >> \delta$, and $\lim_{k\to\infty} f'(k) = 0$. Then this problem has a unique steady state that is easily graphed by splitting the right hand side of (2.1) into two terms that are each plotted separately.
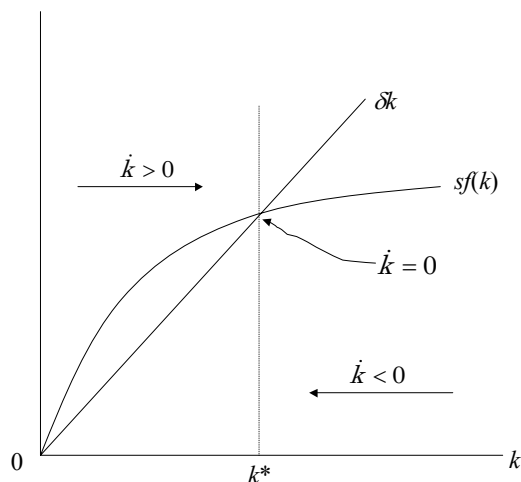

FIGURE 2.1. From 2.1, if $sf(k)<\delta k$, then k will be rising; if $sf(k)<\delta k$, then k will be falling. Define k^* as the point at which k is constant, which is clearly where $sf(k)=\delta k$. The point k^* defines the steady state.

Note also that any initial value of k other than k^* is always associated with a movement toward k^*. That is, the steady state is also **globally stable**.   •

EXAMPLE 2.2 (*Multiple equilibria in the Solow model*).

For most applications, of course, a purely graphical analysis is insufficient. We can usually obtain a useful analytical approximation to the differential equations by **linearizing** around the steady-state value. From a first-order **Taylor expansion**, we have:

=0 in the steady state.

$$\dot{k}(t) \approx sf(k^*) - \delta k^* + \left[sf'(k^*) - \delta \right]\left(k(t) - k^* \right)$$

$$= \left[sf'(k^*) - \delta \right]\left(k(t) - k^* \right)$$

Implies k is increasing when $k(t) < k^*$ and decreasing when $k(t) > k^*$
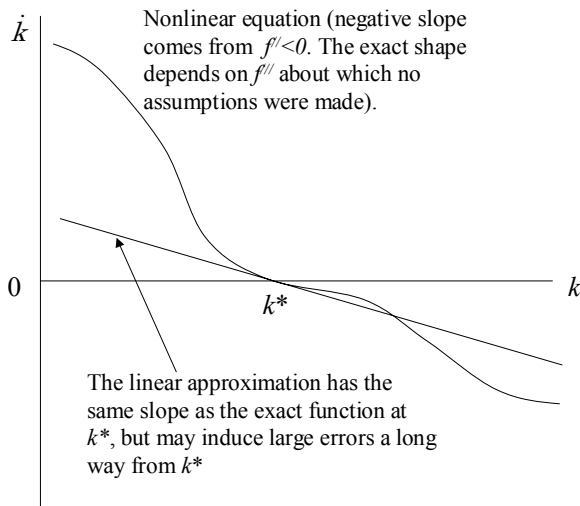
A negative constant: the slope of the curve in Figure 1, $sf'(k^*)$, must be less than the slope of the straight line, δ, at the steady state.

Now let $sf'(k^*) - \delta = b$. Then, the linear approximation to the differential equation is

$$\dot{k}(t) = -bk^* + bk(t). \tag{2.2}$$

This is a constant coefficient linear differential equation, which we can solve in our sleep. The obvious question, however, is how useful this equation will be. That depends upon the researcher's purpose. The nonlinear equation (2.1) and its linear approximation (2,2) give very similar answers about how k behaves for values of k close to k^*, but this similarity weakens the further we move away from k^*. Figure 2.2 illustrates by plotting $\dot{k}(t)$ as a function of $k(t)$ for the nonlinear model and its linear approximation.

Nonlinear equation (negative slope comes from $f'' < 0$. The exact shape depends on f''' about which no assumptions were made).

The linear approximation has the same slope as the exact function at k^*, but may induce large errors a long way from k^*

FIGURE 2.2. Exact and linear approximations to $\dot{k}(t)$.

Because the error induced by linear approximation becomes significant as we move away from the steady state, any results based on the linear approximation can only be said to hold in a small neighborhood around the steady state. Recall our definition of global stability:

- **Global stability** of a unique steady state implies that the system converges onto the steady state from any initial values.

The best we can say when we are limited to linear approximations is that a system is **locally stable**:

- **Local stability** of a unique steady state applies for small perturbations to the steady state. If we are close to the steady state values, the locally stable system will converge onto the steady state. We do not know what happens for initial values that are not in the neighborhood of the steady state.

Clearly, local stability is necessary for global stability, but it is not sufficient. In contrast, global stability is sufficient for local stability.

EXERCISE 2.1 (Stability of nonlinear equations). *For each of the following differential equations, analyze the global stability of the steady state*:

(a) $\dot{x}(t) = b\left(x(t) - a\right)^2$;   (b) $\dot{x}(t) = -b\left(x(t) - a\right)^2$;

(c) $\dot{x}(t) = b\left(x(t) - a\right)^3$;   (d) $\dot{x}(t) = -b\left(x(t) - a\right)^3$

EXERCISE 2.2 (Log-linearization). *Consider the nonlinear capital-stock equation*

$$\dot{k}(t) = sf(k(t)) - \delta k(t)\,.$$

A common analytical approach is to log-linearize the equation. To do so, one substitutes $y(t) = \ln k(t)$, and then linearizes around the steady state value of y. (a) Log-linearize this equation. (b) Interpret the result. Why might log-linearization be preferable to linearization?

Nonlinear Equations with Exact Solutions

One has to be lucky, but sometimes an economic model gives rise to a nonlinear differential equation that has an exact solution. We provide two examples here.

In some special cases, a nonlinear equation can be transformed into a linear differential equation that can always be solved a substitution of variables:

EXAMPLE 2.3 (*A closed-form solution to the Solow Model*).[1] Consider the following version of the Solow model:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} , \qquad \alpha \in (0,1)$$

$$\dot{K}_t = sY_t - \delta K_t , \qquad K_0 > 0 , \quad s \in (0,1)$$

$$L_t = L_0 e^{nt} , \qquad L_0 > 0$$

$$A_t = A_0 e^{gt} , \qquad A_0 > 0$$

where n, g, and δ are non-negative parameters. This model present two difficulties. The first is that A_t and L_t are growing over time. Standard practice is to normalize variables by dividing by effective labor, $A_t L_t$, and then denoting the normalized variables with lower case letters: $y_t = Y_t / A_t L_t$ and $k_t = K_t / A_t L_t$. With this normalization,

$$y_t = k_t^{\alpha} ,$$

$$\dot{k}_t = sk_t^{\alpha} - (n + g + \delta)k_t ,$$

with initial condition $k_0 = K_0 / A_0 L_0$.

The second difficulty is that the equation of motion for normalized capital just derived is nonlinear. Previously, we solved for the steady state in the general case $y_t = f(k_t)$. But we can do better with the particular functional form $y_t = k_t^{\alpha}$. In this case, the equation of motion is known as a Bernoulli equation and can be solved explicitly by a change of variables. Let $z_t = k_t^{1-\alpha}$. Then,

$$\dot{z}_t = (1 - \alpha)k_t^{-\alpha}\dot{k}_t ,$$

which can be used to obtain a differential equation in z:

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[1] This section is taken from Jones (2000).

$$\dot{z}_t = (1 - \alpha)s - \lambda z_t \,,$$

where $\lambda = (1 - \alpha)(n + g + \delta)$. This is a simple linear differential equation, which is readily solved. Moreover, the equation has a nice interpretation. As $z_t = k_t^{1-\alpha} = k_t / k_t^{\alpha} = k_t / y_t$, the equation represents the evolution of the capital-output ratio. The backward solution is

$$z_t = z_0 e^{-\lambda t} + \frac{s}{n + g + \delta}\left(1 - e^{-\lambda t}\right),$$

yielding

$$k_t = \left[k_0^{1-\alpha}e^{-\lambda t} + \frac{s}{n + g + \delta}\left(1 - e^{-\lambda t}\right)\right]^{\frac{1}{1-\alpha}}$$

and

$$y_t = \left[y_0^{(1-\alpha)/\alpha}e^{-\lambda t} + \frac{s}{n + g + \delta}\left(1 - e^{-\lambda t}\right)\right]^{\frac{\alpha}{1-\alpha}}.$$

Income per effective worker is a weighted average of the initial and steady-state values. The parameter λ governs the rate at which the economy converges onto its steady state, and this will be of direct interest in a later application.   •

A second class of nonlinear models are *separable equations* that take the form:

$$\dot{x}(t) = \frac{g(t)}{f(x)}\,.$$

That is, they can be written in terms of two distinct functions, one involving the endogenous variable, the other involving time. These equations are interesting because they can often be solved by writing one side of the equation in terms of x and the other in terms of t. Thus writing the equation in the form

$$f(x)\frac{dx}{dt} = g(t)\,,$$

we can integrate both sides with respect to t

$$\int f(x)\frac{dx}{dt}\,dt = \int g(t)dt\,,$$

or

$$\int f(x)dx = \int g(t)dt \ .$$

Carrying out the integrations allows us to solve for x.

EXAMPLE 2.4 (*Logistic growth*). The logistic growth equation takes the form

$$\dot{x}(t) = \beta x(t)(\alpha - x(t)) \ .$$

Separating the differential equation,

$$\frac{\dot{x}}{x(\alpha - x)} = \beta \ ,$$

and writing the integral yields

$$\int \frac{1}{x(\alpha - x)} dx = \int \beta \, dt \ .$$

Evaluating the integrals, including a constant of integration, k, yields

$$\frac{1}{\alpha} \ln \left| \frac{x}{\alpha - x} \right| = \beta t + k \ ,$$

which can be rearranged to yield a solution for x. Boundary conditions allow one to solve for the constant of integration.   •

EXERCISE 2.3 (Population growth). a) *Suppose $\dot{x} = x(\alpha + \beta x)$. Derive an explicit solution for x and show that it becomes infinite in finite time.*

b) *For the Gompertz growth equation,*

$$\dot{x}(t) = \beta \, x(t)(\alpha - \ln x(t)) \, ,$$

(i) *Solve the equation subject to x(0)=x_0.*

(ii) *Sketch the graph and its associated phase diagram. Derive the steady states and establish their stability or instability.*

EXERCISE 2.4 (R&D-driven growth). *A well-known empirical regularity in industrial economics is that firm R&D is more or less proportional to size. Mak-*

ing use of this regularity write down a simple model of R&D-driven growth in a firm's market share, s, that incorporates the following features: (i) market share is bounded between 0 and 1; (ii) if a firm does no R&D it will lose market share due to the R&D efforts of other firms; (iii) there are n firms; (iv) all firms have the same R&D ability. Solve (if possible) and characterize the solution of the model. What is (are) the steady state(s)?

3. Systems of Differential Equations

In many problems relevant to the study of economic dynamics, we will be concerned with solving two or more differential equations simultaneously. Generally, these systems will also be nonlinear, and we will want to characterize their behavior using graphical techniques. However, just as for single nonlinear equations, local approximations around the steady state can be obtained by linearizing the system. It will therefore be easier to understand what is going on if we first consider the behavior of linear systems.

We will limit ourselves to systems of two equations, which allow for graphical analysis. Systems of three equations can often be reduced to a system of two equations using a technique known as the time-elimination method (Mulligan and Sala-i-Martin [1992]). Systems of more than three equations are, fortunately, almost unheard of in our areas of study, and would usually need to be tackled with numerical methods.

Linear Systems of Two Equations

Consider the following system:

$$\dot{x}(t) = 0.06x(t) - y(t) + 1.4 \,, \tag{3.1}$$

$$\dot{y}(t) = -0.004x(t) + 0.04 \,, \tag{3.2}$$

with boundary conditions $x(0){=}1$ and $\lim_{t\to\infty}(e^{-0.06t}x(t)) = 0$.

To analyze this system, we construct a **phase diagram**, a graphical tool which allows us to visualize the dynamics of the system. The first step in constructing the phase diagram is to use the two equations to construct two curves plotting out **stationary values**

for $x(t)$ and $y(t)$. These curves plot all possible pairs of x and y such that $\dot{x} = 0$ and all possible pairs of x and y such that $\dot{y} = 0$.

Setting $\dot{x}(t) = 0$ in (3.1), yields

$$0 = 0.06x(t) - y(t) + 1.4 \quad ,$$

which gives

$$y(t) = 0.06x(t) + 1.4 \qquad \qquad \text{(3.3)}$$

All pairs of $\{x,y\}$ yielding a constant value of x.

Setting $\dot{y}(t) = 0$ in (3.2) gives

$$x(t) = 10 . \qquad \qquad \text{(3.4)}$$

Only one value of x is consistent with an unchanging y.
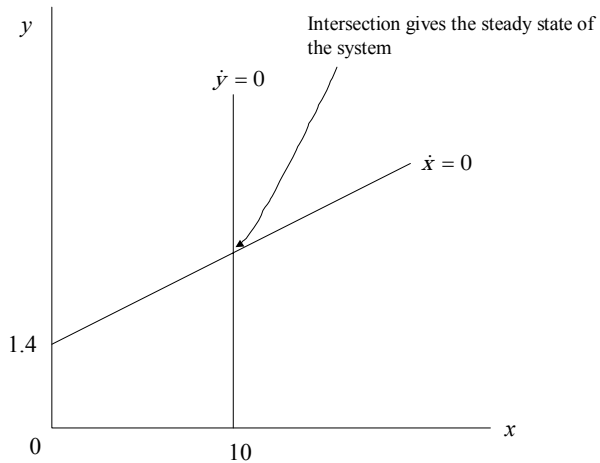
We can then plot these curves:

FIGURE 3.1. Constructing the phase diagram, step 1.

The next step is to think about the behavior of the system when it starts out at some point $\{x(0),y(0)\}$ which is *not* at the steady state. Consider first the $\dot{x} = 0$ locus. If y is greater than the value indicated by the $\dot{x} = 0$ line for any given value of x (i.e given a value of x , y lies above the line), then equation (3.1) tells us that $\dot{x} < 0$ (larger values of y reduce $\dot{x}$). For any point below the $\dot{x} = 0$ locus, in contrast, x must be rising. These movements in $x(t)$ are captured in Figure 3.2 by the horizontal arrows. They point to the left above the $\dot{x} = 0$ locus, and to the right below it. Now consider the $\dot{y} = 0$ locus. At any point on the graph to the right of the locus (i.e. for any $x{>}10$), we have $\dot{y} < 0$; to the left, $\dot{y} > 0$. We can also plot these directions of movement on our emerging phase diagram:
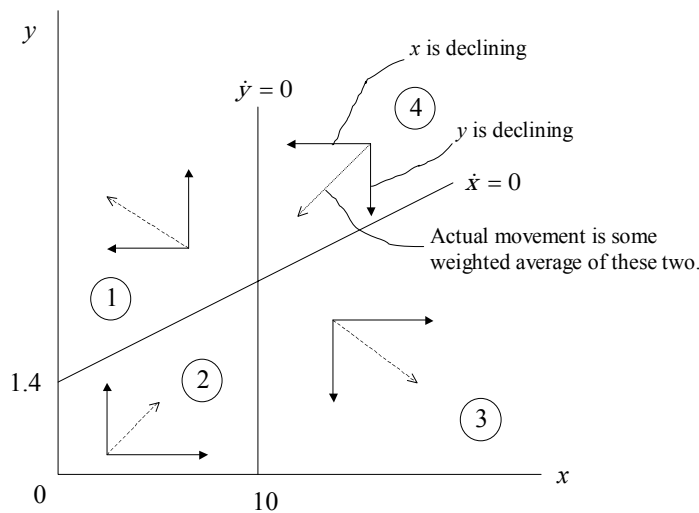

y

x is declining

$\dot{y} = 0$

4

y is declining

$\dot{x} = 0$

Actual movement is some
weighted average of these two.

1

1.4

2

3

x

0

10

We are now in a position to assess the stability of the system. To do this, we ask two questions. First, if the arrows suggest a clockwise or counterclockwise movement in all four quadrants, the system exhibits oscillations. These may be stable oscillations, wherein the steady state is eventually reached as the value of the system spirals in toward it; they may be unstable oscillations where the variables spiral out away from the steady state; or the system may be a stable oscillatory system, wherein the variables cir-

cle around the steady state forever at a distance that depends on initial conditions. To explore further requires analytical techniques which we touch on briefly later. In most economic problems, however, oscillatory behavior is infrequent.

When the arrows do not suggest a clockwise or counterclockwise movement in all four quadrants, we ask the following question: for how many of the four regions indicated do the arrows allow the system to move toward the steady state? If the answer is **none**, the steady state is **unstable** – the steady state cannot be reached from any initial values. One must be careful here, because (a) it is easy to overlook that $\{x=0, y=0\}$ might be a steady state for variables that cannot take on negative values, and (b), the system may exhibit oscillating patterns). If the answer is **four**, the steady state is **globally stable** – it can be reached from any starting values.

Finally, if the answer is **two**, the system is said to be **saddle path stable** (you will not see systems where the answer is one or three). The term saddle path stability comes from an analogy with a marble left on the top of a saddle. There is one point on the saddle where, if placed there, the marble does not move. This point corresponds to the steady state. There is a trajectory on the saddle, running down the highest points from front to back, with the property that if the marble is placed at any point on that trajectory, then it rolls toward the steady state. But if the marble is placed at any other point, it will fall to the ground.

All saddle path stable steady states have two trajectories passing through the steady state. One, the **stable arm** (or the **saddle path**; nerds also like to call it the **stable manifold**), has direction of motion toward the steady state; the other, the **unstable arm**, has movement away from it. Figure 3.3 plots these along with several trajectories for initial values that do not originate on the saddle path. None of these reach the steady state, but rather all go off into space somewhere (again, there may be constraints on the trajectories if the variables cannot become negative). Different trajectories never cross paths.

Saddle path stability is a common feature of many economic models – especially models of economic growth. Note that we cannot reach the steady state unless the starting values lie on the stable arm. If we do not start on the stable arm, either x or y go off to (plus or minus) infinity. This is unlikely to be a good representation of most economies

or industries. So there must be an ability to jump to the stable arm. In most problems, this is feasible because one of the initial values (either x or y) can be freely chosen by, say, the firm or household being modeled. For example, $x(t)$ may be a household's assets, while $y(t)$ may be its consumption at time t. Then, consumption can be freely chosen for any value of $x(0)$. (Note that the boundary conditions in this example include an initial value for $x(0)$ but not for $y(0)$).
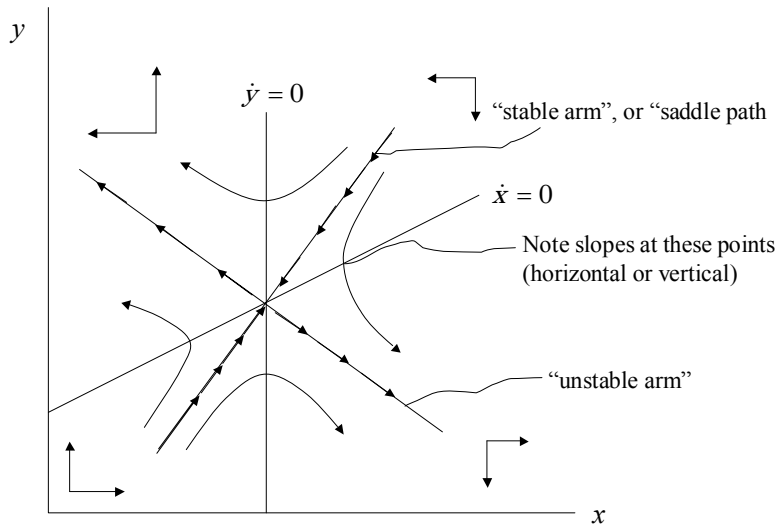

FIGURE 3.3. Constructing the phase diagram, step 3.

The key question is: why would consumption (or whatever variable) be chosen so as to make the model jump to the stable arm? The answer is that any other choice leads us away from the steady state and off toward infinite or zero values, and it turns out that any of these paths violates some important economic constraint in the model. Recall our economic analysis of the forward solution to the household's dynamic budget constraint in Example 1.2. There we required that $\lim_{t\to\infty} a(t) = 0$, and the interpretation was that this was optimal and feasible behavior. In the present example, we have a condition $\lim_{t\to\infty} e^{-0.06t} x(t) = 0$. This type of limiting condition is known as a **transversality condition** (consider the case where the planning horizon is finite at T; the transversality condition tells us how the variables must behave as they cross T – hence the name). In most economic models, the only trajectory in a saddle-path stable system that satisfies a well-

defined transversality condition is the one that follows the stable arm, or saddle path. Violation of the transversality condition invariably involves some sub-optimal or infeasible behavior.

At this point, we will have to limit ourselves with this partial discussion of transversality conditions. In the module on optimal control, where we learn how to analyze explicit optimization problems, we will discuss the various types of transversality conditions that exist, how to select among them for different optimization problems, and how to incorporate them into the solution.

Analytical Tests for Stability

Naturally, there is an analytical way to test for stability. We shall study this now and, as before, we limit our attention to a system of two equations. With more equations the principles are the same but the details are more complex.

Consider the following pair of equations:

$$\dot{x}(t) = a_1 x(t) + b_1 y(t) + p(t) \,,$$

$$\dot{y}(t) = a_2 x(t) + b_2 y(t) + g(t) \,,$$

where a_1, a_2, b_1, and b_2 are constants. Our analysis of the single equation system suggests the forms of the solutions. However, it turns out that the terms $p(t)$ and $g(t)$ have no bearing on the analysis of stability, so let us set them to zero,

$$\dot{x}(t) = a_1 x(t) + b_1 y(t) \,,$$

$$\dot{y}(t) = a_2 x(t) + b_2 y(t) \,.$$

The system that has dropped the terms $p(t)$ and $g(t)$ is known as the homogeneous version of the system. All systems of homogeneous linear equations have solutions that take the form

$$x(t) = A e^{rt} \,,$$

$$y(t) = B e^{rt} \,,$$

for some values of A, B, and r. The task is to find these values.

Substituting the proposed solutions into the differential equations gives

$$rAe^{rt} = a_1 A e^{rt} + b_1 B e^{rt}$$

$$rBe^{rt} = a_2 A e^{rt} + b_2 B e^{rt} .$$

There is a standard methodology for solving a such a pair of equations. Divide through by e^{rt}, collect terms and write in matrix notation.

$$\begin{bmatrix} a_1 - r & b_1 \\ a_2 & b_2 - r \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Now, the values of A and B depend on initial conditions. But as the proposed form of the solution must be true for *any* initial conditions, the coefficient matrix on the left hand side must be singular. Equivalently, its determinant must be zero:

$$\begin{vmatrix} a_1 - r & b_1 \\ a_2 & b_2 - r \end{vmatrix} = 0 .$$

That is,

$$(a_1 - r)(b_2 - r) - a_2 b_1 = 0 ,$$

which can be rearranged to reveal an explicit quadratic equation in r

This is known as the **characteristic equation** of the system.

$$r^2 - r(a_1 + b_2) + a_1 b_2 - a_2 b_2 = 0 .$$

This characteristic equation has two roots,

$$r_1 = \frac{a_1 + b_2}{2} + \frac{1}{2}\sqrt{(a_1 + b_2)^2 - 4(a_1 b_1 - a_2 b_1)} ,$$

and

$$r_2 = \frac{a_1 + b_2}{2} - \frac{1}{2}\sqrt{(a_1 + b_2)^2 - 4(a_1 b_1 - a_2 b_1)} .$$

There are several possibilities for what these roots look like, and each of them has different implications for the stability of the system.

- Both roots are real numbers:

  a) $r_2 \leq r_1 < 0$. The system is **stable**.

  b) $r_1 \geq r_2 > 0$. The system is **unstable**.

    c) $r_1 > 0 > r_2$.   The system is **saddle path stable**.

    d) One of the roots is zero. There may or may not be an equilibrium depending on initial conditions.

- Both roots are complex numbers because $(a_1 + b_2)^2 - 4(a_1 b_1 - a_2 b_1) < 0$ (a complex number has the form $r = a + b\sqrt{-1}$, where a is the real part). The presence of complex roots gives rise to a variety of interesting behaviors

    a) Zero real parts: You get elliptical trajectories that just go around the steady state without approaching it .

    b) Negative real parts. The trajectories spiral in towards the steady state. The steady state is often known as a stable **focus**.

    b) Positive real parts. The trajectories spiral out from the steady state. The steady state is often known as an unstable **focus**.

EXERCISE 3.1 (Stability of a linear system). *Solve and assess the stability of the following differential equations:*

    (a) $\dot{x}(t) = x(t) + y(t)$ *and* $\dot{y}(t) = 4x(t) + y(t)$;

    (b) $\dot{x}(t) = -3x(t) + \sqrt{2}y(t)$ *and* $\dot{y}(t) = \sqrt{2}x(t) + -2y(t)$;

    (c) $\dot{x}(t) = -\dfrac{1}{2}x(t) + y(t)$ *and* $\dot{y}(t) = -x(t) - \dfrac{1}{2}y(t)$.

Non-Constant Coefficients

We have completed an analysis with constant coefficients, but frequently our coefficients take the form $a_1(x,y)$, and so on. In general, one cannot evaluate the global stability of such a problem. However, one can evaluate its **local stability**. To do so for a system such as

$$\dot{x}(t) = a_1(x, y)x(t) + b_1(x, y)y(t)$$

$$\dot{y}(t) = a_2(x,y)x(t) + b_2(x,y)y(t) \, ,$$

we substitute in the steady state values, $\{x^*, y^*\}$,

$$\dot{x}(t) = a_1(x^*, y^*)x(t) + b_1(x^*, y^*)y(t)$$

$$\dot{y}(t) = a_2(x^*, y^*)x(t) + b_2(x^*, y^*)y(t) \, ,$$

and then evaluate in the same way as for a constant coefficient problem.

Systems of Nonlinear Equations

So far we have analyzed single linear equations, single nonlinear equations, and systems of linear equations. When it comes to analyzing a pair of nonlinear equations, we do not come across any new conceptual problems.

- We try to limit ourselves to autonomous problems.

- We generally frame the problem so there is a steady state. Sometimes this requires a bit of reframing. Imagine, for example that the long-run solution to a growth problem has consumption, $c(t)$, and capital, $k(t)$, growing at the constant exponential rate λ. Then, we can define the variables $\overline{c}(t) = c(t)e^{-\lambda t}$ and $\overline{k}(t) = k(t)e^{-\lambda t}$. Rewriting the model in terms of $\overline{c}$ and $\overline{k}$ now gives us a model in which there is a steady state.

- We can analyze the problem graphically using exactly the same methods as for linear systems. Construct the loci of values for which each variable is constant and assess the direction of change for each variable in each section of the graph (there may be more than four distinct sections when the nonlinear loci intersect multiple times; see Exercise 3.2 below).

- We can linearize the two equations and assess analytically the local dynamics and stability of system.

EXERCISE 3.2 (Stability of a nonlinear system). *The following system has two steady states:*

$$\dot{x}(t) - x(t)^2 + y(t)$$
$$\dot{y}(t) = x(t) - y(t) + 1$$

a) *Construct the phase diagram for this system to fully characterize the system's behavior.* b) *Find the roots of the linearized system and verify your graphical characterization of the local properties of the system.*

EXERCISE 3.3 (Solow model with human capital). *Mankiw, Romer and Weil (1992) analyze the following version of the Solow model:*

$$y(t) = k(t)^\alpha h(t)^\beta \,,$$
$$\dot{k}(t) = s_k y(t) - (n + g + \delta)k(t) \,,$$
$$\dot{h}(t) = s_h y(t) - (n + g + \delta)h(t) \,.$$

where y *is output per effective unit of labor,* h *is human capital per effective unit of labor,* k *is physical capital per effective unit of labor, and* s_h *and* s_k *are the savings rates for physical and human capital.* g *is the rate of technical change,* n *of population growth and* δ *the depreciation rate.*

a) *How do the parameters of the model affect the steady state income level,* y*?*

b) *Draw the phase diagram for this model and analyze the stability of the steady state(s).*

c) *Mankiw, Romer and Weil point out that the Solow model makes quantitative predictions about the speed of convergence to the steady state. Specifically, a log linear approximation around the steady state yields*

$$\frac{d \ln(y(t))}{dt} = (n + g + \delta)(1 - \alpha - \beta)\big(\ln(y^*) - \ln(y(t))\big) \,.$$

Derive this expression formally, and interpret it.

Further Reading

Solutions to differential equations are among the first things that I forget with lack of use. It is useful therefore, to have a book or two on hand as a reference. I learnt differential equations from Boyce and DiPrima (1986), and I still think it's a very good book, especially for learning more advanced aspects of differential equations than we have covered in this brief review. For many of you, the material in these lecture notes will see you through your immediate needs. If differential equations become part of your standard toolkit for your own modeling, then studying a book such as Boyce and DiPrima is unavoidable. I also own a copy of *Cliff's Quick Review* of differential equations [Leduc (1995)], which is an excellent little reference to look up things you knew but had forgotten. It is not useful for looking up things you never knew.

References

Boyce, William E., and Richard C. DiPrima (1986): *Elementary Differential Equations and Boundary Value Problems.* 4th edition. New York: John Wiley

Cagan, Phillip (1956): "The Monetary Dynamics of Hyperinflation." In M. Friedman, ed., Studies in the Quantity Theory of Money (Chicago: University of Chicago Press).

Jones, Charles I (2000): "A Note on the Closed-Form Solution of the Solow Model." Stanford: Manuscript.

Leduc, Stephen A. (1995): Differential Equations. Lincoln, NE: Cliff's Notes, Inc.

Mankiw, N. Gregory, David Romer, and David Weil (1992): "A Contribution to the Empirics of Economic Growth." Quarterly Journal of Economics, 107(2):407-437.

Mulligan, Casey B., and Xavier Sala-i-Martin (1992): "Transition Dynamics in Two-Sector Models of Endogenous Growth." Quarterly Journal of Economics, 107:739-773.

Sargent, Thomas J. and Neil Wallace (1973): "The Stability of Models of Money and Growth with Perfect Foresight." *Econometrica*, **41**:1043-1048.