

Final Exam

For the following, make use of results established in class and as homework.

1. Show that the forgetful functor on monoids,

$$U : \mathbf{Mon} \rightarrow \mathbf{Sets}$$

is representable, i.e. that $U \cong \mathbf{Mon}(C, -)$ for a suitable monoid C . Also show that the functor $U \times U$ is representable.

2. Show that the multiplication $m_M : UM \times UM \rightarrow UM$ on any monoid M is the component at M of a natural transformation $m : U \times U \rightarrow U$. Similarly, show that the multiplicative unit $u_M : 1 \rightarrow UM$ is the component of a natural transformation $u : 1 \rightarrow U$, where the terminal object $1 : \mathbf{Mon} \rightarrow \mathbf{Sets}$ in $\mathbf{Sets}^{\mathbf{Mon}}$ is the evident constant functor. Note that this constant functor is also representable.
3. Show that $\mathcal{U} = (U, u, m)$ is a monoid in $\mathbf{Sets}^{\mathbf{Mon}}$. Moreover, any monoid M (in \mathbf{Sets}) is a functorial image of \mathcal{U} , in the sense that:

$$M \cong M^*(\mathcal{U})$$

for a functor $M^* : \mathbf{Sets}^{\mathbf{Mon}} \rightarrow \mathbf{Sets}$ that preserves monoids. (Hint: let M^* be the functor “evaluation at M ”.)

4. Show that \mathcal{U} is in the image of the Yoneda embedding,

$$y : \mathbf{Mon}^{\text{op}} \rightarrow \mathbf{Sets}^{\mathbf{Mon}}$$

Infer that there must be a “comonoid”,

$$I \xleftarrow{u} C \xrightarrow{m} C + C$$

in the category of monoids, where I is an initial object, and $C + C$ is a coproduct, in \mathbf{Mon} (a *comonoid* in a category \mathbf{C} is just a monoid in \mathbf{C}^{op}). Describe (C, u, m) explicitly in terms of specific monoids and homomorphisms.

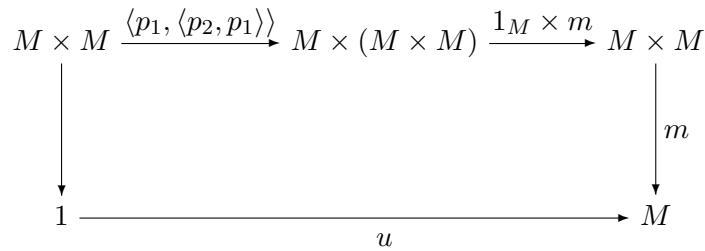
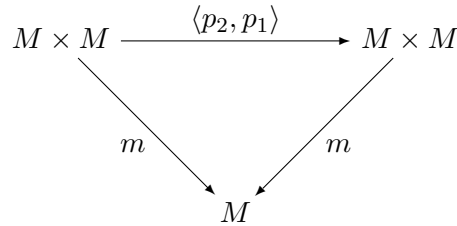
5. * Consider equations $s = t$ between terms s and t consisting of variables and the monoid operations 1 and $x \cdot y$, such as:

$$x \cdot y = y \cdot x, \quad 1 = x \cdot (y \cdot x)$$

A monoid M (in any category \mathbf{C}) is said to *satisfy* such an equation $s = t$, written:

$$M \models s = t$$

if the condition it expresses holds in that monoid, in the sense that the corresponding diagram constructed from product projections and monoid operations commutes in \mathbf{C} . For instance, the above displayed equations are satisfied in a monoid (M, u, m) if the following diagrams commute.



Show that the only equations satisfied by the monoid (C, m, u) in \mathbf{Mon}^{op} are those satisfied by all monoids in **Sets**.

The completeness of equational logic with respect to models in **Sets** says that an equation is provable $T \vdash s = t$ in the theory T of monoids iff it is satisfied by all models of T (i.e. monoids) in **Sets**. Infer from this and the foregoing that (C, m, u) models exactly the *provable* equations,

$$C \models s = t \quad \text{iff} \quad T \vdash s = t$$

We have thus shown that (for monoids), “syntax” is the categorical dual of “semantics”, in the sense that there is a copy of provability in the opposite of the category of models.