Homework 8

1. Complete the proof from the text of Kripke completeness for the positive fragement of IPC as follows:

   (a) Show that for any poset I, the exponential poset $\mathbf{2}^I$ is a Heyting algebra. (Hint: the limits and colimits are "pointwise", and the Heyting implication $p \Rightarrow q$ is defined at $i \in I$ by $(p \Rightarrow q)(i) = \top$ iff for all $j \leq i, p(j) \leq q(j)$).

   (b) Show that for any poset CCC $\mathbf{A}$, the map $y : \mathbf{A} \to \mathbf{2}^{\mathbf{A}^{\mathrm{op}}}$ defined in the text is indeed (i) monotone, (ii) injective, and (iii) preserves CCC structure.

2. Verify the claim in the text that the products $A \times B$ in categories $\mathbf{Sets}^I$ of I-indexed sets (I a poset) can be computed "pointwise". Show, moreover, that the same is true for all limits and colimits.

3. Prove that every functor $F : \mathbf{C} \to \mathbf{D}$ can be factored as

   $$\mathbf{C} \xrightarrow{E} \mathbf{E} \xrightarrow{M} \mathbf{D}$$

   in the following two ways:

   (a) $E : \mathbf{C} \to \mathbf{E}$ is bijective on objects and full, and $M : \mathbf{E} \to \mathbf{D}$ is faithful;

   (b) $E : \mathbf{C} \to \mathbf{E}$ surjective on objects and $M : \mathbf{E} \to \mathbf{D}$ is full and faithful.

   When do the two factorizations agree?

4. Using the fact that representable functors preserve limits, show that for any sets A, B, C:

   $$A^{B+C} \cong A^B \times A^C.$$

   Conclude that for any sets A, B with power sets $P(A)$, $P(B)$, etc.:

   $$P(A + B) \cong P(A) \times P(B).$$

5. * Consider the (covariant) composite functor,

   $$\mathcal{F} = \mathcal{P}^{\mathbf{BA}} \circ \mathrm{Ult}^{\mathrm{op}} : \mathbf{BA} \to \mathbf{Sets}^{\mathrm{op}} \to \mathbf{BA}$$

taking each Boolean algebra B to the powerset algebra of sets of ultrafilters in B. Note that,

$$\mathcal{F}(B) \cong \mathrm{Hom}_{\mathbf{Sets}}(\mathrm{Hom}_{\mathbf{BA}}(B, \mathbf{2}), 2)$$

is a sort of "double-dual" Boolean algebra. There is always a homomorphism,

$$\phi_B : B \to \mathcal{F}(B)$$

given by $\phi_B(b) = \{\mathcal{V} \in \mathrm{Ult}(B) \mid b \in \mathcal{V}\}$. Show that for any Boolean homomorphism $h : A \to B$, the following square commutes.