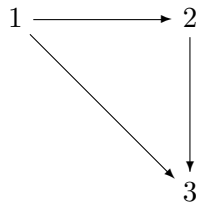
Homework 6

1. Give four different presentations by generators and relations of the category **3**, pictured:



   Is **3** free?

2. Given a congruence $\sim$ on a category $\mathbf{C}$ and arrows in $\mathbf{C}$ as pictured below,

$$A \underset{f'}{\overset{f}{\rightrightarrows}} B \underset{g'}{\overset{g}{\rightrightarrows}} C$$

   show that $f \sim f'$ and $g \sim g'$ implies $g \circ f \sim g' \circ f'$.

3. (a) Consider the sequence of posets $[0] \to [1] \to [2] \to \ldots$, where

$$[n] = \{0 \leq \cdots \leq n\},$$

   and the arrows $[n] \to [n+1]$ are the evident inclusions. Determine the limit and colimit posets of this sequence.

   (b) Do the same for the sequence of powerset boolean algebras,

$$\mathcal{P}0 \leftarrow \mathcal{P}1 \leftarrow \mathcal{P}2 \leftarrow \ldots,$$

   where the maps are determined by inverse image along the inclusions $0 \subseteq 1 \subseteq 2 \subseteq \ldots$.

4. Consider sequences of monoids,

$$M_0 \to M_1 \to M_2 \to \ldots$$
$$N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \ldots$$

   and the following limits and colimits, constructed in the category of monoids:

$$\varinjlim_n M_n, \quad \varprojlim_n M_n, \quad \varinjlim_n N_n, \quad \varprojlim_n N_n.$$

(a) Suppose all M_n and N_n are abelian groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. is also an abelian group.

(b) Suppose all M_n and N_n are finite groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. has the following property: for every element x there is a number k such that $x^k = 1$ (the least such k is called the *order* of x).

5. * Given functors $F, G : \mathbf{C} \to \mathbf{D}$ such that for all $C \in \mathbf{C}$, $FC = GC$, define a congruence on $\mathbf{D}$ by the condition:

$$\begin{aligned} f \sim g \quad \text{iff} \quad & \mathrm{dom}(f) = \mathrm{dom}(g) \\ & \& \ \mathrm{cod}(f) = \mathrm{cod}(g) \\ & \& \ \forall \mathbf{E}, H : \mathbf{D} \to \mathbf{E}.\ HF = HG \Rightarrow H(f) = H(g) \end{aligned}$$

Prove that this is a congruence.

Prove that $\mathbf{C}/\!\sim$ is the coequalizer of F and G.