

Homework 5

1. (a) Complete the proof that the groups in the category of groups are exactly the abelian groups by showing that any abelian group admits homomorphic group operations.
 - (b) Use the Eckmann-Hilton argument to prove that every monoid in the category of groups is an internal group.
2. Let \mathbf{C} be a category with pullbacks.
 - (a) Show that an arrow $m : M \rightarrow X$ in \mathbf{C} is monic if and only if the diagram below is a pullback.

$$\begin{array}{ccc}
 M & \xrightarrow{1_M} & M \\
 \downarrow 1_M & & \downarrow m \\
 M & \xrightarrow{m} & X
 \end{array}$$

Thus as an object in \mathbf{C}/X , m is monic iff $m \times m \cong m$.

- (b) Show that the pullback along an arrow $f : Y \rightarrow X$ of a pullback square over X ,

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & X
 \end{array}$$

is again a pullback square over Y . (Hint: draw a cube and use the 2-pullbacks Lemma). Conclude that the pullback functor f^* preserves products.

- (c) Conclude from the foregoing that in a pullback square

$$\begin{array}{ccc}
 M' & \longrightarrow & M \\
 \downarrow m' & & \downarrow m \\
 A' & \xrightarrow{f} & A
 \end{array}$$

if m is monic, then so is m' .

3. Let \mathbf{C} be a locally small category, and $D : \mathbf{J} \rightarrow \mathbf{C}$ any diagram for which a limit exists in \mathbf{C} . Show that for any object $C \in \mathbf{C}$, the representable functor

$$\text{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \rightarrow \mathbf{Sets}$$

preserves the limit of D .

4. (Pushouts)
 - (a) Dualize the definition of a pullback to define the “copullback” (usually called the “pushout”) of two arrows with common domain.
 - (b) Indicate how to construct pushouts using coproducts and coequalizers (proof “by duality”).
5. Suppose the category \mathbf{C} has limits of type \mathbf{J} , for some index category \mathbf{J} . For diagrams F and G of type \mathbf{J} in \mathbf{C} , a morphism of diagrams $\theta : F \rightarrow G$ consists of arrows $\theta_i : Fi \rightarrow Gi$ for each $i \in \mathbf{J}$ such that for each $\alpha : i \rightarrow j$ in \mathbf{J} , one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square). This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).

Show that taking the vertex-objects of limiting cones determines a functor:

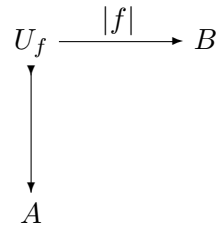
$$\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \rightarrow \mathbf{C}$$

Conclude that for any set I , there is a product functor,

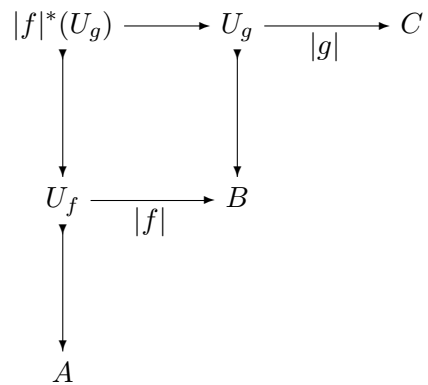
$$\prod_{i \in I} : \mathbf{Sets}^I \rightarrow \mathbf{Sets}$$

for I -indexed families of sets $(A_i)_{i \in I}$, generalizing the binary product functor $\times : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$.

6. * (Partial maps) For any category \mathbf{C} with pullbacks, define the category $\mathbf{Par}(\mathbf{C})$ of partial maps in \mathbf{C} as follows: the objects are the same as those of \mathbf{C} , but an arrow $f : A \rightarrow B$ is a pair $(|f|, U_f)$ where $U_f \rightarrow A$ is a subobject (an equivalence class of monomorphisms) and $|f| : U_f \rightarrow B$ (take a suitably-defined equivalence class of arrows), as indicated in the diagram:



Composition of $(|f|, U_f) : A \rightarrow B$ and $(|g|, U_g) : B \rightarrow C$ is given by taking a pullback and then composing to get $(|g \circ f|, |f|^*(U_g))$, as suggested by the follow diagram.



Check to see that this really does define a category.