Homework 4

1. Consider the category of proofs of a natural deduction system with disjunction introduction and elimination rules. Identify proofs under the equations

$$[p, q] \circ i_1 = p, \qquad [p, q] \circ i_2 = q$$
$$[i_1, i_2] = 1_{A \vee B}$$

   by passing to equivalence classes of proofs with respect to the equivalence relation generated by these equations (i.e. two proofs are equivalent if you can get one from the other by removing all such "detours"). Show that the resulting category does indeed have coproducts.

2. Show that in any category with coproducts, the coproduct of two projectives is again projective.

3. In the text it is shown that any monoid M has a specific presentation $T^2M \rightrightarrows TM \to M$ as a coequalizer of free monoids. Show that coequalizers of this particular form are preserved by the forgetful functor **Mon** $\to$ **Sets**.

4. Consider the category of sets.

   (a) Given a function $f : A \to B$, describe the equalizer of the functions $f \circ p_1, f \circ p_2 : A \times A \to B$ as a (binary) relation on A and show that it is an equivalence relation (called the *kernel* of f).

   (b) Show that the kernel of the quotient $A \to A/R$ by an equivalence relation R is R itself.

   (c) Given *any* binary relation $R \subseteq A \times A$, let $\langle R \rangle$ be the equivalence relation on A generated by R (the least equivalence relation on A containing R). Show that the quotient $A \to A/\langle R \rangle$ is the coequalizer of the two projections $R \rightrightarrows A$.

   (d) Using the foregoing, show that for any binary relation R on a set A, one can characterize the equivalence relation $\langle R \rangle$ generated by R as the kernel of the coequalizer of the two projections of R.

   (e) Prove that **Sets** has all coequalizers by constructing the coequalizer of a parallel pair of functions,

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \longrightarrow Q = B/(f = g)$$

by quotienting B by a suitable equivalence relation R on B, generated by the pairs $(f(x), g(x))$ for all $x \in A$.

5. * Show that the category of monoids has all coequalizers as follows.

   1. Given any pair of monoid homomorphisms $f, g : M \to N$, show that the following equivalence relations on N agree:

      a) $n \sim n' \Leftrightarrow$ for all monoids X and homomorphisms $h : N \to X$, one has $hf = hg$ implies $hn = hn'$,

      b) the intersection of all equivalence relations $\sim$ on N satisfying $fm \sim gm$ for all $m \in M$ as well as:

      $$n \sim n' \text{ and } m \sim m' \Rightarrow n \cdot m \sim n' \cdot m'$$

   2. Taking $\sim$ to be the equivalence relation defined in (1), show that the quotient set $N/\sim$ is a monoid under $[n] \cdot [m] = [n \cdot m]$, and the projection $N \to N/\sim$ is the coequalizer of f and g.