Homework 3

1. In the category of types $\mathbf{C}(\lambda)$ of the λ-calculus, determine the product functor $A, B \mapsto A \times B$ explicitly, and verify directly that it is a functor. Also show that, for any fixed type A, there is a functor $A \to (-) : \mathbf{C}(\lambda) \to \mathbf{C}(\lambda)$, taking any type X to $A \to X$.

2. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ from monoids to sets is representable. Infer that U preserves all (small) products.

3. Dualize the notion of projectivity to define an *injective* object in a category. Show that a map of posets is monic iff it is injective on elements. Give examples of a poset that is injective and one that is not injective.

4. In any category $\mathbf{C}$, show that

$$A \xrightarrow[c_1]{} C \xleftarrow[c_2]{} B$$

is a coproduct diagram just if for every object Z, the map

$$\begin{aligned} \mathrm{Hom}(C, Z) &\longrightarrow \mathrm{Hom}(A, Z) \times \mathrm{Hom}(B, Z) \\ f &\longmapsto \quad \langle f \circ c_1,\ f \circ c_2 \rangle \end{aligned}$$

is an isomorphism. Do this by using duality and the corresponding fact about products, which you may take as given.

5. Show in detail that the free monoid functor M preserves coproducts: for any sets A, B:

$$M(A) + M(B) \cong M(A + B) \qquad \text{(canonically)}$$

Do this as indicated in the text by using the UMPs of the coproducts $A + B$ and $M(A) + M(B)$ and of free monoids.

6. * Verify that the construction given in the text of the coproduct of monoids $A + B$ as a quotient of the free monoid $M(|A| + |B|)$ really is a coproduct in the category of monoids.