Homework 12

1. For any set I with powerset $P(I)$, consider the functor $i : P(I) \to \mathbf{Sets}/I$ that takes a subset $U \subseteq I$ to its inclusion function $i(U) : U \to I$. Show that this is indeed a functor and that is has a left adjoint

$$\sigma : \mathbf{Sets}/I \longrightarrow P(I).$$

   Does it have a right adjoint (in general)? Determine the units and counits.

2. Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by setting: $A \leq B$ if and only if there is an arrow $A \to B$. Show that the functor P is (left? right?) adjoint to the evident inclusion functor of preorders into categories. Determine the units and counits. Does the inclusion also have an adjoint on the other side?

3. (a) A *coHeyting algebra* is a poset P such that P^{op} is a Heyting algebra. Determine the coHeyting implication operation a/b in a lattice L by adjointness (with respect to joins), and show that any Boolean algebra is a coHeyting algebra by explicitly defining this operation a/b in terms of the usual Boolean ones.

   (b) In a coHeyting algebra, there are operations of *coHeyting negation* $\sim p = 1/p$ and *coHeyting boundary* $\partial p = p \wedge \sim p$. State the logical rules of inference for these operations.

   (c) A *biHeyting algebra* is a lattice that is both Heyting and coHeyting. Give an example of a biHeyting algebra that is not Boolean. (Hint: consider the lower sets in a poset.)

4. (a) Let $\mathbf{C}$ be a category and $T : \mathbf{C} \to \mathbf{C}$ an endofunctor. A T-*algebra* consists of an object A and an arrow $a : TA \to A$ in $\mathbf{C}$. A morphism $h : (a, A) \to (b, B)$ of T-algebras is a morphism $h : A \to B$ in $\mathbf{C}$ such that $h \circ a = b \circ T(h)$.

$$\begin{array}{ccc} TA & \xrightarrow{Th} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[h]{} & B \end{array}$$

Let **C** be a category with a terminal object 1 and binary coproducts. Let $T : \mathbf{C} \to \mathbf{C}$ be the functor

$$T(C) = 1 + C$$

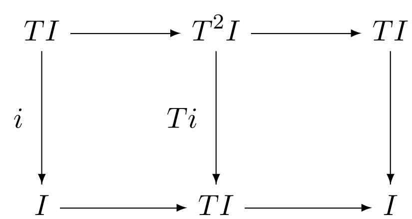
Show that the category of T-algebras in this sense is equivalent to the category of $\mathbb{T}$-algebras (i.e. models) for the algebraic theory $\mathbb{T}$ with just one nullary and one unary operation,

$$T\text{-}\mathbf{Alg} \simeq \mathbb{T}\text{-}\mathbf{Alg}.$$

Conclude that free T-algebras exist in **Sets**, and that an initial T-algebra is the same thing as an NNO.

5. ("Lambek's Lemma") Show that for any endofunctor $T : \mathbf{C} \to \mathbf{C}$, if $i : TI \to I$ is an initial T-algebra, then i is an isomorphism.

   Hint: Consider a diagram of the following form, with suitable arrows.



   Conclude that for any NNO N in any category, there is an isomorphism $N + 1 \cong N$.

6. * (Freyd's characterization of NNOs)
   Let $1 \xrightarrow{0} N \xrightarrow{s} N$ be a natural numbers object in **Sets** (for your information, however, the following holds in any topos).

   (a) We've already shown that $N \cong 1 + N$, i.e. the following is a coproduct diagram:

   $$1 \xrightarrow{\;0\;} N \xleftarrow{\;s\;} N$$

   (b) Prove that the following is a coequalizer:

   $$N \underset{1_N}{\overset{s}{\rightrightarrows}} N \longrightarrow 1$$

   (c) Show that any structure $1 \xrightarrow{0} N \xrightarrow{s} N$ satisfying the foregoing two conditions is a natural numbers object.