Homework 11

1. Show that if P is a poset and $A : \mathrm{P}^{\mathrm{op}} \to \mathbf{Sets}$ a presheaf on P, then the category of elements $\int_{\mathrm{P}} A$ is also a poset and the projection $\pi : \int_{\mathrm{P}} A \to \mathrm{P}$ is a monotone map.

   Show, moreover, that the assignment $A \mapsto (\pi : \int_{\mathrm{P}} A \to \mathrm{P})$ determines a functor,
   $$\int_{\mathrm{P}} : \mathbf{Sets}^{\mathrm{P}^{\mathrm{op}}} \longrightarrow \mathbf{Pos}/\mathrm{P}.$$

2. Show that every slice category $\mathbf{Sets}/X$ is cartesian closed. Calculate the exponential of two objects $A \to X$ and $B \to X$ by first determining the Yoneda embedding $y : X \to \mathbf{Sets}^X$, and then applying the formula for exponentials of presheaves. Finally, observe that $\mathbf{Sets}/X$ is a topos, and determine its subobject classifier.

   The next problems give examples of the notion of *adjoint functor*.

3. Let $\mathbf{C}$ be any cartesian closed category. Show that for any three objects A, B, C the isomorphism
   $$\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C^B)$$
   is natural in both A and C. This means that the functor $(-) \times B : \mathbf{C} \to \mathbf{C}$ is left adjoint to the functor $(-)^B : \mathbf{C} \to \mathbf{C}$.

4. Let $F : \mathbf{Sets} \to \mathbf{Mon}$ be the free monoid functor and $U : \mathbf{Mon} \to \mathbf{Sets}$ the forgetful functor. Show that for any set X and monoid M, there is an isomorphism
   $$\mathrm{Hom}(F(X), M) \cong \mathrm{Hom}(X, U(M))$$
   which is natural in both X and M. This means that the functor F is left adjoint to the functor U.

5. Let $\mathbf{C}$ be any category and consider the "diagonal functor"
   $$\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}.$$
   Show that Δ has a right adjoint if $\mathbf{C}$ has binary products. Thus consider the isomorphism,
   $$\mathrm{Hom}(\Delta(X), (A, B)) \cong \mathrm{Hom}(X, A \times B),$$
   which, if natural in both X and (A, B), says that Δ is left adjoint to $\times$.

6. * Show that the contravariant powerset functor $\mathcal{P} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is self-adjoint.