Homework 10

1. Let $\mathbf{C}$ be a locally small category with binary products. Show that the Yoneda embedding

$$y\colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

preserves them. (Hint: this involves only a few lines of calculation.)

2. (a) Let $\mathbf{C}$ be a locally small, cartesian closed category. Use the Yoneda embedding to show that for any objects A, B, C in $\mathbf{C}$:

$$(A \times B)^C \cong A^C \times B^C$$

   (b) Show that if $\mathbf{C}$ also has binary coproducts, then:

$$A^{(B+C)} \cong A^B \times A^C$$

3. Let $\mathbf{C}$ be a small category. Prove that the representable functors *generate* the diagram category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, in the following sense: given any objects $P, Q \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and natural transformations $\varphi, \psi : P \to Q$, if for every representable functor yC and natural transformation $\vartheta : yC \to P$ one has $\varphi \circ \vartheta = \psi \circ \vartheta$, then $\varphi = \psi$. Thus the arrows in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ are determined by their effect on generalized elements based at representables.

4. (a) Let $\mathbf{C}$ be any category and $\mathbf{D}$ any complete category. Show that the functor category $\mathbf{D}^{\mathbf{C}}$ is also complete.

   (b) Use duality to show that the same is true for cocompleteness in place of completeness.

5. * Let $\mathbb{T}$ be a theory in the lambda calculus. For any type symbols σ and τ let,

$$[\sigma \to \tau] = \{M : \sigma \to \tau \mid M \text{ closed}\}$$

be the set of closed terms of type $\sigma \to \tau$. Suppose that for each type symbol ρ, there is a function,

$$f_\rho : [\rho \to \sigma] \to [\rho \to \tau]$$

with the following properties:

- for any closed terms $M, N : \rho \to \sigma$, if $\mathbb{T} \vdash M = N$ (provable equivalence from $\mathbb{T}$), then $f_\rho M = f_\rho N$,
- for any closed terms $M : \mu \to \nu$ and $N : \nu \to \sigma$,

$$\mathbb{T} \vdash f_\mu(\lambda x : \mu.N(Mx)) = \lambda x : \mu.(f_\nu(N))(Mx)$$

Use the Yoneda embedding of the cartesian closed *category of types* $\mathbf{C}_\mathbb{T}$ of $\mathbb{T}$ to show that there is a term $F : \sigma \to \tau$ such that f_ρ is induced by composition with F, in the sense that, for every closed term $R : \rho \to \sigma$,

$$\mathbb{T} \vdash f_\rho(R) = \lambda x : \rho.F(Rx)$$

Show that, moreover, F is unique up to $\mathbb{T}$-provable equivalence.