Homework 1

1. The objects of **Rel** are sets, and an arrow $f : A \to B$ is a relation from A to B, that is, a subset $f \subseteq A \times B$. The equality relation $\{\langle a, a\rangle \in A \times A |\ a \in A\}$ is the identity arrow on a set A. Composition in **Rel** is to be given by

$$g \circ f = \{\langle a, c\rangle \in A \times C \mid \exists b\,(\langle a, b\rangle \in f \ \&\ \langle b, c\rangle \in g)\}$$

for $f \subseteq A \times B$ and $g \subseteq B \times C$.

Show that **Rel** is a category. Show also that there is a functor G : **Sets** $\to$ **Rel** taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a)\rangle \in A \times B \mid a \in A\}.$$

2. Let $2 = \{a, b\}$ be any set with exactly 2 elements a and b. Define a functor F : **Sets**/2 $\to$ **Sets** $\times$ **Sets** with $F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analogous situation with a one element set $1 = \{a\}$ instead of 2?

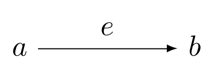
3. Any category **C** determines a preorder $P(\mathbf{C})$ by defining a binary relation $\leq$ on the objects by:

$$A \leq B \text{ if and only if there is an arrow } A \to B$$

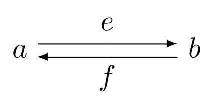
Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and checking the required conditions. Show that P is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

4. Describe the free categories on the following graphs by determining their objects, arrows, and composition operations.

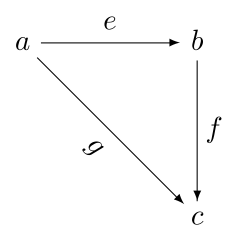
(a)



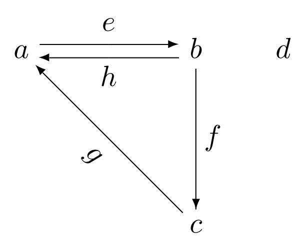
(b)



(c)



(d)



5. Show that the free monoid functor

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists, in two different ways:

(a) Assume the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 \dots a_k) = f(a_1) \dots f(a_k), \qquad a_1, \dots a_k \in A.$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Reflect on how these two approaches are related.

6. * Use the Cayley representation to show that every small category is isomorphic to a "concrete" one, i.e. one in which the objects are sets and the arrows are functions between them.