Practice Midterm Exam

This practice test will not be collected or scored. You may use one page (front and back) of notes in solving the following problems.

1. Let A be the subset of natural numbers defined inductively by the following clauses:

   - $1 \in A$
   - If $n \in A$, then $2n \in A$
   - If $n \in A$, then $2n + 6 \in A$

   (a) Show that 32 is an element of A.

   (b) Show that 87 is not an element of A.

   (c) Is A freely generated? Justify your answer.

2. Remember that if φ and ψ are any propositional formulas and p is a propositional variable, then $\varphi[\psi/p_i]$ is defined by recursion, as follows:

   - If φ is a propositional variable p, then
   $$\varphi[\psi/p_i] = \begin{cases} \psi & \text{if } p_i = p \\ p & \text{otherwise} \end{cases}$$
   - If φ is of the form $(\theta \,\square\, \eta)$, then
   $$\varphi[\psi/p_i] = (\theta[\psi/p_i] \,\square\, \eta[\psi/p_i]).$$
   - If φ is of the form $(\neg\theta)$, then
   $$\varphi[\psi/p_i] = (\neg\theta[\psi/p_i]).$$

   (a) Show that if φ and ψ are any two formulas, and v and v' are truth value assignments such that
   $$v'(p_j) = \begin{cases} [\![\psi]\!]_v & \text{if } i = j \\ v(p_j) & \text{otherwise} \end{cases}$$
   then
   $$[\![\varphi[\psi/p_i]]\!]_v = [\![\varphi]\!]_{v'}.$$

(b) What does it mean for a propositional formula φ to be *valid*?

(c) Show (using parts a and b) that if φ is valid then so is $\varphi[\psi/p_i]$.

3. Use algebraic means to put the following formula into disjunctive and conjunctive normal forms:

$$\neg((\neg p \to q) \wedge (r \to (q \vee s)))$$

Is the formula valid? Why or why not?

4. Use truth tables or a semantic argument to show the following.

   - $((p \vee r) \to (q \to s)) \to ((p \wedge q \to s) \wedge (r \wedge q \to s))$ is a tautology.
   - It is not the case that $p \vee q \models p \to q$

5. Give natural deduction proofs of the following:

   - $\vdash (\varphi \vee \neg\varphi)$
   - $(\varphi \vee \psi) \vdash \neg(\neg\varphi \wedge \neg\psi)$
   - $(\neg\varphi \vee \psi) \vdash \varphi \to \varphi \wedge \psi$

6. State the Soundness and Completeness Theorems (and say which is which).

7. What does it mean to say that a set Γ of propositional formulas is *maximally consistent*?

   Show that if Γ is maximally consistent and $\Gamma \vdash \varphi$, then φ is in Γ.

8. Show that the following set of sentences is consistent:

$$\{p \to q, \neg r \to p, \neg p\}$$

$\star$ 9. Suppose that for formulas φ and ψ some truth valuation v makes both of them true. Show that then it is not the case that $\varphi \vdash \neg\psi$.