Practice Midterm Exam

This practice test will not be collected or scored. You may use one page (front and back) of notes in solving the following problems.

1. Let $A$ be the subset of natural numbers defined inductively by the following clauses:
   - $1 \in A$
   - If $n \in A$, then $2n \in A$
   - If $n \in A$, then $2n + 6 \in A$

   (a) Show that 32 is an element of $A$.
   (b) Show that 87 is not an element of $A$.
   (c) Is $A$ freely generated? Justify your answer.

2. Remember that if $\varphi$ and $\psi$ are any propositional formulas and $p$ is a propositional variable, then $\varphi[\psi/p_i]$ is defined by recursion, as follows:
   - If $\varphi$ is a propositional variable $p$, then
     $$\varphi[\psi/p_i] = \begin{cases} \psi & \text{if } p_i = p \\ p & \text{otherwise} \end{cases}$$
   - If $\varphi$ is of the form $(\theta \square \eta)$, then
     $$\varphi[\psi/p_i] = (\theta[\psi/p_i] \square \eta[\psi/p_i]).$$
   - If $\varphi$ is of the form $(-\theta)$, then
     $$\varphi[\psi/p_i] = (-\theta[\psi/p_i]).$$

   (a) Show that if $\varphi$ and $\psi$ are any two formulas, and $v$ and $v'$ are truth value assignments such that
   $$v'(p_j) = \begin{cases} [\psi]_v & \text{if } i = j \\ v(p_j) & \text{otherwise} \end{cases}$$

   then
   $$[\varphi[\psi/p_i]]_v = [\varphi]_{v'}.$$
(b) What does it mean for a propositional formula \( \varphi \) to be valid?
(c) Show (using parts a and b) that if \( \varphi \) is valid then so is \( \varphi[\psi/p_i] \).

3. Use algebraic means to put the following formula into disjunctive and conjunctive normal forms:

\[-((\neg p \to q) \land (r \to (q \lor s)))\]

Is the formula valid? Why or why not?

4. Use truth tables or a semantic argument to show the following.
   - \(((p \lor r) \to (q \to s)) \to ((p \land q \to s) \land (r \land q \to s))\) is a tautology.
   - It is not the case that \( p \lor q \models p \to q \)

5. Give natural deduction proofs of the following:
   - \( \vdash (\varphi \lor \neg \varphi) \)
   - \( (\varphi \lor \psi) \vdash \neg (\neg \varphi \land \neg \psi) \)
   - \( (\neg \varphi \lor \psi) \vdash \varphi \land \psi \)

6. State the Soundness and Completeness Theorems (and say which is which).

7. What does it mean to say that a set \( \Gamma \) of propositional formulas is maximally consistent?
   Show that if \( \Gamma \) is maximally consistent and \( \Gamma \vdash \varphi \), then \( \varphi \) is in \( \Gamma \).

8. Show that the following set of sentences is consistent:

\[ \{p \to q, \neg r \to p, \neg p\} \]

9. Suppose that for formulas \( \varphi \) and \( \psi \) some truth valuation \( v \) makes both of them true. Show that then it is not the case that \( \varphi \vdash \neg \psi \).