Homework 9

Due Thursday, November 9

1. The language of *additive monoids* has a constant symbol 0 and a binary function symbol, written $x + y$. The axioms are associativity $\forall x \forall y \forall z\ (x+(y+z) = (x+y)+z)$; commutativity $\forall x \forall y\ (x+y = y+x)$; and 0 is a (two-sided) unit $\forall x\ (0 + x = x)$ and $\forall x\ (x + 0 = x)$.

   Define the *even* elements by

   $$E(x) =_{\mathsf{df}} \exists z(x = z + z).$$

   Use deduction to show that the sum of two even elements is even.

2. Fix a language, L, with two, one-place predicate symbols A and B.

   (a) For each of the following formulas, find a (simple!) interpretation that makes it true and one that makes it false. Justify your answers.

      i. $\forall x(A(x) \vee B(x))$
      ii. $\forall x A(x) \vee \forall x B(x)$
      iii. $\exists x(A(x) \vee B(x))$
      iv. $\exists x A(x) \vee \exists x B(x)$

   (b) Do the same for $\wedge$ in place of $\vee$.

3. Let L be a language with two unary predicates, A and B. Consider the equivalence

   $$\forall x(A(x) \vee B(x)) \leftrightarrow \forall x A(x) \vee \forall x B(x).$$

   (a) Show that one direction is valid. In particular, your answer should make it clear that you know what "valid" means!

   (b) Show that the other direction is not valid.

4. Let L be any language. Which of the following statements are true and which are false? Justify your answers.

   (a) If φ is any sentence, either $\models \varphi$ or $\models \neg\varphi$.

(b) If φ is any sentence and $\mathcal{A}$ is any L-structure, either $\mathcal{A} \models \varphi$ or $\mathcal{A} \models \neg\varphi$.

(c) If φ is any sentence and Γ is any set of sentences, then either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$.

(d) If φ and ψ are any sentences, $\models \varphi \wedge \psi$ implies $\models \varphi$ and $\models \psi$.

(e) If φ and ψ are any sentences, $\models \varphi \vee \psi$ implies $\models \varphi$ or $\models \psi$.

$\star$ 5. Show that the definition of existential quantifier in terms of negation and universal quantification is semantically valid by proving

$$\models \exists\varphi \leftrightarrow \neg\forall\neg\varphi$$

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