

HOMEWORK 9

Due Tuesday, November 29

1. Prove that any set A is at most the size of the set A^A of all functions from A to A , and that it is the same size in exactly one case (which?).
2. Let $L = (R, \dots, f, \dots, c, \dots)$ be a fixed language. Two L -structures \mathcal{A} and \mathcal{B} are *isomorphic* if there is a bijective function $F : A \rightarrow B$ such that:
 - (a) $a \in R^{\mathcal{A}}$ if and only if $F(a) \in R^{\mathcal{B}}$ for all $a \in A$ (and that for all relation symbols R , and similarly for R of higher arity than one).
 - (b) $F(f^{\mathcal{A}}(a)) = f^{\mathcal{B}}(F(a))$ for all $a \in A$ (and that for all function symbols f , and similarly for f of higher arity than one).
 - (c) $F(c^{\mathcal{A}}) = c^{\mathcal{B}}$ for all constant symbols c .

Show that for isomorphic \mathcal{A} and \mathcal{B} and any L -sentence φ ,

$$\mathcal{A} \models \varphi \quad \text{iff} \quad \mathcal{B} \models \varphi.$$

Conclude that first-order logic cannot distinguish between isomorphic structures.

3. A theory \mathbb{T} is said to be *categorical* if all of its models are isomorphic. Show that a categorical theory can have only finite models. Give an example of such a theory.
4. The language of *linear orders with endpoints* has two constant symbols $0, 1$ and a binary relation symbol, written $x \leq y$. The axioms for linear orders with endpoints are:
 - reflexivity: $\forall x (x \leq x)$,
 - transitivity: $\forall x, y, z ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$,
 - antisymmetry: $\forall x, y ((x \leq y \wedge y \leq x) \rightarrow x = y)$,
 - linearity: $\forall x, y (x \leq y) \vee (y \leq x)$,
 - endpoints: $\forall x (0 \leq x) \wedge (x \leq 1)$.

Consider the following models of the theory of linear orders with endpoints:

$$\mathcal{I} = ([0, 1] \subseteq \mathbb{R}, 0, 1, \leq)$$

(the usual ordering of unit interval in the reals)

$$\mathcal{N} = (\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$$

(the usual ordering of the natural numbers,
but with a new element ∞ added at infinity)

$$\mathcal{Z} = (\mathbb{Z} \cup \{-\infty, +\infty\}, -\infty, +\infty, \leq)$$

(the usual ordering of the integers,
but with new elements $-\infty, +\infty$ added at both ends)

- (a) Show that these models are distinguishable in first-order logic by producing three sentences, each of which satisfied by one, but not the others.
- (b) Can there be a model that satisfies all the same first-order sentences as \mathcal{N} and is uncountable? Justify your answer!
- (c) Can there be a model that satisfies all the same first-order sentences as \mathcal{I} and is countable? Justify your answer!
- ★(d) (for Grad Students) Prove that \mathcal{N} and \mathcal{Z} are not isomorphic.