

HOMEWORK 6  
Practice Midterm Exam

1. Let  $A$  be the subset of natural numbers defined inductively by the following clauses:

- $1 \in A$
- If  $n \in A$ , then  $2n \in A$
- If  $n \in A$ , then  $3n + 1 \in A$

- (a) Show that 28 is an element of  $A$ .
- (b) Show that 87 is not an element of  $A$ .
- (c) Is  $A$  freely generated? Justify your answer.

2. Remember that if  $\varphi$  and  $\psi$  are any propositional formulas and  $p$  is a propositional variable, then  $\varphi[\psi/p_i]$  is defined by recursion, as follows:

- If  $\varphi$  is a propositional variable  $p$ , then

$$\varphi[\psi/p_i] = \begin{cases} \psi & \text{if } p_i = p \\ p & \text{otherwise} \end{cases}$$

- If  $\varphi$  is of the form  $(\theta \square \eta)$ , then

$$\varphi[\psi/p_i] = (\theta[\psi/p_i] \square \eta[\psi/p_i]).$$

- If  $\varphi$  is of the form  $(\neg\theta)$ , then

$$\varphi[\psi/p_i] = (\neg\theta[\psi/p_i]).$$

(a) Show that if  $\varphi$  and  $\psi$  are any two formulas, and  $v$  and  $v'$  are truth value assignments such that

$$v'(p_j) = \begin{cases} \llbracket \psi \rrbracket_v & \text{if } i = j \\ v(p_j) & \text{otherwise} \end{cases}$$

then

$$\llbracket \varphi[\psi/p_i] \rrbracket_v = \llbracket \varphi \rrbracket_{v'}.$$

- (b) What does it mean for a propositional formula  $\varphi$  to be *valid*?
- (c) Show (using parts a and b) that if  $\varphi$  is valid then so is  $\varphi[\psi/p_i]$ .

3. Use algebraic means to put the following formula into disjunctive normal form:

$$\neg((\neg p \rightarrow q) \wedge (r \rightarrow (q \vee s)))$$

4. Use truth tables or a semantic argument to show the following.

- $((p \vee r) \rightarrow q) \rightarrow ((p \rightarrow q) \wedge (r \rightarrow q))$  is a tautology.
- It is not the case that  $p \vee q \models p \rightarrow q$

5. Give natural deduction proofs of the following:

- $\vdash (\varphi \vee \neg\varphi)$
- $(\varphi \vee \psi) \vdash \neg(\neg\varphi \wedge \neg\psi)$
- $(\neg\varphi \vee \psi), \vdash \phi \rightarrow \phi \wedge \psi$

6. State the Soundness and Completeness Theorems (and say which is which).

7. What does it mean to say that a set  $\Gamma$  of propositional formulas is *maximally consistent*?

Show that if  $\Gamma$  is maximally consistent and  $\Gamma \vdash \varphi$ , then  $\varphi$  is in  $\Gamma$ .

8. Show that the following set of sentences is consistent:

$$\{p \rightarrow q, \neg r \rightarrow p, \neg p\}$$

- ★ 9. Suppose that for formulas  $\varphi$  and  $\psi$  some truth valuation  $v$  makes both of them true. Show that then it is not the case that  $\varphi \vdash \neg\psi$ .